A UNIFIED APPROACH TO MATHEMATICS EDUCATION CREATED AROUND THE COMPUTER PARADIGM

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> [E]ach generation must define afresh the nature, direction and aims of education [...] For there are changes both in circumstances and in knowledge that impose constraints on and give opportunities to the teacher in each succeeding generation. (Bruner, 1966, p. 22)

> In keeping with the spirit of the age, researchers can think of the laws of physics as computer programs and the universe as a computer. (Lloyd and Ng, 2004, p. 53)

Electronic computing devices as the key phenomenon of our age

What is the most important change, in the present circumstances, which could make an impact on mathematics education? The appearance of various electronic computing devices comes to mind. One way to see these instruments from an educational perspective is as useful tools. Thus, the two most popular applications of computing machinery in mathematics education are educational software and utilizations of new opportunities that these devices provide for various mathematical activities – such as function explorations by graphic calculators and computer-aided numerical solutions of equations.

A lot of literature of diverse kinds is dedicated to these uses. However, the impact of computers is not limited to their applications. A new paradigm has emerged from the computer revolution. This article discusses different aspects of the impact of this paradigm on mathematics education. [1]

While the computer revolution obviously has serious implications for mathematics education, the nature of these implications is far from clear. For example, the ubiquity of calculators and computers raises the question of relevance for many traditional topics of educational mathematics. Some researchers reach rather extreme conclusions, for example, Keitel (1989), associating mathematics with the traditional skills (numeric and algebraic manipulations), discerns a "demathematization" trend in contemporary society. She notes that

> the overwhelming majority of people in a modern society can and do live quite well *while doing hardly any mathematics*. (p. 9, original emphasis)

And so an inevitable conclusion is:

> Does mathematics education in any way contribute to the preparation of students to face reality, so as to improve their chances of meeting this challenge competently? I do not think so. (p. 12, [2])

On the other hand, the computer revolution led to huge changes in the work market. One important aspect of these changes is a large *increase* in the number of high school graduates who continue their professional education in mathematics-related subjects:

> Forty years ago [...] it could safely be said that mathematics was an elitist subject. Society's demand for the mathematically fluent was quite small, and those few who survived the mathematics curriculum were more than sufficient to fill the need for teachers, researchers, and mathematical scientists [...]
>
> There is now a broad perception that what we do does *not* work. Society now demands a technologically literate work force, and the elitist teaching methodology developed by earlier generations of mathematicians is no longer adequate to the job. (Gavosto *et al.*, 1999, Preface, p. xiii)

The following three topics are explored in the next three sections of this article: *New skills*, *Cultural influences of computerization on mathematics education* and *The computer metaphor as a conceptual framework*.

New skills

Keitel's conclusion could be interpreted as meaning that some of the major assumptions behind traditional mathematics education do not hold. Indeed, work in the computerized environment demands peculiar mathematics-related skills, bringing mathematics educators to new terrains, virtually unexplored from the pedagogical perspective.

Mathematical abilities needed to succeed in computerized workplaces

It is not easy to discern the mathematical needs of people working at computerized workplaces:

> mathematics is not always visible: it lies beneath the surface of practices and cultures. (Noss, 1998, p. 3)

Moreover, computer-related mathematics is of a peculiar kind:

> The sort of mathematics that arises in a computing context is not necessarily what most people would consider to be mathematics at all. Its character may seem like that of 'mere' organization, symbol management, or data manipulation. (Truss, 1999, Preface, p. v, [3])

Many school mathematics teachers are not familiar with the 'weird' mathematics relevant to computers and, thus, are unaware of the most basic future mathematical needs of their students. As a result, school graduates are often not prepared adequately for further studies to become competent professionals in fields of their choice. Many avoid entering computer-related areas altogether, reducing the scope of their employment possibilities on the one hand, and creating shortages of experts in economically important occupations on the other hand.

In many developed countries, there is no lack of first-rate mathematicians – candidates for academic positions, or brilliant teachers in privileged schools, or software developers – who are prepared by professors from the best universities. There are, however, shortages in the second echelon of experts, those providing competent routine support and maintenance for various public systems (such as programmers, system administrators, teachers in average schools).

The reason for this discrepancy lies in the quantitative difference in the needs for specialists of these two types. There is only a limited demand for professional mathematicians and inventors of new technologies. These needs are normally supplied by individuals who are naturally inclined to the fields of their expertise and graduate from the elite schools. This is in contrast to a huge demand for professionals who support and maintain various essential systems.

The modern society is dependent on the professional level of these people, because they are responsible for covering the widening gap between the researchers and developers, on the one hand, and the society at large, on the other hand. Thus, the discrepancy between the initial state of these individuals (*i.e.*, natural inclinations and typical mathematics-related abilities at the end of the secondary school) and their needs as professionals invites the scrutiny of researchers in mathematics education.

The peculiarity of the mathematical needs of workers in the computerized environment is highlighted by the fact that two main mathematical activities on which most mathematics education has been focused, namely problem solving in the traditional sense of providing answers to the given questions, and proving, are of secondary importance to many computer-related applications of the support and maintenance level [4]. The work of a computer professional could be seen as incessant transitions between formal (human-computer) and informal (human-human) communications. Such a specialist has to have working abilities to translate intuition to a formal language, creating definitions of new abstract objects (functions, structures, classes). More generally, work in a computerized environment demands abilities in reading and writing succinct and comprehensible program comments and system descriptions, technical requests, manuals and specifications.

Practising programmers usually do not prove the correctness of their programs, they have to have intuition to understand the meaning of code for preparing good samples of inputs to check their programs. At non-advanced levels they are rarely concerned with the development of new algorithms solving an ingenious mathematical problem, rather they translate informally formulated problems to a form suitable for existing software packages. Thus the main mathematics-related activity in the computer context is to comprehend and to create computer-compatible formulations.

Mathematics education for non-mathematicians has been developed mainly around the creation of intuitive concept images (Tall and Vinner, 1981), while the formal side of mathematics (including definitions) played at best a secondary role and was often ignored. Work with computers eliminates the possibility of being satisfied with descriptions on the intuitive level: computers can only be engaged in formal talk!

When nineteenth century thinkers such as Boole and de Morgan developed what is now called mathematical logic, they believed that the subject of their study was the basic laws of human thought. This statement appeared explicitly in the title of Boole's ground-breaking book (Boole 1854/1958). In the late nineteenth century, it was natural to state that logic "is nothing if not the physics of thought" (Chase *et al.*, 1998, p. 206). As recently as 1943, McCulloch and Pitts tried to explain the assumed normative logic of thought, reducing it to the rules that govern behavior of the nervous system cells. Their classic paper was among those that laid theoretical foundations for constructing the first computers. The hardware was supposed to simulate normative human thought.

Research on human reasoning in the second part of the twentieth century showed that the belief in the normative character of formal logic for human thought is unfounded:

> Beyond the simple problems […], it makes unrealistic demands on the mind. In the real world, matters are more complicated than the simple content-blind norms [associated with formal logic]. (Chase *et al.*, 1998, p. 206)

Meanwhile, computers, whose behavior *is* governed by the laws of formal logic, became an important part of our social landscape. And now humans, in particular those who seek computer-related occupations, have to be educated to communicate in this formal language [5] – the normative language of any computerized environment. The wheel has come full circle!

An additional assumption of rationalists in the last several centuries was that humans are used to following explicit rules in terms of which they formulate their intentions. Recent research on human reasoning showed that most people do not think naturally in precise terms, preferring the fuzziness of imprecise intuition (*e.g.*, Stanovich and West, 2000) [6]. For many people, rule following is difficult and acquiring this ability demands a lot of practice.

Mastering these two skills – the ability to follow strict rules and the application of logical reasoning – amounts to acquiring the computer-compatible mind-set. It is remarkable that neither of these topics appears in the list of major parts that compose the new National Council of Teachers of Mathematics (NCTM, 1999) school curriculum. It is usually claimed that logical and algorithmic abilities of students are developed implicitly through other mathematical topics, but experience contradicts this assumption. Indeed, the main conclusion from my work of teaching these skills, to college freshmen striving to work in 'high-tech' industries, was that lack in these two abilities presents the major obstacle in the professional preparation of this population [7].

'Dirty' mathematics

'Mathematics as a science' and 'mathematics relevant to education of non-mathematicians' are two different subjects, united by common features, with no clear line separating them. They have different criteria for importance such as aesthetics or difficulty. Care has to be taken if criteria accepted in the science of mathematics are to be transferred to its educational counterpart. It is especially true in our age of drastic changes in educational demands.

Theory of Meetings

A meeting is an order set (M, P, c, s, C_1, C_2, b, i_1, i_2, S, i_3) consisting of
a bounded part M of Euclidean space; a finite set P, that of participants;
two elements c and s of P called chairman and secretary;
a finite set C_1, called the chairs; a finite set C_2, called the cups of coffee;
an element b, called bell; an injection i_1 of P into C_1; a mapping i_2 of C_2 into P;
an ordered set S, the speeches;
a mapping i_3 of S into P with the property that c belongs to the image of i_3.
If i_3 is a surjection, it is usual to say that everybody has had the floor.

Figure 1: An illustrative example

The "theory", illustrated in Figure 1, was given by Freundenthal (1978) to demonstrate "grotesque misunderstandings" (by some educators) of what a mathematical model is:

> Contriving such models and presenting them used to be an amusement for people who organized an institute ball with a cabaret. In the last few years it has become a serious concern for model makers and an ornament to educational research. As a mathematician I am ashamed of it. Science, in whatever area, is never so cheap that it requires no more that mathematical jargon. (pp. 136-137)

This example was then reprinted by Davis and Hersh (1981) as a kind of symbol of bad mathematics. On the other hand, as an educator of future experts in computer-related specializations, I use it to explain the essence of my pedagogical activity. For my students, whose expertise should include modeling pieces of reality on computers, this is an example of bad modeling only in the sense that before dealing with the bell (which was obviously mentioned by Freundenthal to exacerbate the irony of the whole "theory") a model of a meeting has to include an element d, called door, in the boundary of "a bounded part M of Euclidean space" to enable the members of P to enter M.

Moreover, from my experience of working in computer departments of various academic colleges, I can testify that 'good' mathematical structures frequently appear in the curricula on a FIFO (first in, first out) basis: an *Algebraic structures* course is typically included in all initial curricula – as a large course, it is then the first to be sacrificed to time limitations and the total excessive burden of mathematical courses. Classical mathematical structures (such as groups and rings) rarely arise in practical computer applications. At the same time, creating "grotesque" structures is the central activity in object-oriented programming. The main problem of my students, in this context, is their difficulty in adequately formulating these models. So, this is a new problem posed to the science of mathematics education [8].

Cultural influences of computerization on mathematics education

Another result of the computer revolution is the simplification of many basic mathematical operations, leading to changes in the relative difficulty of various topics and in the mathematical abilities of typical students on all educational levels. Cultural changes stemming from this development imply that the structure of mathematics, as it is presented to students on different levels, should be reconsidered.

The feeling that mathematics curricula should be re-examined is not limited to teachers of experts in computerized industries. There is a recent trend among elementary mathematics educators against teaching several of the most basic mathematical subjects in the traditional fashion. This sometimes includes the practising of algebraic manipulation skills, but mostly it concerns arithmetic. Some arguments of the proponents of such innovations are based on theoretical considerations related to constructivist education, but the strongest support for change stems from the ubiquity of electronic calculators, and the feeling of the demathematization of our society.

Stated bluntly, the proponents of this view claim that the introduction of calculators made drilling children in arithmetic unnecessary. Some educators, who are unwilling to make such extreme statements, invite derision from the radicals, since there is no sound basis for their position:

> It is either beyond justification ("you simply must know how to…"), or founded in nostalgia ("at least once in one's life one should have done…"). (Keitel, 1989, p. 11)

On the other hand, many mathematicians and mathematically educated parents feel strongly against this trend. The main argument of the opponents of drastic changes in arithmetic curricula is that the topics taught in the elementary school are not isolated subjects. There is a long tradition of developing the mathematical culture of students banking on their knowledge and abilities acquired during primary mathematics education. Many of the opponents of traditional teaching are unaware of the use of the abilities acquired in the elementary school during more advanced mathematics education (for a discussion of this point, see Klein and Milgram, 2000).

The vast majority of the population is unconcerned with this problem. The existence of the mathematically unaware majority implies that many parents would not like their children to be subjected to what they perceive as unnecessary senseless drilling for the sake of some 'dubious' benefits in their future education. No convincing arguments from scientists will help. For the time being, the system continues through inertia, but it cannot be sustained in the long run any more than the teaching of classical languages, which were abolished when education became egalitarian.

The conclusion is that educators who are aware of the contribution of elementary school mathematics to advanced education have to search for alternative ways of inculcating these skills, while helping delay the changes until an alternative has been created. In the secondary school humanities curricula, ancient literature written in Greek and Latin have been substituted successfully by works of recent authors writing in modern languages. There is no reason to believe that it is impossible to update in mathematics.

Concerning the secondary-tertiary mathematics education interface, the relevance of school mathematics curricula is frequently questioned by college educators. On the one hand, it does not provide the needed abilities:

> It is now accepted that A-level no longer provides what it used to as a preparation for university study. (Kent and Noss, 2002, p. 9)

On the other hand, there are unnecessary demands:

> There is a feeling expressed by some practitioners and academics that the A-level mathematics requirement is acting as a block against allowing a more diverse group of people to enter into the civil and structural engineering profession. (p. 14)

Looking from the opposite direction, some of what seem to be essential topics appear in the college curricula for traditional reasons, *e.g.*, as a result of the custom of creating mathematics courses for non-mathematicians as watered-down versions of professional mathematics courses. Some of the learning chains (chains of skills or concepts), which justify the teaching of elementary topics because they provide the basis for advanced subjects, now could end in questionable 'maximal elements'. As an example, consider the *long division* → *division of polynomials* → *integration of rational functions* chain. Is it really essential to teach various intricate integration methods to different populations learning calculus? The importance of these techniques lies in the fact that they are a tool to obtain a result in terms of elementary functions. Many important integrals do not have such features (*e.g.*, elliptical functions or Bessel functions). Someone who needs to make a computation in which such integrals are involved uses approximations, consults tables, or turns to calculators and computers.

So, why bother if one more family of functions is added to this list? As long as most students had the required algebraic skills this question was not worth considering, now one may claim that the ends of teaching some techniques do not justify the means. Just compare the situation with that of polynomial equations. Similarly, there is no analytical solution for most equations (those of high powers). The analytical solution for cubic equations obtained by Cardano has been available since the sixteenth century. However, for many years, almost nobody, including professional mathematicians, has studied this solution. If such an equation is encountered it is usual either to use some approximation or turn to handbooks for the formula.

Research programs studying mathematics from the educational point of view, constructing alternative chains of needed competences, could be useful both for the immediate purpose and also for a better understanding of the structure of mathematics [9]. There is no ultimate hierarchy from simple to difficult – it depends on tradition. Actually, the history of the position of arithmetic in mathematics is the best example: for many centuries the best scientists had no dexterity in division and even multiplication, using tables in their computations. In the Hellenic tradition, logic could be considered more elementary than arithmetic.

The computer metaphor as a conceptual framework

In addition to a demand for changes, the computer revolution creates new conceptual opportunities. A viable metaphor is known to be helpful to the learning process (*e.g.*, Sfard, 1994) [10]. It is one of the contributions of science and technology to give us metaphors that we can use to organize and express our experience in life (Kadanoff, 2002). Most mathematical topics taught to non-mathematicians developed before the computer age. So, computers did not appear among the metaphors around which mathematics has grown. What is the potential of this metaphor?

Virtual computers

A computer is a tangible object and the model of computation associated with it can be understood in concrete terms. Thus, it seems fruitful to reformulate various mathematical topics in terms inspired by the computer metaphor. This can help students cope with the abstractness of mathematics (similarly, for example, to various metaphors that made negative numbers teachable in primary schools [11]).

The abstract nature of mathematics (and, in particular, that of algebra) is a perpetual cause of negative feelings towards mathematics among generations of high school students. This abstractness stems from the platonic model of mathematics, which *is* natural to mathematics teachers [12]. The lack of tangible references plays an important part in creating a feeling of the irrelevance of mathematics in many students, including those who like to work with computers.

Recent developments in mathematics education show that reification, *i.e.*, thinking in terms of objects, is an important part in absorption of new material (*e.g.*, Sfard, 1994). The nature of mathematical objects is a serious philosophical problem. Most working mathematicians (and certainly the vast majority of mathematics teachers) are not interested in this question, being satisfied with the vague Platonism mentioned above. It is only natural that schoolchildren have difficulties in creating models of unknowns and parameters, for instance. So, it is worthwhile to attempt to create concrete models of mathematical objects [13].

It is important to stress the difference between a computer as a tool (real computers) and as a concept (virtual computers). The use, and even the existence, of real computers is neither a necessary nor sufficient condition for the use of computers as a conceptual framework. This use is similar to the various formalisms aimed to conceptualize the meaning of algorithm (such as the Turing machine). They were invented before the appearance of computational hardware, and inspired the first computer designers. The problem of all these formalisms from the educational perspective is intricate technical details. Since the present purpose is of a different nature we can discard these details in educational applications [14].

Here is a prime use of the computer metaphor [15]. It is built around a simple model of a computer (see Figure 2), which could be called a virtual computer. This model includes various computer-related notions that naturally arise when one thinks about mathematics in terms of this metaphor. A virtual computer could be seen as a device somewhat like a food processor: the basic frame with modular hardware. The hardware is composed of container cells and processing units. Inputs are inserted in special input cells. An output is returned at the outlet (*e.g.*, special output cells) and can be taken (copied) for further use. For each type of object, there are cells of a suitable structure. For each processing unit, there is a specific type of allowed input, so that only objects of this type can serve as inputs. No two units can be used simultaneously. New units can be added to a device. *To solve the problem amounts to designing a suitable internal structure of the processing unit.*

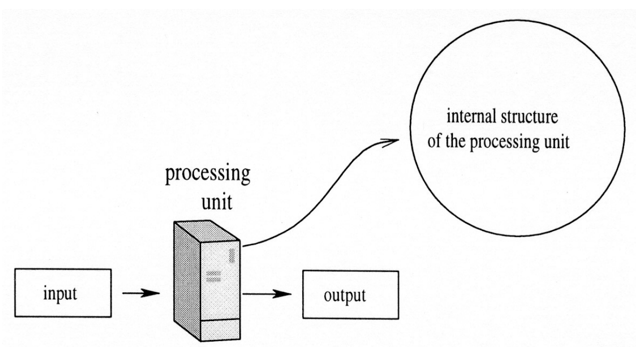

Figure 2: The generic metaphor

The computer metaphor invites the writing of mathematics in an algorithmic style, emphasizing the algorithmic nature of mathematics. Teaching algebra along these lines transfers the stress from numerical problems to solutions of generic (parameterized) problems. A natural way to do mathematics in algorithmic style is writing in simple pseudocode. Such pseudocode, used below, is based on just three standard structures (assignment, conditioning, and the WHILE-loop).

As an example, consider the solution of a linear equation, ax = b (see Figure 3).

Apparently the two representations are very similar, however, there are important differences in the meanings assigned to various symbols. The algorithmic style representation helps create tangible models for mathematical objects:

1. Parameters *a* and *b* in *SimpleLinearEquation* (a,b) [*SLE*] are inputs inserted in specific cells.

2. The computed value of the unknown is the output.

Regular style

1. a=0

1_a b=0: x any real number

1_b b ≠0: No Solution

2. a ≠0: x=b/a

Algorithmic style

```
SimpleLinearEquation(a,b)
 IF a=0 THEN IF b=0 THEN RETURN          (−∞,∞)
             ELSE RETURN "No Solution"
 ELSE RETURN b/a
```

Figure 3: Contrasting regular and algorithmic styles of solution of the equation ax = b.

The command "RETURN b/a" leads to a delivery of the result at the outlet. Thus, a clear distinction between parameters (as inputs) and unknowns (as outputs) is established.

3. *SLE* is an algorithm *object* supporting reification.

4. The functional form of writing *SLE* (a,b) emphasizes the dependence of the solution on parameters.

5. The algorithmic style naturally focuses attention on special cases, encouraging use of the "IF" clause, while in the traditional way the unconditional "x = b/a" answer is a common mistake.

Now, the algorithmic style solution can be associated with the internal structure of the processing unit, depicted as a flowchart (see Figure 4).

Algorithm (SLE) can be used in the solution of other equations, turning attention to reduction as an essential tool of mathematics. The "return to the previous problem" line does not usually appear in standard mathematics textbooks, while using a previously written procedure is natural in algorithmic style.

Consider the equation $a_1x + b_1 = a_2x + b_2$. There are two possibilities to describe the solving procedure:

1. inserting "$a_1 - a_2$" into the A cell and "$b_2 - b_1$" into the B cell (see Figure 6a);

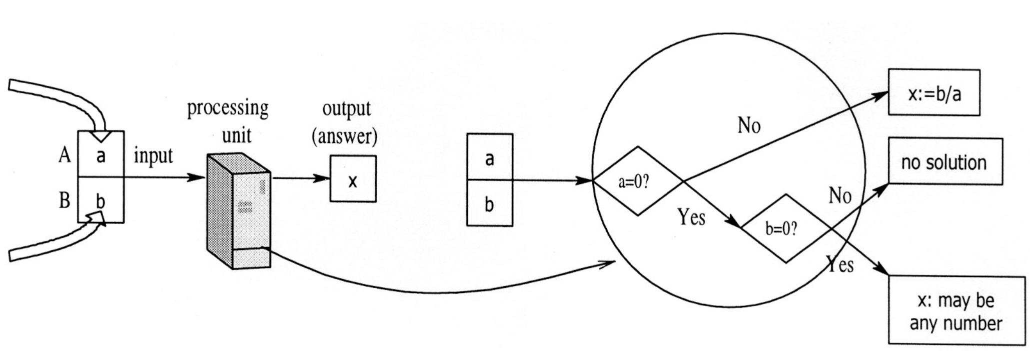

Figure 4a: The external view

Figure 4b: Internal structure of the solving unit

Figure 4: The solving unit for the linear equation ax = b.

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Regular style
a1x+b1=a2x+b2
(a1-a2)x=b2-b1
return to the previous problem
```

```
Algorithmic style
LinearEquation(a1,b1,a2,b2)
   a:=a1-a2; b:=b2-b1;
   Sol:=SLE(a,b);
   RETURN Sol;
```

Figure 5: Contrasting regular and algorithmic styles of solution of the equation $a_1x + b_1 = a_2x + b_2$.

2. connecting to the A and B cells an additional unit which prepares correct input for the basic unit solving *SLE* (see Figure 6b).

There are striking similarities between the constructivist tendency in mathematics education and the object-oriented approach in computer science. While the former emphasizes the construction of 'cognitive' mathematical objects, the latter demands that a computer specialist define formal objects for computer applications. Thus, in addition to being less abstract, the cognitive reification *explicitly taught as a method*, prepares mathematics students to work in an object-oriented environment [16].

Computer-referred definition of mathematics

Truss's description of computer-related mathematics (see page 43 this article) provokes a question: how can we talk about mathematics that "most people would not consider to be mathematics at all"? Suppose someone says: "This is a chair, but most people would not consider it to be a chair." What could such a statement possibly mean? A reasonable interpretation is that this piece of furniture was built to be sat on, but it is not obvious from its look. Such understanding comes from the dictionary definition of a chair. So what is its analogue for mathematics?

Khait (2005, p. 145) suggested the following answer to this question. A topic/ an object/ a statement belongs to the scope of mathematics whenever words (symbols and diagrams included) that appear in it could be assigned with precise meanings, otherwise mathematics cannot be applied in a natural way. *Mathematics is a linguistic activity, which is characterized by the association of words with precise meanings.*

Concerning the meaning of "precise meanings", the situation can be compared with that of the concept of algorithm. The latter is delimited by the Church thesis that essentially states

> that all serious proposals for a model of computation have the same power. (Hopcroft *et al.*, 2001, p. 318)

If we limit ourselves to finite mathematics, to which most mathematics relevant for computer-related applications belongs, then computers can serve as the precision criterion: a precise formulation is one that can be translated for a computer. The importance of such a tangible reference is most clear when we consider infinite mathematics. Indeed, concerning infinite structures and theories there is no such referee except the public opinion of colleagues [17].

Concluding remarks

Returning to the quotations at the beginning of this article, mathematics educators should reconsider their subject in the light of changes in circumstances. The arrival of a new conceptual paradigm associated with computing devices, which substituted the nineteenth century machine as the basic metaphor used in description of a wide range of phenomena, suggests a direction for such work. Creating a unifying view of educational mathematics around the computer paradigm can serve as a unifying umbrella for a wide range of diverse topics.

Changes in mathematics education that stem from the computer revolution have the potential of making an impact on mathematics as a science, similarly to that which happened about 200 years ago when technological development led to drastic changes in mathematics education [18]. Thus, it becomes a fascinating subject from the purely mathematical perspective.

Notes

[1] In different sections of the article "mathematics education" is interpreted according to the context as general school education or as preparation of experts for computer-related specializations. These two aspects are strongly connected both because obligatory education has to enable graduates to pursue careers of their choice, and because mathematics-related abilities needed for the general population living in computer-infested environments

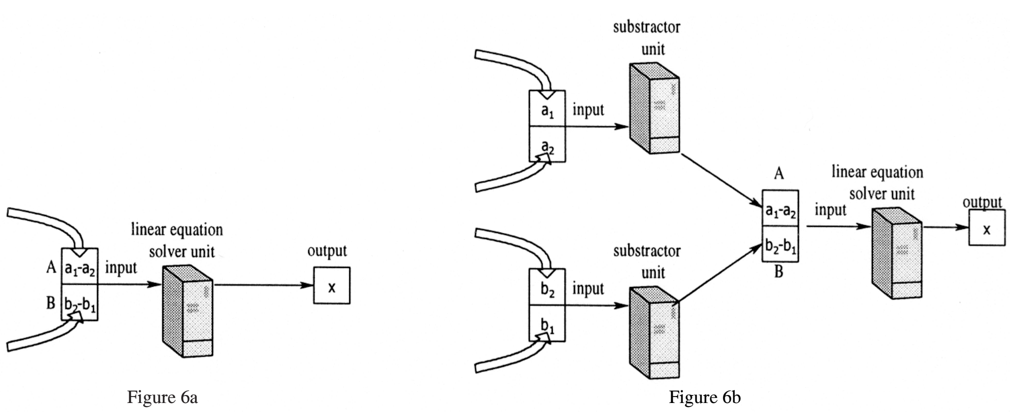

Figure 6: Two possible solving units for the equation $a_1x + b_1 = a_2x + b_2$.

are similar (albeit less advanced) to that of professionals in computerized workplaces [3].

[2] Since the time these remarks were published, the feeling of dispensability of traditional mathematics education, which then was more a subject of academic discussion, has penetrated the society as a whole. Children are less inclined to learn mathematics in schools. Thus,

> the problem of a gap between the expectations and reality of the mathematical competence of new undergraduate students has been a concern across all numerate degree subjects since the mid-90s. (Kent and Noss, 2002, p. 9)

[3] Note the similarity to the mathematics-related expectations of an average citizen to react adequately to the deluge of information bearing data of various types.

[4] This is in contrast to that of the research and development level where the mathematical demands are on par with those of professional mathematicians.

[5] By this I mean a generic language of communication in the formal context, independent of specific devices or software. The means of formal communications in mathematics is a prototype of formal languages.

[6] One may hypothesize that the main problem of learning mathematics stems from the gap between the natural fuzziness of human thought and the need for precision in doing mathematics (for a discussion, see Khait, 2005).

[7] Khait, A. (2004) 'Advanced mathematical thinking in computerized environment', presented in *Topic Study Group 13: Research and development in the teaching and learning of advanced mathematical topics* ICMI10, http://www.icme-organisers.dk/tsg13, last accessed, 29th June, 2005.

[8] In addition to its direct utility, the analysis of mathematical contents of such 'dirty' problems from the educational perspective can lead to deeper understanding of the nature of basic mathematical structures in the context of the mathematics/average-humans interface.

[9] Khait (2003a) argued that the urgency of research in teaching elementary subjects could be downgraded, while the highest research priority should receive inculcation of new advanced topics (needed for work in the computerized environment) into students who are not inclined to mathematics. This is because there is no tradition of teaching a new kind of mathematics to such a population, in contrast to the long experience of teaching arithmetic successfully to most children attending reasonable schools. It seems that the cultural aspect has to be taken into account and the conclusion concerning the elementary subjects has to be amended along the lines presented in this section.

[10] Moreover, emerging from Lakoff and Núñez (2000), all our thinking, including the most abstract, is bound to our bodily experience and based on physical metaphors.

[11] To appreciate the non-trivial nature of this fact, it is sufficient to note that Augustus de Morgan argued in 1830s that it is wrong to teach children such a "nonsensical" subject as negative numbers (Pycior, 1983).

> A basic and often implicit underlying philosophy of mathematics in teaching is that of 'sufficiently liberal Platonism', which teachers have acquired during their university studies and then taken with them to school. (Seeger and Steinbring, 1994, p. 151).

> One of the many features of computer programs […] is that they present objects. Thus they can be used to support reification. (Mason, 1989, p. 6).

[14] Without entering into the arguments, note that the positive effect of extensive use of computers in teaching mathematics is controversial. So, Dubinsky (2000) advocates using programming to facilitate mathematics learning, while Kent and Noss (2002) are less enthusiastic:

> Two negative factors are the high cost of moving 'chalk and talk' mathematics teaching out of lecture rooms and into computer laboratories, and the lack of a common grounding in mathematical technology in school mathematics curricula. There are significant dangers in losing the teaching of pen-and-paper mathematical techniques to 'button pressing'. (p. 10)

[15] A development of an idea first presented in Khait (2003b).

[16] A unified approach introducing students to mathematics and computer science could be fruitful for both. Khait (2003b) considers the eventual convergence of mathematics education with computer science education.

[17] This was clearly demonstrated by the history of the acceptance of Cantor set theory by the mathematical community (*e.g.*, Dauben, 1990).

[18] See Grabiner (1974/1986) for a historical description of this episode.

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