Two Theories of "Theory" in Mathematics Education: Using Kuhn and Lakatos to Examine Four Foundational Issues

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Meaningful inquiry is always guided by a theory The theory may be a refined, highly predictive calculus, as it is in physics, or it may be a rough, tentative collection of hunches, as it often is in education. When a mathematics educator studies the effects of lax and restrictive learning environments on children of different anxiety levels, she presumably has a theory that relates achievement to both anxiety and the structure of the learning environment. Or when a cognitive psychologist examines classification and seriation tasks in the learning of early number concepts, the psychologist most likely has a hunch as to how these tasks are related Or, when a doctoral candidate designs an experiment in which children are taught several different problem solving heuristics, she presumably has a theory that predicts which of these treatments will be the most effective.

A good theory specifies how evidence for that theory is to be interpreted Problems arise, however, when different theories interpret what appears to be the same evidence in different ways Assume, for example, that a researcher finds "significant results" when comparing an instructional program that diagnoses and remediates students' errors with a program that just reteaches correct procedures Proponents of one theory may interpret these results as showing that the diagnosis helped students rebuild or "debug" their cognitive structures. Another researcher might reject this interpretation The observed "improvement" was merely a side effect of the fact that the instruction helped students to forget their faulty procedures Unlearning is not a matter of rebuilding cognitive structure; " 'unlearning' ... is a matter of extinction." "This means that teachers would best ignore the incorrect performances and set about as directly as possible teaching the rules for correct ones " [Gagné, 1983a, p 12]

When different theories interpret what appears to be the same evidence in different ways, it becomes difficult to make a *rational choice* between them. Consider a second scenario: A researcher conducts several teaching experiments in which children discover how to perform subtraction problems using structured rods After several weeks of work, these children are then taught the symbolic algorithm. These children show "greater understanding" of the algorithm, as compared to children who are taught without using the manipulatives. This result is interpreted to support the theory that "children develop constructive thinking long before analytical thinking." In other words, "although children may not be ready to make logical judgements, they are well able to build mathematical concepts." [Dienes, 1960, p 30] Proponents of a different theory might interpret these results differently. Of course children *can* be taught "the additional translation task of turning ... numerals into arrays of sticks or piles of blocks." However, manipulating blocks is more abstract than manipulating symbols. There is no advantage in turning a "concrete computational task into an abstract one, so that it can be properly 'understood'." [Gagné, 1983a, p 12]

The problem of rationally choosing between theories has been addressed in at least two ways by philosophers of science. T. Kuhn [1962] argues that rational choice is often just not possible. Proponents of different theories "interpret the world differently." Criteria that might be used to judge a theory are themselves part of a biased paradigm A neutral ground does not exist. A second solution, proposed by I. Lakatos [1970], holds out the possibility for rational choice between theories Though the different theories may represent different paradigms,there still exist criteria which enable one to judge a program as either "progressive" or "degenerative "

In the following discussion, the ideas of Kuhn and Lakatos will be used to study four foundational issues in mathematics education. These issues are related to matters of values, units of analysis, theory of mind, and nature of mathematical entities. Proponents of current theories in mathematics education (information processing psychology, constructivism) tend to make assumptions regarding these four issues This will be pointed out below.

The goal of this paper is not to determine which assumptions regarding the four issues are "right" It is rather to determine whether the differences between the assumptions are best understood in Kuhnian or Lakatosian terms.

The discussion is divided into three parts Part I will explain the four foundational issues These issues will be introduced using a "thought experiment", a hypothetical research problem. Part II will explain the "theory" that will be used to analyze the four foundational issues Here the notions of "paradigm" and "progressive research program" will be made as precise as possible In Part III, the theory explained in Part II will be used to examine the issues adduced in Part I

I. The four foundational issues

The following hypothetical situation will help introduce

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the four foundational issues

> *Transfer of training: a thought experiment*
> Susie, an algebra I student, has just completed a unit on trinomial factoring. By a series of questions, it is determined that Susie also knows the following: If a product of two terms equals zero, then at least one of these terms must be equal to zero. (However, she does not express herself in quite this language.) Assume now that Susie is given the following problem: Find those values of x that makes $x^2+3x+2=0$. Could she solve this problem?

Whether or not Susie can solve this problem depends, in part, on whether she can "transfer" her knowledge of factoring to this new situation. A proponent of a conservative theory of transfer might argue that she probably could *not* solve the problem. The concepts of a "factored trinomial" and of a "solved quadratic equation" are sufficiently different. There are various prerequisites that must be mastered before a child will see the connection. Knowledge about one skill (factoring trinomials) transfers to a new situation (solving quadratic equations) only to the extent that components of the old are included in the new. Susie might factor the left side of $x^2+3x+2=0$ and then become stuck. There is no "component" in the factoring skill that would indicate what she should do next.

A proponent of liberal theory of transfer, on the other hand, would argue that Susie might be able to solve the quadratic equation. Assuming that her teachers were careful, then Susie would have acquired, not just the skills for factoring trinomials. She would also have learned the concepts and principles that make these skills "meaningful." In other words, she would have learned (or constructed) a "cognitive structure" that will transfer to many different problem solving situations. If Susie has acquired this cognitive structure, she might be expected to factor the left side of $x^2 + 3x + 2 = 0$, set $(x+2)$ and $(x+3)$ each to zero, and thus solve the quadratic equation.

Which theory of transfer is correct? It might appear that this question could be decided by an "experiment." Find two groups of first year algebra students with comparable mathematics preparation and ability. Teach one group the prerequisite components of factoring trinomials, as identified by (say) a learning hierarchy. Teach the other group the structural principles behind factoring. After several weeks of treatment, administer a posttest and compare the results.

Consider, now, how the all-deciding posttest might look. Would it probe whether students can transfer their knowledge to novel situations (eg., solving quadratic equations by factoring?) If the posttest was one of simple computational efficiency, then one would expect the learning hierarchy group to outperform the group taught the structural principles. The learning hierarchy group was taught all the necessary components for "getting the right answer." On the other hand, if the posttest required the students to apply their knowledge to new situations, the group taught with an emphasis on structural principles would probably outperform the learning hierarchy group. The structural principles group would have been taught to search for similarities between old and new situations.

The problem with designing the posttest is a symptom of at least three other problems. The most obvious is the following:

1) Advocates of conservative theory of transfer want a child to master procedures for performing correctly on certain tasks. Mathematics is a "tool" that enables children to perform correctly on these problems. Advocates of liberal theory, on the other hand, want a child to grasp the underlying significance of her mathematical skills. Mathematics is a structured discipline possessing a certain internal beauty. In other words, advocates of the different theories of transfer might attach different *values* to mathematics instruction.

Two other differences are related to the theories of transfer.

2) An identical component theorist attempts to break a complex skill down into smaller, fundamental parts. Transfer is successful only if the two skills have these smaller parts in common. An advocate of the structural principles view of transfer, on the other hand, teaches a child to search for patterns and similarities between different situations. These patterns are presumably much "larger" than the "identical components" of the other theory. In other words,the researchers who study the two theories are likely to adopt different *methodologies*.

3) The identical component theorist attempts to teach a complex skill by specifying an instructional sequence that the child should follow. The mind is viewed as a passive entity which is subject to an active environment (the instructional sequence). The advocate of structural principles view of transfer, on the other hand, sees the child as searching for and constructing her own ties between one bit of knowledge and another. The mind is more active. In other words, the theorists are apt to hold different views as to *how the mind works*.

Proponents of the conservative and liberal views of transfer might also tend towards different views on the nature of mathematical entities.

4) Those who try to dissect a skill using a learning hierarchy focus on the child's performance in manipulating symbols to get the "right answer." A right answer is the result of symbol manipulation behavior. Advocates of structural principles view of transfer, on the other hand, hypothesize the existence of cognitive entities that give meanings to the symbols. A right answer results from the child's having constructed certain cognitive structures that enable her to correctly manipulate the symbols. In other words, proponents of the two views are apt to hold different theories regarding the *nature of mathematical entities*.

The next four subsections will examine these differences related to values, methodology, theory of mind, and theory of mathematical entities in more detail.

Different values attached to mathematics instruction

An advocate of a conservative theory of transfer tends to focus on the product of mathematics instruction. She wants to insure that a particular skill, such as the addition of two digit whole numbers, is properly learned. She might point out that addition of two digit whole numbers is one of many mathematical skills that are of great value to society. Children must be able to perform these skills to meet the demands put on them in everyday life A careful, methodical approach to transfer appears to insure the mastery of these skills.

Though an advocate of the liberal view of transfer is not "anti-utilitarian," she is likely interested in more than just the social or economic advantages of mathematics Dienes speaks of the power of mathematics to develop an "integrated person." Such a person ...

> will take a wider, as opposed to a personal or sectional view in most questions ... try to unite things rather than separate them ... will seek connections rather than differences ... will have made a good adjustment to his environment by establishing a fundamental identity of interest between himself and his fellows [Dienes, 1960, pp. 12-13]

Different methods for studying mathematics learning
Advocates of the conservative theory of transfer make use of a relatively "small" unit of analysis in their research Thorndike's [1922] basic unit was a stimulus response "bond," whereas the early Gagné [1963, 1970] argued that the basic unit was a simple skill. The common idea here is that knowledge of a complex principle can be reduced to simple facts or concepts. Advocates of the liberal view of transfer, on the other hand, tend to use much "larger" units of analysis in their explanations Piaget's structures are a system of internalized, mental operations Dienes' structures include both the "relationships between concepts connected with numbers" and the "applications [of these relationships] to problems arising in the real world ... "[Dienes, 1960, pp. 19-20] Both theorists tend to treat structures as "wholes."

Are cognitive structures reducible to simple concepts or facts, or are they best studied as systems or "wholes?" This question is similar to the biological question: Is the human body a sum of atoms and molecules, or is it part of a larger system of peoples, cities, states, etc.? The answer to each question depends upon the *purpose* of the study, which, in turn, influences the *method of investigation*. A cell biologist and an identical component theorist want to understand the smaller parts into which a given body or skill can be reduced A sociologist and an advocate of teaching structural principles wants to understand the larger systems into which the smaller components fit As a result, the different theorists use different methods.

However, there is at least one important difference between the psychological and the biological question. The cell biologist and the sociologist would admit that their research has different purposes. Each is trying to answer a different set of questions. However, the identical component theorist and the advocate of structural principles are *both* trying to understand mathematics learning. It is not clear whether mathematics learning is best understood using a reductionistic or a holistic method of research.

Different theories of mind
Advocates of the conservative theory of transfer argue that only individual, identical parts of one skill transfer to a different task The child needs to have the various parts of a complex skill presented to her. As a result, conservative transfer theorists try to determine the parts of each complex skill and teach these individually. The relationships between the simple and complex skills are determined *before* they are presented to the child In other words, the work of finding the connections between a complex skill and its simple parts is done by the instructional designers, not the child The child is viewed as a passive recipient of information and relationships.

Advocates of the liberal theory of transfer, on the other hand, emphasize that the mind will actively search for connections and patterns in different situations The connections and patterns are not imposed on the child from the environment. They are constructed within the child's mind. This construction assumes that the mind has some preexisting organizational principles within itself to begin the constructive process. Chomsky solves this problem by attributing to the child primitive, innately existing structures Piaget, on the other hand, argues that primitive structures develop by a process analogous to biological growth

Different views on the nature of mathematical entities
Advocates of a conservative theory of transfer tend to believe that a complex skill can be described as a series of related "bonds," or links between stimuli (problems) and responses (answers). Though a bond is presumably "inside a child's mind," the conservative transfer theorist is not interested in mental constructs. Her focus is on the observed stimuli and the response, both of which can be described as symbol manipulation behavior When a conservative transfer theorist says that a child has learned how to multiply integers, she *means* the following: If the child is given a multiplication problem, she will manipulate the symbols to produce a certain answer In other words: "Talk about bond formation" is just another way to talk about symbol manipulation behavior.

Advocates of a liberal theory of transfer, on the other hand, believe that performance on a complex skill is the result of having the right cognitive structures in place. Mathematical symbols are used to represent relationships within and among these structures The significant point is this: Though evidence for the existence of the structures is provided by symbol manipulation behavior, the structures are not *equivalent* to the symbols. Whereas the symbols are "external," the structures are presumably "internal." Symbol manipulation behavior is only a *manifestation* of the internal structures.

Are mathematical entities just chimeras resulting from symbol manipulation behavior? Or do these entities have some counterpart "inside" the human mind? Quine has argued that this problem is a variant of the classical "problem of universals." [Quine] The problem of universals can be illustrated by considering the relationship between a numeral and a number.

The numeral, "3", is somehow related to all those collections in the world that contain three elements "3" is the name for whatever property that these collections have in common. But where does this property reside? Is it part of the world, in the same sense that the elements of a three-member set are part of the world? Or is the property an illusion arising from the fact that humans use the symbol "3" to name certain collections for social or survival reasons? Or is the property a concept that users of the symbol "3" construct in their minds?

Plato was one of the first thinkers to consider this question systematically He argued that each universal term, such as the numeral "3", refers to a separate entity that exists apart from all the collections that contain three elements This separate entity is *not* perceived by the senses. It is intuited by the intellect However, this entity is *not* "constructed" by the mind It exists in a separate, unchanging world of "Forms" that can be seen by the Eye of Reason.

Plato's position might seem to be farfetched, but it often appeals to mathematicians. When two mathematicians are talking, it is almost as if the one can "see" the abstraction that the other is talking about. This supports the view that mathematical entities exist in some substrate or domain independent from the human mind.

Plato's position (which will be called "realism") was not without its challengers. During the Middle Ages, some thinkers put forward a view that is diametrically opposite to Plato's Whereas realism populates the world with many things that are not apparent to the senses (eg , platonic Forms), the Medieval position strikes from the world many things that *do* seem to be apparent to the senses For example, the common property shared by all sets with three elements is just an illusion, according to proponents of this position "3" is just a *name* which humans use when talking about certain collections. Accordingly, this position is often called "nominalism."

Nominalists argue that mathematical symbols are useful in making predictions about, say, the path of a missile or the movement of a weather front. However, though the symbols have utilitarian value, they have no "meaning" apart from that derived from the context in which they are used. Mathematics is just a play of symbols Learning mathematics is learning how to manipulate symbols, just as learning how to ride a bicycle is learning how to manipulate the pedals and handle bars. Conservative transfer theorists may lean towards this nominalist position.

Both realism and nominalism appear to be extremes. Not surprisingly, some thinkers attempted to steer a compromise between these two positions. They argued that the common property shared by instances of a universal term, though more than a word, is less than an independently existing Form. It is an *idea* or concept that is constructed by the mind of the term's user. For example, sets of three elements are perceived as similar because users of the term "3" have constructed similar concepts in their minds Advocates of the liberal theory of transfer probably lean towards this view, which is often called *conceptualism*

At first blush, the conceptualist position appears to be the most plausible. Problems arise, however, when one tries to *explain* the fact that users of a term such as "triangle" construct similar concepts It would be highly unlikely if the concepts are similar by chance. But if one argues that there is something in the world to account for the similarity, one is back to a variant of Plato's solution Namely, users of the term "triangle" construct similar concepts because they perceive a common, platonic Form

Summary

Four issues pertinent to "theory" in mathematics education have been explained These issues stem, in part, from the notion "transfer of training" First, advocates of the different transfer theories tend to ascribe different values to mathematics education. Whereas a conservative transfer theorists may stress the utilitarian value of mathematics, a liberal transfer theorist may argue that mathematics can also "develop the personality." Second, advocates of each theory tend to make different methodological decisions regarding the basic "unit of analysis" Whereas the conservative theorist tends to reduce complex skills to simple facts, the liberal theorist posits larger structures that are either inborn or which develop biologically. The third assumption relates to the theory of mind. Whereas the conservative theorist views the mind as a blank, passive recipient of information, the liberal theorist views the mind as an active, creator of cognitive structures A fourth difference pertains to the nature of mathematical entities Whereas the conservative theorist tends towards the view that mathematics is a formal "play of symbols," the liberal theorist is likely to believe that mathematical symbols refer to conceptual entities.

The next section will explain the "theory" that will be used to study the differences between positions on each of these four issues

II. The theory: Kuhn and Lakatos

Philosophers of science have often considered the problem of different theories allegedly explaining the same phenomenon. Perhaps it is to be expected that the philosophers disagree as to how this question of disagreeing theories is best explained. T. Kuhn [1962] and I. Lakatos [1970], for example, offer very different explanations of "theory clash"

Thomas Kuhn: competing paradigms

Kuhn describes a community of scientists as working within a shared set of beliefs that determines the way in which they perceive the world. Physicists, for example, are "expose[d] to a series of exemplary problem-solutions [which] teaches them to see different physical situations as like each other "[Kuhn, 1970, p. 273] Kuhn describes the physicists as seeing the world in a Newtonian or Einsteinian "gestalt" or "paradigm." Scientists from one paradigm cannot discuss the ideas from another paradigm in an impartial, rational manner As Kuhn puts it, the paradigms are *incommensurable*. [Kurn, 1962, p. 103]

An example from the history of physics is often used to illustrate incommensurable paradigms Before the turn of the century, physicists saw the world through a Newtonian point of view. The mass of an object and its energy were two distinct concepts. After Einstein's work, physicists came to view mass, velocity, and energy as all related. Kuhn argues that the Newtonian and Einsteinian views are incommensurable. "Though an out-of-date theory can always be viewed as a special case of its up-to-date successor, it must be transformed for the purpose."[Kuhn, 1962, p. 103] The Newtonian world must be "transformed" to fit into the Einsteinian world. As a result of this transformation, fundamental concepts like "mass" and "energy" change their meaning.

Proponents of the Kuhnian model find it difficult to judge one paradigm as "better" than another. They might explain the success of Einstein's paradigm over Newton's by pointing to historical trends, political events, or the personalities of individual researchers. Scientists change paradigms for reasons that are often not "rational." As Kuhn puts it, the choice between competing paradigms:

> proves to be a choice between incompatible modes of community life. Because it has that character, the choice is not and cannot be determined merely by the evaluative procedures characteristic of science, for these depend in part upon a particular paradigm, and that paradigm is at issue. When paradigms enter, as they must, into a debate about paradigm choice, their role is necessarily circular. Each group uses its own paradigm to argue in that paradigm's defense. [Kuhn, 1962, p. 94]

Kuhn's position on scientific change is debatable. Some philosophers contend that, when one theory becomes more acceptable than another, there are good (rational) reasons for preferring it. For example, the new theory may be closer to the truth [Popper, 1935, 1963], or it may be able to explain everything the other theory explains and more [Lakatos, 1970]. Different theories may account for the same phenomena, but this does not mean that the theories hail from incommensurable paradigms. It just means that proponents of one theory have not yet demonstrated the superiority of their explanations.

The question as to whether or not scientific change is "rational" can be illustrated using an analogy from anthropology. Assume one is trying to determine whether or not a Third World tribe is making progress. Some would argue that, in order to make this judgement, one would first have to become a *member* of the tribe. This is because it is not possible to understand the culture from the "outside." Other anthropologists would regard this conclusion as overly charitable. It *is* possible to find a neutral perspective from which the progressivity of a culture can be judged. Pursuing the analogy, some argue that a paradigm is like a tribe and can only be assessed using its own standards. Others contend that it is possible to find a neutral ground from which the rationality of scientific change can be determined. I. Lakatos belongs among this latter group.

Imre Lakatos: research programs

Lakatos [1970] argues that, though research programs may be based on incommensurable assumptions, these assumptions may have different "fruits" in the world of science. In other words, research programs can be compared on the basis of their *progress*. Thus, much hinges on this notion of "progress." In Lakatos' model, a theory is more progressive than another if it meets three criteria: First, the new theory makes some predictions that its predecessor did not. Second, some of these predictions must have been corroborated. Third, the new theory must explain all the facts that its predecessor could explain.[Lakatos, 1970]

Einstein's and Newton's theories can be used to illustrate these three criteria. Einstein's theory predicts that light will bend in a gravitational field, which Newton's theory does not. This prediction was, in part, confirmed by Eddington's photographs of starlight bending as a result of the sun's gravitational pull in the 1917 eclipse. Finally, because Newton's theory can be viewed as a special case of Einstein's, the latter can explain all the facts that the former could. Thus, using Lakatos' three criteria, Einstein's theory is more progressive than Newton's.

In addition to these three criteria, which define the notion of "progress," Lakatos provides a useful analysis of scientific research programs. A research porogram has two parts: foundational, or "hard core" assumptions, which are never questioned, or auxiliary or "protective belt" assumptions, which may be changed to accommodate negative evidence. Examples of hard core assumptions are "action at a distance" in Newton's theory and "development occurs in stages" in Piaget's theory. The protective belt, in Newton's theory, consisted of a series of models of planetary systems which are successively more and more in accord with his basic laws of motion.[Lakatos, 1970, p. 135] In Piaget's program, the protective belt would include stipulations that children might sometimes regress in their development or straddle more than one stage.

The hard core assumptions behind a research program are often not testable. However, it *is* possible (at least in principle) to determine whether a program meets the three criteria for "progress." This enables one to assess the rationality of scientific change.

Kuhn and Lakatos offer different perspectives on the nature of disagreement about fundamental issues. The next section will attempt to determine whether a Kuhnian or a Lakatosian point of view is "most appropriate" for understanding the differences, explained in Part I above.

III: A closer look at the four issues

Researchers in mathematics education may hold differing views on questions of values, units of analysis, theory of mind, and the nature of mathematical entities. Are positions on these issues incommensurable, or is it possible to argue that one position is "more progressive" than another? This question will be considered for each of the issues, explained in Part I.

Different values attached to mathematics instruction

Researchers who lean towards either a utilitarian or a "develop the personality" value to mathematics education are apt to study different things. For the former,the key issue is whether or not a child learns particular, socially useful mathematical skills. These skills can probably be assessed by an objective test and might be studied over a relatively short period of time Character development, on the other hand, is difficult to describe without specifying *how* the development occurs. A study into character development is based on a much more complex inferential chain A study into this area might require several years or more to complete.

Comparing these different value assumptions based on their success is tricky. If "success" is measured by "popular appeal," then perhaps the utilitarian assumption has prevailed Even Dienes admits: "Most of us frankly come down in favor of economic arguments when trying to justify the learning of mathematics " [Dienes, 1960, p 8] In a sad sense, the utilitarian value can also be a boon to educational decision makers. Since emphasis on this value leads to the view that mathematics can be assessed by objective tests, decision makers can demonstrate their accountability by producing test score gains.

However, Lakatos' criterion for "success" hinges, not on the popularity of the assumption, but on the *arguments* in support of it. Here it can be argued that the "develop the personality" value is supported by better arguments. In particular, instruction that attempts to develop a child's personality also has the potential to enhance the child's mastery of basic skills. "Personality development" would appear to be facilitated by a program that pointed out connections between mathematics and other elements of a child's life But these connections will also tend to help the child in recreating skills that have been forgotten As Wachsmuth argues: "If a rote skill has not been employed for a while, decay is possible; understanding the skill would allow one to *reconstruct* the procedure from declarative knowledge." [Wachsmuth, 1983, p. 208]

Different methods for studying mathematics learning

In part, the reductionistic and holistic assumptions are incommensurable. Those who opt for the former are likely to design experiments to, say, validate a learning hierarchy The purpose of the research is to determine in what way a given skill can be decomposed into simpler skills. The major criterion for evaluating this research would be the success of the reductionism On the other hand, a researcher who believes that the basic unit of knowledge is the "cognitive structure" is apt to design a different type of study She might look at the effect of different types of instruction on the creation of cognitive structures "Success," for research based on this hard core assumption, would mean being able to describe those variables which make for the construction of these structures.

Has research framed on either of these assumptions been more successful? This question is difficult to answer because much of the evidence is still out The claim that all of the component parts in a reduction must be "observable behaviors" is no longer popular However, some theorists try to reduce more complex mental phenomena, not into observable behaviors, but into simpler *mental* processes. This information processing approach does not require that theorists eschew all reference to mental entities. Thus an information processing theorist can "practice" her reductionist assumption and suggests ways that this assumption can be advanced This is one mark of a "progressive research program."

Different theories of mind

Researchers who hold different theories of the mind also tend to work within incommensurable paradigms. Though proponents of the passive and active theories might each study a child's performance when given certain tasks, the nature of these tasks might be very different. Advocates of the passive theory, for example, might study the ways in which a child links simple skills together to answer more complex problems. If the "right" responses do not occur, the researcher will probably redesign her experiment rather than change the assumption that the mind is passive. Those who stress that the mind is active might study whether a child can build structural ties between tasks which are very different. If the connections are not made, the implication would probably *not* be that the mind is passive Again, it would be that the experiment needs to be redesigned

Though these two theories of mind may appear to be incommensurable, one can argue that the active theory is more progressive, in a Lakatosian sense. Support for this argument comes from recent work in the theory of knowledge During the past 50 or 60 years, thinkers have come to near unanimous agreement on one principle: The distinction between "fact" and "theory" is difficult if not impossible to maintain This principle is in stark contrast to the views of classical empiricists, such as Locke and Hume. These classical thinkers argued that sensory experience provided a neutral, "data base" which could be used to establish theoretical claims. In the early part of this century, some philosophers of science tried to further advance the views of Locke and Hume. Schlick [1936], for example, argued that the *meaning* of a theoretical claim was tied to its *method of verification* in experience.

These attempts to distinguish fact from theory fell victim to a variety of logical traps. [Quine, 1953b] Most researchers have now concluded that there is no theory-neutral, observational base from which the truth or falsity of scientific statements can be justified. The significance of this conclusion for the present discussion is this: The passive theory of the mind is intimately connected with the classical, empiricist view. Those who hold that the mind is a passive receptor of information, like the classical empiricists, try to remove any contamination of human inference from "the facts." The passive theory of the mind supports, and is supported by, the distinction between fact and theory In contrast, proponents of the active theory of the mind have no need for this distinction In a sense, "cognitive processes" are both fact *and* theory.

Different views on the nature of mathematical entities

The distinction between nominalism and conceptualism is most evident in the way proponents of each account for the *meaning* of *mathematical symbols* The nominalist argues that the meaning of mathematical symbols is derived from the context in which the symbols are used. If a child is asked to calculate the area of a rhombus, for example, the meaning of the symbols "A," "1," and "w" would be derived from the area formulas in which these symbols appear. There is no need to claim that the symbols refer to postulated cognitive entities. The meaning of a symbol is exhausted by however the symbol is used. Mathematical concepts just *are* symbol use

For the conceptualist, on the other hand, the meaning of a mathematical symbol cannot be totally specified by describing the behavior of those who use the symbol. When a child is asked to calculate the area of a rhombus, for example, the meaning of the symbols "A," "l," and "w" is derived, not just from the area formulas that the child manipulates, but also from the mathematical ideas to which these symbols refer. Symbols refer to cognitive constructs. The concept is more than just the symbol use.

So stated, it appears that nominalism and conceptualism are incommensurable. Nominalists and conceptualists almost "speak different languages." The conceptualist uses terms such as "cognitive structures," "conservation," "INRC group," etc. Though the nominalist would try to translate these "chimeras" into behavioral terms,the conceptualist accepts them as they are. This difference in language would make it difficult to compare nominalist and conceptualist research programs from a neutral perspective.

Given the role of "construction" in the conceptualist view, together with the popularity of the active theory of mind, one would think that the conceptualist view is more progressive. However, this supposition can be challenged by looking more closely at the information processing model. Proponents of this model will readily grant that the mind is active. They will also grant that the mind contains cognitive structures. However, the nature of these structures is not clear.

From a logical point of view, a computer can be described using three, functional notions: inputs from the environment, outputs to the environment, and internal configurations or "states." [Putnam, 1960] Both the inputs and outputs are symbolic. A state can also be specified symbolically. Namely, a state can be described by specifying the actions (go to a different state, output a certain symbol, etc.) that the computer takes when subject to a given input. The upshot of this is that the workings of a computer can be described in terms of how the computer manipulates symbols. Put differently: The computer's "cognitive structures" can be translated into symbol manipulation behavior.

The information processing model can be interpreted in at least two ways. On the one hand, it can be interpreted as merely a function *description* of how the mind manipulates information. Terms such as "register," "buffer," and "node" have no meaning beyond the symbol manipulation behavior that they are used to describe. This "formal" interpretation of the model is consistent with nominalism. On the other hand, the model can be interpreted as a substantive or causal *explanation* of how the mind works. Terms such as "register," "buffer," and "node" refer to hypothetical conceptual entities. These terms play a role in the explanation of cognitive processes and thus have meaning beyond the symbol manipulation behavior. This "substantive" interpretation of the information processing model is not consistent with nominalism.

Though the information processing model might be part of a "progressive research program," users of this model do not always specify their interpretation. Thus, it is difficult to say whether the nominalist or the conceptualist view of mathematical entities is more progressive.

Final conclusion: Kuhn or Lakatos?

Theories in mathematics education often rest on different assumptions regarding four issues: values, units of analysis, theories of mind, and nature of mathematical entities. The outstanding question is this: Is it more rational to accept any particular assumptions regarding these issues?

From one point of view, assumptions regarding these issues are incommensurable. Advocates of the different assumptions tend to ask different questions and study different things. As Kuhn might put it: The choice between these assumptions cannot be determined merely by the evaluation procedures characteristic of science, for these criteria would depend upon the paradigm. [Kuhn, 1962, p. 94]

However, if choices between the assumptions is based on the consequences of these assumptions in mathematics education research, perhaps there *are* some grounds for rational decisions. More specifically, two of the four assumptions appear to be more "progressive" than their counterparts. The first is the "develop the personality" value. This assumption can embrace both itself and the utilitarian value. The second is the active theory of the mind. Given the context of recent intellectual history, this assumption appears to be clearly more progressive than the passive theory.

To compare the assumptions regarding units of analysis and nature of mathematical entities, however, is more difficult. The reductionistic methodology holds promise in certain information processing models, but the holistic strategy is alive in the constructivist models. The nominalist view of mathematical entities is viable in a certain interpretation of the information processing model. However, the information processing model could also be interpreted in a conceptualist fashion. Furthermore, most if not all constructivists lean towards a conceptualist view of mathematical entities.

One way to sum up the analysis to this point is with the following question: Is it more rational to follow a research program based on a formalist, reductionist information processing model or conceptualist, holistic constructivist model? Both of these models would share the active view of the mind; however, they would differ on their assumptions regarding methodology and theory of mathematical entities. They might also differ on their "values" assumption. Constructivists are likely to lean towards the "develop the personality" value. However, with information processing theorists, the matter is not as clear. Gagné, who now ascribes to an information processing model, stresses the importance of skill automaticity. [Gagné, 1983a] Does Gagné stress automaticity because it tends to build up a child's personality or because it helps insure that a child will produce "correct answers?" Some might suspect the latter.

Kuhn or Lakatos? Because different views on research methods and theory of mathematical entities are both defensible and incommensurable, a Kuhnian model is most appropriate for understanding the differences between the information processing and the constructivist models. When the "fruits" of these programs are harvested, Lakatos' model may provide the better description. However, this story remains to be told.

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of students and play the important role of managing the debates between them

Techniques of managing debates have to be learned Some of them have already been partially validated [Alibert, Grenier, Legrand & Richard, 1986], others are still being studied. The first sessions of an experimental course must be very carefully organised: they consitute a critical period in the negotiation of new customs with students whose previous experiences have been somewhat different This particular point will be the subject of another article

The problem of transmitting this method to other teachers, not to other didacticians, has to be solved. A didactical-engineering must be developed. But we believe that we must simultaneously deepen the theoretical study of the system in order to understand better what is truly fundamental to it, we must try to justify scientifically some of our empirical choices, and we must determine which aspects of our experiment are reproducible We have built a first model for this purpose and it is now being tested

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