STUDENTS WORKING WITHIN AND BETWEEN REPRESENTATIONS: AN APPLICATION OF DIENES'S VARIABILITY PRINCIPLES

SERIGNE GNINGUE

This article describes an attempt to introduce algebra concepts to sixth and seventh grade (11- and 12-year old) students by applying Dienes's "perceptual and mathematical variability principles."

Zoltan Dienes, a Hungarian mathematician who immigrated to England then to Canada, contributed greatly to the 'new mathematics,' which was developed in reaction to the Soviet launch of Sputnik in 1957. His principal interest was in the early learning of mathematical concepts. In fact, he can justifiably be viewed as one of the fathers of the extensive use of manipulatives and play in teaching those concepts. His genius lies in designing experiences that can expose children to higher-order mathematical concepts (Wisthoff, 2001).

Dienes believes that each child may see the world differently, approach it differently, and understand it differently. Therefore, to ensure that all children learn a concept with understanding, along with children's active participation in its development, Dienes prescribes the use of various representations of the concept rather than a single representation (Dienes, 1971). In particular, he postulates a system of teaching mathematics based upon two principles of learning, upon which to base concept development:

perceptual variability principle or *multiple embodiment principle*, which suggests that conceptual learning is maximized when children are exposed to a concept through a variety of physical contexts or embodiments (Dienes, 1971). The provision of multiple experiences (not the same experience many times), using a variety of materials, is designed to promote abstraction of the mathematical concept. For instance, when learning about the regrouping procedures used in the process of addition, a child might be presented with chips, counters, abacus, or with the Multibase Arithmetic Blocks (MAB, also called base-ten blocks). In this way, the child is more likely to perceive that the regrouping procedures are independent of the different types of materials used. Likewise, when learning about fraction concepts, children might be presented with fractions represented in various forms such as pattern blocks, fraction bars, fraction circles, or Cuisinaire rods. In that way, they will make connections between the manipulatives and fractions and will not relate fractions to only one physical model.

The mathematical variability principle implies that children need to experience many variations of "irrelevant attributes" (complicating factors) linked to the concept structure, in order to single out the general mathematical concept which is constant to all manipulations. For instance, if one wishes to promote an understanding of a parallelogram, this principle suggests that one should vary as many of the irrelevant attributes as possible: the size of the angles, the length of the sides, the position of the parallelogram on the paper, while the relevant attribute – opposite sides parallel – remains constant. In teaching the concepts of simplification of algebraic expressions, varying the signs and the coefficients of the variables using whole numbers, decimals or fractions, and keeping constant the relevant attributes (same variable to the same exponents) will make students become conscious of what happens to different numbers in the similar situations while ensuring an understanding of like and unlike terms. The generalization of the mathematical concept will consequently derive from the many variations of those elements.

Dienes considers the learning of a mathematics concept to be difficult because it is a process involving abstraction and generalization. He suggests that the two variability principles be used together since they are designed to promote the complementary processes of abstraction and generalization, both of which are crucial aspects of conceptual development.

The results that are discussed in this study pertain to the application of Dienes's variability principles for the solving of first-degree equations of one variable only. I sought to determine the extent to which the use of the variability principles, implemented with manipulatives, enabled middle school students to perform the algebraic processes of solving equations. Differences in the effects of using the variability principles and manipulatives related to age were analyzed.

Background

The study took place during the spring of 1998 and lasted about four weeks. It involved students in a middle school located in the Bronx in New York City. A sample of four classes, two seventh-grade and two sixth-grade, which I taught, participated in the study. Eight students who had poor attendance or whose parents did not consent to the study were not taken into consideration but were taught in the same manner. All 53 participating students in the seventh-grade classes were aged 12 or slightly over, while all 53 participating students in the sixth-grade classes were aged 11 or slightly over. There was no control group given that research describes 'show-and-tell' approaches not to be appropriate to teach high algebraic concepts, especially to novice students.

The two experiments

Preliminary activity: A preliminary unit on integers using base-ten blocks of different colors (blue and yellow cubes) was first designed and implemented with all sixth and seventh grade students participating in the study. Familiarity with the behavior of integers was needed because an algebra is implicitly present in the generalized procedures for dealing with and producing integers (Wheeler, 1996). In fact, working with integers using the cubes allowed students to form mental representations they would use later when investigating equation concepts in the second experiment.

Following the preliminary unit on integers, units on algebraic expressions and variable multiplication were also implemented to allow students to develop a good understanding of the concepts of variable, coefficient, like and unlike terms. Algebra tiles and base ten blocks were utilized to develop such concepts.

Perceptual and mathematical variates: Behr, Harel, Post and Lesh (1992), in an experiment with the variability principles on fraction and ratio concepts, used the term "perceptual variates" to describe the different materials used, and the term "mathematical variates" to describe the many complicating factors whose variations do not change the general mathematical concept. I used the same terminology as Behr *et al*. to describe those features.

Figure 1 shows the first perceptual variates used in the first experiment. They consisted of numbered cubes representing numbers, circular-shaped counters representing variables, a flat scale made from construction paper, and a fixed physical scale used by the teacher for demonstration purposes. The model described in this first experiment is a variation of a model of representing equations used by Borenson (1986). Figure 2 shows the second perceptual variates used in the second experiment. They consisted of base-ten blocks, blue and yellow cubes representing positive and negative numerals respectively, and blue and yellow bars representing positive and negative variables respectively. The flat scale and a fixed physical scale were also used in the second experiment.

Perceptual Variates	Mathematical Variates and Weight
- Square numbers showing numbers 1 – 10 1 2 3 4 5 6 7 8 9 10	**MV1** - One- or two-step equations (**A0**). **MV2** - More than two-step equations (**A1**) **MV3** - The equation contains like terms (**A1**)
- Circular Shape Counters as variables "x" - Flat Scales drawn on students' worksheets	**MV4** - The equation has the variable on both sides (**A1**) **MV5** - Use of decimals as coefficients (**N2**) **MV6** - Use of fractions as coefficients (**N2**)
- Physical Stationary Scale for teacher for demonstration purposes	
0, **1**, or **2** = Weight **A1** = Algebraic Variate of Weight 1	**MV1, 2,3…** = Mathematical Variate 1,2,3… **N2** = Numerical Variate of Weight 2
Processes: **1)** Elimination by removing as in x + 4 = 6, or by splitting (dividing) as in 2x =8)	

Figure 1: Perceptual and mathematical variates used for the first equation experiment.

Perceptual Variates	Mathematical Variates (Complicating factors)
-Blue and Yellow Base Ten Blocks +1 -1 +x -x Blue yellow blue bar yellow bar	**MV1** - One- or two-step equations (**A0**) **MV2** - More than two-step equations (**A1**) **MV3** - The equation contains like terms (**A1**) **MV4** - The equation has the variable on both sides (**A1**)
-Blue blocks are drawn shaded -Yellow blocks are drawn no shaded - Flat scales drawn on students' worksheets	**MV5** - Use of decimals as coefficients (**N2**) **MV6** - Use of fractions as coefficients (**N2**) **MV7** - Use of negative integers (**N1**) **MV8** - The equation contains parentheses (**A2**)
- Physical stationary scale for teacher for demonstration purposes	
New processes: **1)** Renaming process (adding the opposite in a subtraction) **2)** Elimination (Cancellation) is done by use of the multiplicative or additive inverse 3) Elimination using zero-pairs.	
0, **1**, or **2** = Weight **A1** = Algebraic Variate of Weight 1	**MV1, 2,3…** = Mathematical Variate 1,2,3… **N2** = Numerical Variate of Weight 2

Figure 2: Perceptual and mathematical variates used for the second equation experiment.

The first perceptual variates were utilized to introduce the concept of equation solving without reference to negative integers, whereas the second perceptual variates were designed to incorporate the use of both positive and negative integers. The learning of integers presents difficulties that, when added to those intrinsic to learning new algebraic processes, could make learning those algebraic processes more difficult for younger students. Consequently, the use of the two materials in the two activities allowed me to disassociate the difficulties that come with the use of negative integers from those of understanding the different processes involved in solving equations.

Numerical and algebraic variates: Two types of mathematical variates were identified in equation solving: numerical variates and algebraic variates. A numerical variate is a decimal as coefficient, a fraction, a negative integer, an exponent, or any number other than a positive integer – a number that has the potential to add a degree of difficulty to a problem. For instance, students may find equations such as $(2/5)x + 8 = 4/5$, $.25x + .7 = 8.625$, or $-4x + 17 = -35$ more difficult to manipulate and solve, than equations like $2x + 8 = 4$ because of the presence of the fractions, decimals or negative integers as coefficients. An algebraic variate is defined as a process conceptually linked to the algebraic process to be performed that is not necessary for understanding the concept. Four such variates were identified for equation solving; they were determined by whether the equation contained like terms on the same side of the equation; parentheses, $2(3x + 2) = 22$; a variable on both sides, $3x + 5 = 2x + 13$, $8n = 60 - 4n$; or by the number of steps necessary to solve the equation (see Figures 1 and 2).

Weight of equations and difficulty levels: Each mathematical variate (numerical and algebraic) was given a 'weight' of 1 or 2. Except for fractions, decimals, and parentheses, which were given a weight of 2, all other variates were given a weight of 1. Decimals and fractions were considered as possibly having an additional effect that was not linked to the use of manipulatives. A student could, for example, correctly carry out the process of solving an equation, but could get the solution wrong because of failing to divide by the decimal or fraction correctly. This study did not involve manipulatives in the use of fractions and decimals, whereas algebraic variates are features whose variations have a lesser effect on problem solutions.

Difficulty Level 0 (Weight = 0)

1. $3n + 13 = 43$ (MV1)
2. $3y = 3$ (MV1)
3. $10 = 4x + 10$ (MV1)
4. $y + 8 = 8$ (MV1)

Passing score: 3 / 4 correct

Difficulty Level 1 (Weight = 1)

5. $4x + 15 = 9x$ (MV4)
6. $4x + 5x = 18$ (MV3)
7. $10 = 4x - 10$ (MV7)
8. $-18 = 3m$ (MV7)

Passing score: 3 / 4 correct

Difficulty Level 2 (Weight = 2)

9. $3x + 5 = 2x + 13$ (MV2,4)
10. $4x + 9x + 8 = 47$ (MV2,3)
11. $8n = 60 - 4n$ (MV4,7)
12. $\frac{2}{5}x = 8$ (MV 1,6)
13. $0.3d + 2 = 8$ (MV 1,5)

Passing score: 4 / 5 correct

Difficulty Level 3 (Weight = 3)

14. $4h + 3 = 23 - h$ (MV 2,4,7)
15. $9m + 2 + 6 = 4m + 20 + m + 16$ (MV 2,3,4)
16. $9x - 40 + x = 0$ (MV 2,3,7)
17. $\frac{3}{5}p - 4 = 26$ (MV 6,7)
18. $2(3x + 2) = 22$ (MV 2,8)

Passing score: 4 / 5 correct

Difficulty Level 4 (Weight = 4 or more)

19. $3y = 2(10 - y)$ (MV 2,4,7,8)
20. $3(2x + 1) + 5 = 3x - 10$ (MV 2,4,7,8)
21. $\frac{2}{3}t - 11 = 64\ 4\frac{1}{3} - t$ (MV 2,4,6,7)

Passing score: 2 / 3 correct

MV1, 2,3… = Mathematical Variate 1,2,3… present in the equation as defined in **Tables 2** and **3**

Figure 3: Difficulty levels of equations (passing score: at least 2/3 correct).

Each equation was consequently assigned a weight equal to the sum of the weights of the mathematical variates it contained. The weight of an equation measured the 'difficulty level' in which it belonged. The idea of identifying difficulty levels originated from a design similar to that of the CSMS team (Hart *et al.*, 1981), but I assigned weights to the variates.

In the different difficulty levels, the more variates an equation contained, the harder the equation was and, thus, the higher the level of difficulty. An equation was considered to be at difficulty level 0 if its weight was 0. Equations in difficulty levels 1, 2, and 3 had a weight of 1, 2, and 3, that is, any combination of mathematical variates that added up to 1, 2, and 3 respectively. Equations in difficulty level 4 had a total weight of 4 or more, that is, any combination of mathematical variates that added up to 4 or more. The different mathematical variates and their weights (MV1, 2, 3 …) are described in Figures 1 and 2.

Post-test instrument: I administered a 21-question post-test that I had designed to measure performance. The questions were a collection of standard equation problems taken from different New York State examinations, from years prior to the study, for ninth grade and advanced eighth grade students. The twenty-one questions were sorted into five groups of different difficulty levels (Figure 3).

Scoring procedures: One point was given for each correct answer, and 0 point for each incorrect answer. No partial credit was given even though students had to show their work. The level of understanding displayed by a child was then assessed by determining the most difficult group of items on which he/she showed competency. To do this, the criterion of correctly answering at least two-thirds (about 67%) of the items within a difficulty level was used. This criterion-referenced interpretation is commonly used in testing procedures to determine if students have attained a certain level of proficiency in a cluster of items that measure an objective (Küchemann, 1981; Frykholm, 1994; Carney and Schattgen, 1994).

Data analysis: For each difficulty level, the percentages of 6th and 7th grade students who performed above the passing score of the different difficulty levels were studied. A look at textbooks used in the district revealed that level 0 and level 1 were the most basic equation-solving levels for sixth grade, while level 3 was the most basic equation solving level for seventh grade. Consequently, performance was deemed satisfactory at level 1 or beyond for sixth graders, at level 3 or beyond for seventh graders.

Teaching sequence: The first experiment was implemented through a sequence of four activities (see Figure 4) which lasted two to three days each. However, students spent more time with the first two activities than with the last two. The activities progressed from the solving of simple one-step equations, to the solving of multi-step equations with the variable on both sides. Equations containing decimals and fractions (new mathematical variates) were introduced, in each activity, once students passed from the pictorial to the symbolic manipulations, and after the concept of leaving the variables alone on one side had been assimilated. Students needed to familiarize themselves with the processes of translating manual and pictorial manipulations into symbolic ones, so that they could understand and explain the more difficult equations that would come later in Activities 3 and 4 (see Figure 4).

Figure 4 shows and describes examples of the types of problems that were studied in Activities 1, 2, 3 and 4, using circular counters as the variable *x*, and cubes numbered 1-10. The second experiment, which was also implemented through a sequence of four activities (see Figure 5), intro-

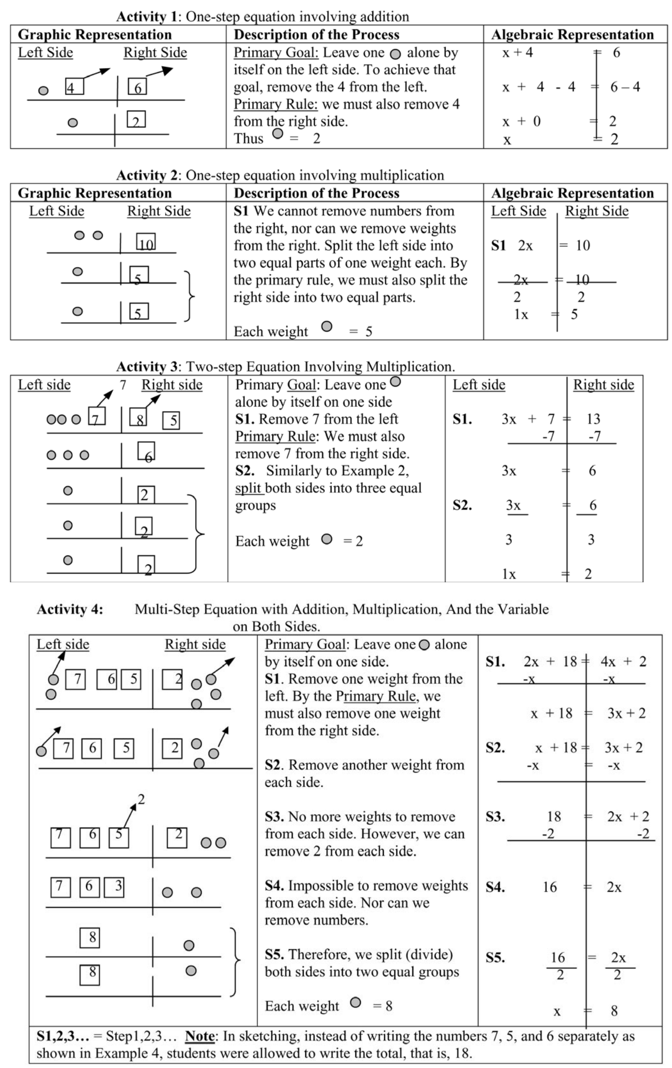
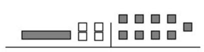


Figure 4: Description of activities 1, 2, 3 and 4 of the first experiment

duced students to solving equations containing negative integers and negative variables. The base-ten blocks were used for that purpose. The variable x was represented by a blue (shaded in drawing) tens-bar and the variable $-x$ was represented by a yellow (not shaded in drawing) tens-bar . Unnumbered blue (positive, shaded in drawing) and yellow (negative, non-shaded)cubes represented the integers. It took less time to implement the second experiment because of students' familiarity with the translation of hands-on and drawing manipulations to symbolic ones.

Two key ideas drove this experiment. The first was the concept of renaming all subtraction problems 'addition of the opposite'; that concept was studied in the preliminary unit on integers. When solving the equation $x - 4 = 9$, for example, students were led to discover that the variable x needed to be represented using one x-bar, and four zero pairs to make it possible to remove 4 (four blue cubes) from x. After removing four blue cubes from this representation of x, the blue bar and four yellow cubes was left. The expression $x - 4$ was thus written as $x + (-4)$ and the equation $x - 4 = 9$ rewritten as $x + (-4) = 9$:



The second key idea is the concept of elimination using the additive inverse to form zero pairs. Now, in solving the equation $x + (-4) = 9$, to leave x alone by itself, the four yellow cubes that represent (-4) are cancelled by *adding four blue cubes to the left* (and to the right side by the Primary Rule) to form zero pairs. Such actions enabled me to introduce the concepts of elimination by using the additive inverse or the multiplicative inverse.

Primary Goal and Primary Rule. In each experiment, a flat scale was given to students represented by two intersecting lines forming two sections. Each section represented one side of the scale: a left side and a right side. Students were given two simple laws to follow: a *Primary Goal* and a *Primary Rule*. The Primary Goal was stated as follows: "to find the value of an unknown weight, we must achieve the goal of getting the unknown weight alone by itself on one side of the scale." To satisfy that goal, the Primary Rule must be applied. The Primary Rule was stated as follows: "Whatever is done on one side of the scale must be done on the other side of the scale to keep it balanced." Symbolic representations were introduced once the processes of finding an unknown weight manually and pictorially seemed assimilated. In the symbolic representations, the Primary Goal and Primary Rule were restated by using the word variable instead of weight, and the word equation instead of scale.

In both experiments, each activity was also followed by a series of related equation-solving exercises. Students were not required to use the manipulatives in these exercises, but were encouraged to do so whenever possible if they encounter difficulties. They were, however, required to sketch each step and describe the process in writing. Describing the processes through writing allowed students to be aware of difficulties they encounter along the way, and find ways to resolve them.

Figure 6 compares the percentages of 6th and 7th grade students who reached the different difficulty levels, seeming

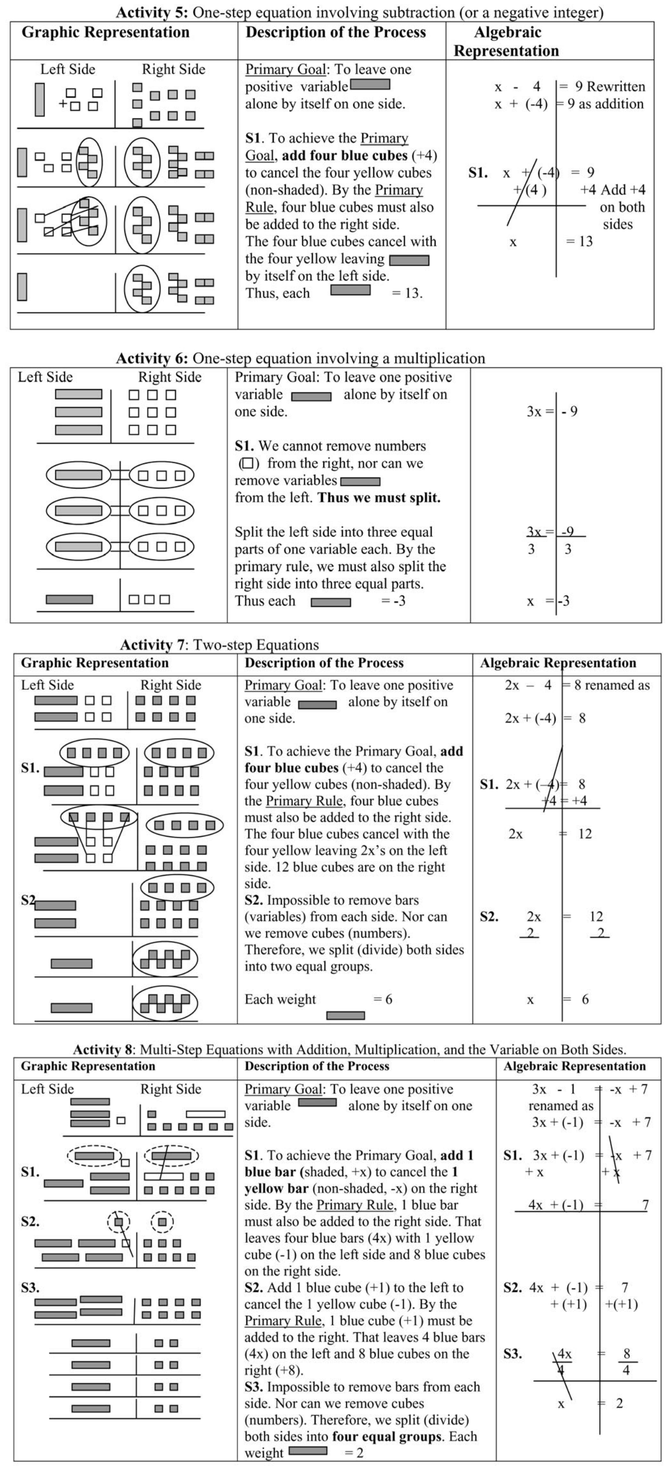


Figure 5: Description of activities 5, 6, 7 and 8 of the second experiment.

to indicate that 7th grade students performed better than 6th grade students at all five levels.

Level 0 and level 1 equations were well mastered by most students in the 6th and 7th grades: 82% of 6th graders compared to 88% of 7th graders reached level 0, while 84% of 6th graders compared to 96% of 7th graders for level 1. For level 2, many 6th graders easily solved the first three equations, but

Difficulty Levels	Age 11 (6^{th} graders) N = 49	Age 12 (7^{th} graders) N = 51
Level 0	82	88
Level 1	84	96
Level 2	39	67
Level 3	29	63
Level 4	22	55

Figure 6: Percentages of 6th and 7th grade students who reached the difficulty levels.

had difficulties solving the equations $(2/5)x = 8$, and $0.3d + 2 = 8$. They carried out the process properly, but failed to perform correctly a division by 0.3 or by 2/5. The presence of some mathematical variates (parentheses, fractions, decimals) in the equations seemed to present cognitive difficulties for them more than for the 7th graders. Only 39% of 6th grade students compared to 67% of 7th grade students reached level 2. Such differences were even more evident on level 3 and level 4: 29% *vs* 63% on level 3, and 22% *vs* 55% on level 4. The percentages of 7th grade students who reached the three highest levels (2, 3, and 4) are almost double the percentages of 6th grade students who reached them. The application of the variability principles was perhaps more successful with 12-year olds than with 11-year olds students.

Discussion

The application of Dienes's variability principles to teach equation concepts and processes can be deemed a success for both groups for many reasons. Firstly, levels 0 and 1, the most basic equation solving levels for sixth grade students in most curricula, were reached by more than 80% of that age group. Even though results seem to indicate that 7th grade students seem to perform better than 6th grade students, it is not my belief that 6th graders are not ready to study these concepts and processes. Throughout the study, it was clear that processes of elimination and cancellation used to solve equations were fairly mastered by most students from both groups. The main difference between the two groups lies in difficulties related to the presence of mathematical variates. Seventh graders had had the chance to do more review and practice on concepts involving the use of fractions, decimals, and parentheses through the distributive property. The use of manipulatives that embody these variates and enable the separation of cognitive difficulties (as done with the integers) could enable 6th graders to close the gap.

Secondly, students learned and internalized the difficult concepts and processes of equation solving not through teacher explanation and rote memorization alone, but for understanding, through exploration, visualisation, verbalization, and discovery. They were able to make connections between the manipulatives and the concepts, represent pictorially and symbolically the processes involved, and establish themselves an appropriate sequence of steps needed to solve equations. For instance, when representing two-step equations such as $2x + 5 = 17$ using the materials, students were automatically removing the 5 blue cubes from both sides before splitting each side by 2. The reason given by one student (agreed upon by the others) was that he could not 'see' how to he could divide equally $2x + 5$ on one side and 17 on the other side. That led to the finding that when two numerals are on the same side of the variable, the additive inverse is used first before the multiplicative inverse. That discovery was fundamental in the sense that students found the order in which to cancel numbers to be natural when using the manipulatives.

Finally, I was able to design and apply an effective teaching strategy, and discovered ways of alleviating some of the many different cognitive difficulties students have in regard to the concept of equation. For instance, at the beginning of the experiment, when discussing the equation $x + 4 = 7$, students' obvious knowledge that the value of x was 3, interfered with the learning of the new process of elimination and with the concept of equality that were being taught. The use of the scale model to represent an equation associated with the defining of a *primary rule* and *primary goal* forced them to focus on the manipulations involved rather than on trying to guess the value of the variable. This conception of the scale, (the idea of a value on the left side that is always equal to a value on the right side) used as the basis of equation manipulations, led thus, to alleviate one source of difficulty referred to by Booth (1988) as students' conceptions of side of an equation.

The passage from the process of elimination by removing to the process of elimination by cancelling represented another discovery. In the first experiment, in which negative numbers were not used, equations of the type $x + 5 = 11$ were solved by removing (subtracting) 5 from both sides, and equations of the type $4x = 12$ were solved using a division on both sides. Solving an equation evolved around the idea of using the inverse operation (subtraction undoes addition, division undoes multiplication, and vice versa). In the second experiment, the second set of manipulatives made it possible to introduce the process of elimination using the additive inverse. In the equation $x + 5 = 9$ for instance, the numeral 5 was cancelled by adding -5 on both sides of the equation. The passage from the process of elimination by removing to the process of elimination by cancelling is not often emphasized in the teaching of solving processes for equations. For the new algebra learner, such transition presents cognitive obstacles that may not be sufficiently surmounted through teacher explanation alone.

Similarly, in solving equations such as $3x = 21$, the multiplicative inverse of 3 (1/3) was introduced to cancel the "3." The use of the controversial term "cancel" had the advantage of making the teaching of the process of solving equations homogenous. No matter what operation is present in an equation, the term "cancel" can be used. If the equation is $x + 3 = 7$, the additive inverse of 3, that is -3, is added to both sides. If the equation is $x - 4 = 9$, after renaming it to $x + (-4) = 9$, the additive inverse of -4, that is $+4$, is added to both sides. Equations such as $(2/3)x = 8$ could also be solved by multiplying both sides by the multiplicative inverse of 2/3, that is, 3/2. Such equations could not be solved using the materials, which did not embody decimals and fraction concepts. The use of the additive or multiplicative inverse, however, seemed sufficient to explain how to solve an equation.

Reflections

The results obtained by applying Dienes's variability principles to middle school algebra can reinforce the argument that middle-school children (aged 11 and 12) can learn higher concepts in algebra if appropriate methods are used.

For Usiskin (1987), certain attributes of algebra make it difficult, if not impossible, for most students to learn algebra before the age of fourteen. The cognitive demands involved in the structural aspects of algebra, being quite different from those of arithmetic (Kieran, 1992), cause many students to have the perception that algebra and arithmetic are different (Chalouh and Herscovics, 1988; Davis, 1975). Children give multiple meanings to symbols such as the equal sign of equation solving (Kieran, 1981, 1992), to conjoint terms such as xy or $7x$ (Matz, 1980), to non-conjoint terms of the form $8 + x$ (Davis, 1975), to conventions such as those for the use of brackets and the order of operations, and to conceptions of a term or side of an equation as well. Elementary algebra students in the US are not usually ready to accept the idea that the equals sign indicates an equivalence relation rather than a "write down the answer" signal (Booth, 1988). Most tend to view the 'equals' sign as an operator symbol, that is, a "do something signal," rather than a relational symbol of equivalence, meaning "the same as" (Baroody and Ginsburg, 1983; Behr, Erlwanger and Nichols, 1980; Kieran, 1981, 1992; Selitto [1]). Such belief makes it difficult for them to understand the balancing of equations later on in school, and to make sense of equations such as $23 = 5x + 8$ or $5x + 8 = 3x + 10$ (Kieran, 1992).

Yet, in other countries, students are being introduced to the language of algebra as early as the primary grades (Usiskin, 1987, 1997). In Japan, for example, algebra drives the mathematics curriculum and Japanese students are expected to solve complicated linear equations involving non-integral coefficients (*e.g.*, $.125x + 2/5 = 7/8$) in their very first year of middle school.

In teaching equation solving in the US, teachers and textbook authors rarely engage in sufficient discourse about the process, its justification, and the meaning of equation and solving an equation. The extent of the vocabulary of written mathematics seems to be limited to "solve the equation" or "simplify the equation," with few alternative verbalizations or visualisations that describe the procedures (Lee, 2000).

Because of the dread that students generally have for algebra, the use of more than one set of manipulatives to introduce tedious and abstract concepts and processes can help reduce students' anxiety, and make them discover that algebra rules make sense and are no different than arithmetic rules. I believe that students who study these concepts at this level will be ready to take algebra as a course in 8th grade and enter high school ready to take on more advanced mathematics courses.

Implications for teaching: The results confirm the importance of introducing algebra first with the teaching of integers, as suggested by Wheeler (1996). The processes of solving equations imply a certain familiarity with the behavior of integers. The use of the first set of materials that embodied positive values only played a crucial role in developing students' understanding and confidence by lessening the difficulties they had to face in learning the new algebraic concept. The difficulties associated with the use of negative values did not interfere with those associated with the concepts to be learned. Middle school students need that dissociation at this stage of basic elementary algebra.

The number of variates identified in equation solving concepts played a crucial role in determining the differences observed between the two age groups. The fewer the number of variates present in solving an equation, the better the 6th grade students performed. Seventh graders showed a greater mastery in solving problems that had numerical variates such as negative integers, fractions or decimals, and algebraic variates such as the presence of parentheses (implying the use of the distributive property) or a greater number of steps required to solve an equation. When designing activities for 6th graders, one should allow students more time to review concepts containing these mathematical variates, or design manipulative activities that embody these concepts.

The application of Dienes's variability principles should not necessarily be limited to the use of just two types of materials, as was done in this study. Had I chosen a third type of manipulative embodying the use of decimals and fractions with the algebraic processes studied, perhaps students would have felt more comfortable in doing related problems and could have reached level 4, the highest level of achievement. At this level of basic introduction to elementary algebra, however, simply understanding and performing these algebraic concepts and processes can be considered a notable achievement for 11- and 12-year-old students. Adding another level of difficulty, by associating decimal and fractional concepts to already tedious concepts, can have negative effects on how middle school students perceive algebra and how confident they are in doing mathematics in general. That is not to say that decimal and fractional concepts are to be avoided when teaching algebra to middle school students. The teaching of algebraic processes could represent an opportunity to reinforce the mastery of decimal and fractional concepts. In addition, the early learning of algebraic processes could lead to the realization by middle school students that using decimals and fractions is governed by the same laws that govern whole numbers.

Notes

[1] Selitto, G. (1997) *An identification of problematic algebraic concepts and the understanding possessed by students in elementary algebra*, unpublished PhD Dissertation, Columbia University, NY.

References

Baroody, A. and Ginsburg, H. (1983) 'The effects of instruction on children's understanding of the "equal" sign', *The Elementary School Journal* **84**, 199-212.

Behr, M., Erlwanger, S. and Nichols, E. (1980) 'How children view the equals sign', *Mathematics Teaching* **92**, 13-15.

Behr, M., Harel, G., Post, T. and Lesh, R. (1992) 'Rational number, ratio and proportion', in Grouws, D. (ed.), *Handbook of research on mathematics teaching and learning*, New York, NY, Macmillan Publishing Company, pp. 296-333.

Booth, L. (1988) 'Children's difficulties in beginning algebra', in Coxford, A. (ed.), *The ideas of algebra, K-12 (1988 Yearbook)*, Reston, VA, NCTM, pp. 20-32.

Borenson, H. (1986) *The hands-on-equations learning system*, Allentown, PA, Borenson and Associates.

Carney, R. and Schattgen, S. (1994) '[Review of] California Achievement Tests, fifth edition', in Keyser, D. and Sweetland, R. (eds), *Test Critiques 10*, Austin, TX, Pro-Ed, Inc., pp. 110-119.

Chalouh, L. and Herscovics, N. (1988) 'Teaching algebraic expressions in a meaningful way', in Coxford, A. (ed.), *The ideas of algebra, K-12*, Reston, VA, NCTM, pp. 33-42.

Davis, R. (1975) 'Cognitive processes involved in solving simple equations', *Journal of Children's Mathematics Behavior* **1**, 7-35.
Dienes, Z. (1971, fourth edition) *Building up mathematics*, London, UK, Hutchinson Educational Ltd.
Frykholm, B. (1994) *The significance of external variables as predictors of van Hiele levels in Algebra and geometry students*, University of Wisconsin Research Report, ERIC Document Reproduction Service No. ED372 924.
Hart, K., Kerslake, D., Brown, M., Ruddock, G., Küchemann, D. and McCartney, M. (1981) *Children's understanding of mathematics: 11-16*, London, UK, John Murray.
Kieran, C. (1981) 'Concepts associated with the equal symbol', *Educational Studies in Mathematics* **12**, 317-26.
Kieran, C. (1992) 'The teaching and learning of school algebra', in Grouws, D. (ed.), *Handbook of research on mathematics teaching and learning*, New York, NY, Macmillan Publishing Company, pp. 390-419.
Koirala, H. and Goodwin, P. (2000) 'Teaching algebra in the middle grades using mathmagic', *Mathematics Teaching in the Middle School* **5**, 562-6.
Lawson, D. (1997) 'The problem, the issues that speak to change', in Edwards Jr., E. (ed.), *Algebra for everyone*, Reston, VA, NCTM, pp. 1-6.
Lee, M. (2000) 'Enhancing discourse on equations', *Mathematics Teacher* **93**, 755-756.
Matz, M. (1980) 'Towards a computational theory of algebraic competence', *Journal of Mathematics Behavior* **3**(1), 93-166.
Usiskin, Z. (1987) 'Why elementary algebra can, should, and must be an eighth-grade course for average students', *The Mathematics Teacher* **80**(9), 428-38.
Usiskin, Z. (1997) *Doing algebra in grades K-4*, *Teaching Children Mathematics* **3**, 346-56.
Wheeler, D. (1996) 'Backwards and forwards: reflections on different approaches to algebra', in Bednarz, N., Kieran, C. and Lee, L. (eds), *Approaches to algebra*, Dordrecht, The Netherlands, Kluwer Academic Publishers, pp. 317-325.
Wisthoff, J. (2001, book review) 'Memoirs of a maverick mathematician', *Mathematics Teacher* **94**(7), 616.

If you understand something in only one way, then you do not really understand it at all. This is because if something goes wrong you get stuck with a thought that just sits in your mind with nowhere to go. The secret of what anything means to us depends on how we have connected it to all the other things we know. This is why, when someone learns "by rote," we say that they do not really understand. However, if you have several different representations, when one approach fails you can try another. Of course, making too many indiscriminate connections will turn a mind to mush. But well-connected representations let you turn ideas around in your mind, to envision things from many perspectives until you find one that works for you. And that is what we mean by thinking!

Marvin Minsky, *The society of mind*