On Strategies for Teaching

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Although mathematics is very old, at least 25 centuries and perhaps much more, the teaching of mathematics on a large scale is recent: it appears only in the last century when going to school became compulsory And for many, many years, up to the fifties, there has been only one way to teach mathematics: as it has been taught by previous teachers. Nobody took care of the students who didn't succeed: they just weren't able to learn mathematics, that's all.

For many reasons — our purpose here is not to analyse them — teachers, as well as researchers in many areas (psychology, sociology, artificial intelligence, etc.) asked: why do so many students not succeed in learning mathematics? Can we be sure we are teaching mathematics in the right way? And, by the way, how does one teach mathematics? On what principles is the teaching of mathematics based?

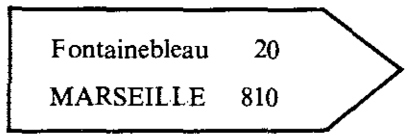
Our goal, in the following pages, is to show how the teaching of mathematics is based on some principles; how it emphasizes mainly two of them and neglects others. How can I know how I teach and how can I change it to be as I wish?

This paper developed from a lecture I gave last July in a training session organized by the French Foreign Ministry for teachers, professors, and instructors coming from a dozen French-speaking countries. It is based on my experience in France So readers will have to appreciate some differences from their own experiences. To make my examples easier to understand, I'll give from time to time details on how mathematics is taught in France.

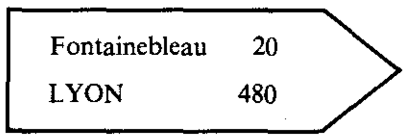
1. The skill principle

Let me begin with a metaphor I live in a suburb of Lyon, which is a large city (population more than 1 million). Recently I had to drive from my home in the southwest of the city to a small village called Heyrieux, located to the southeast of Lyon As is often the case for this size city, there is a motorway circling Lyon.

To drive in North America or in England, it helps to know the numbers of the roads you will take. In France there is usually no information about the numbers of the roads. The road signs are not numbered and, on the highways, the exits are not numbered. Instead, on the road signs are written the name of one large city in that direction and the names of one or two small cities or villages on the way to that large city. For instance, if you go from Paris to Marseille, taking the motorway in Paris you could find a sign like the following:



or even:



because Lyon is a large city on the way from Paris to Marseille.

Let us return to my story. I followed the motorway watching for the sign with "Heyrieux" on it But after a few minutes I saw the sign for Grenoble, which is a town fifty miles east of Lyon. I had gone too far and missed the right exit Later, I knew the mistake I had made. On the sign before the exit I should have taken is written "Saint Priest," the name of another village just before Heyrieux.

Nevertheless, I knew where I wanted to go and it was important for me to be there on time So I left the highway. I took several small roads and a few minutes later I arrived in Heyrieux on time.

For students learning mathematics it is essential for them to know the objectives they are supposed to reach. Teachers who are convinced of this, and who teach using skills, try to describe the targets in the most precise and explicit ways. To do this in a more and more precise way, they state smaller and smaller objectives. Then they try to organize the set of objectives they have found.

Having done this, they try to answer the following questions: Among these objectives, which are the easiest ones to reach? Are there any which might be reached before some others? Which ones? They try to build, following Bloom's ideas [BL], some taxonomies, perhaps based on three families of objectives: cognitive, psycho-motor and affective. (As a matter of fact, many math teachers pay attention only to the first family and are not worried about the other two.) This strategy is described and analysed deeply in [B_1], while different particular examples at the secondary level are given in [B_4]. In this strategy, the main goal of the teacher becomes *evaluation*. To prepare his classes, the teacher first has to find the activities which will lead to the

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accumulation of small pieces of knowledge and produce precise mathematical behaviour; he has secondly to produce the right tools for evaluating what his students learn. Indeed, people who actually practice skill teaching and pedagogy-through-objectives know that they spend the main part of their time before classes in preparing means of evaluation and then, during classes, they spend time helping their students to evaluate themselves. (See [B_2] and [B_4].)

In my story my goal was Heyrieux. Someone might have told me to take St. Priest as an intermediate objective. But suppose, for some reason, I wasn't able to find St. Priest. Then, instead of trying to discover a way of going directly to Heyrieux, I would have wasted all my time first finding a way to St. Priest — and I would have been late for my appointment. I do not claim that information about St. Priest would have been — *a priori* — useless; but it would only have been one fact among others and not a compulsory objective. Once I know where I really want to go, even if I make some mistakes along the way, I will know when I have arrived. *When I know my goal, I can recognize when I have reached it. Mistakes are part of the process.* No one has to give low grades. What kind of marks could help me to find my way more easily next time: a bad mark or a good one?

Why do we use grades in the school system? In no way can they improve the learning process (except for very good marks, perhaps). If their utility is only to enable us to have relationships with the families of the students, then let us try to find other ways to talk together; and if they are only used to force students to be quiet, let us try to find why they do not enjoy our teaching and let us change it. Knowledge of learning objectives by students is a very important step in the learning process. That does not mean that teachers must provide detailed, long and abstract statements such as, "By the end of the week you must be able to solve in two minutes four equations like $5x + 3 = 0$". First of all the teacher has to find some attractive activities, to *challenge* students and to *give some sense* of the goals that he wants to present. Then, with the students, he can discuss these goals and *decide, with them,* which tests will prove that the target has been reached.

In this way of looking at the learning process *it is less important that the teacher identifies his students' knowledge than that the students understand clearly the goals and want to reach the target.* So a lot of time must be spent on this. Teaching following these principles is not easy. Math teachers in France who want to be prepared this way can follow some inservice training sessions organized throughout the academic year. (See [B_4] for more details.) People who organize this kind of training session claim it takes, at least, two years for teachers to feel comfortable teaching in this way.

Note: In France, inservice training sessions are organized during school time and paid for by the State. Teachers who want to attend some of them ask for some days off. They can have four or five such days a year (in a very few cases they can get up to six weeks for math and one year for computer science). They can also join meetings organized by local or national teachers associations or by some unions and ask for extra days off for those. At least some "summer-universities" are organized for them during the summer vacation. For instance, for two years I organized a summer university where 50 teachers (from secondary and primary schools) came in teams to think about research and teaching and to meet other groups as well as different kinds of researchers.

Though the ideas presented in this section can improve teaching skills a lot, it is also true, as we shall see in the next sections, that this strategy of teaching ignores others. I shall introduce now two other theories before ending with a research-action strategy for teaching math that I have called *didactic-action* following a lecture I gave in Montreal [B_2] two years ago.

2. The curriculum principle

Actually, for math teachers, the best known strategy for guiding the learning process is a standardized development of math concepts, more often called a *curriculum* strategy. Pupils at the primary level, students at the secondary level and at universities, all have their studies governed by a curriculum. And even when teachers follow some inservice training sessions or participate in math meetings, they still spend their time talking, criticizing, and comparing various curriculums.

The main characteristic of curriculum strategy is seen in the linear way that the mathematical concepts are listed. For people planning this strategy, the set of mathematical concepts they present must be totally ordered in the following way, where each c_n is a concept:

$$\ldots \rightarrow c_{n-2} \rightarrow c_{n-1} \rightarrow c_n \rightarrow c_{n+1} \rightarrow c_{n+2} \rightarrow \ldots$$

As a consequence, for a given curriculum, there is one and only one way to teach it: starting from c_1, one must introduce one-by-one each c_n in the order of the curriculum. So when two teachers discuss a curriculum, they take the same set of concepts and only compare the different total orderings which can be imposed:

$$\ldots \rightarrow c_{17} \rightarrow c_{38} \rightarrow c_{44} \rightarrow c_{22} \rightarrow c_{35} \rightarrow \ldots$$

and

$$\ldots \rightarrow c_{38} \rightarrow c_{29} \rightarrow c_{17} \rightarrow c_{22} \rightarrow c_{46} \rightarrow \ldots$$

or possibly

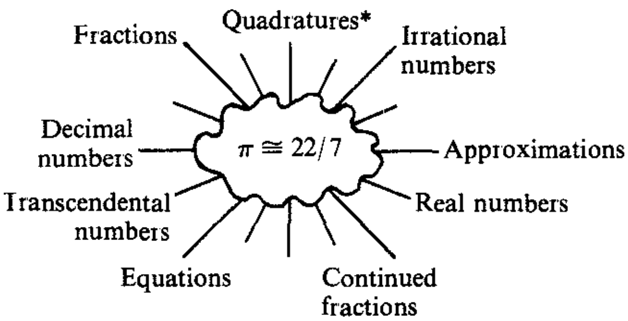
$$\ldots \rightarrow c_{35} \rightarrow c_{66} \rightarrow c_{12} \rightarrow c_{17} \rightarrow c_{29} \rightarrow \ldots$$

For instance, French math teachers will talk about these kinds of questions:

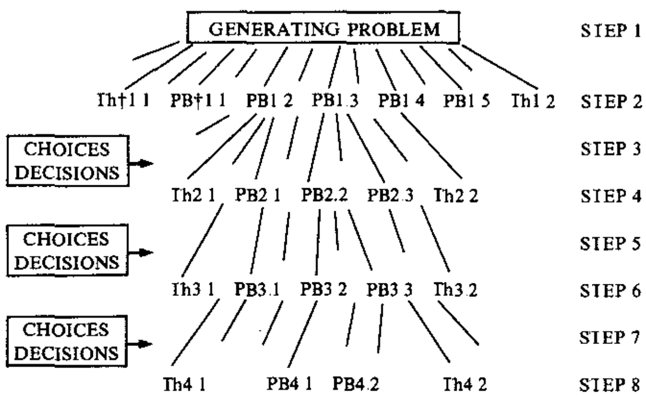
> What must we teach first in grade 9: affine geometry or vectorial geometry? Shall we teach linear algebra before geometry or the converse? In what order shall we introduce set theory vocabulary? Shall we present Thales theorem as the theorem or as an axiom?

But, first of all, science is concerned with problem solving. Now many people following Popper's ideas [P], or Lakatos' ideas [L], think there are really no *disciplines;* there are only *problems.* Even if this statement seems, at a first glance, too radical, the ideas behind it are clear. *There is no science without problems,* new scientific discoveries always come from good questions. Now, for teachers practicing the curriculum strategy, what are problems for? When do they give problems? What kind of problems?

Let us start from the following question asked by a group of secondary math teachers during an inservice training session: Why are the two numbers π and 22/7 "close"? The study of such a problem had led them to discover or to introduce, or to use many concepts:



To portray concept relationships this pattern is too poor and not precise enough. Many more precise pictures could be drawn. What is sure, for people who are willing to study the question for several hours or for several weeks, is that many concepts would be useful; but nobody can predict which ones will actually be used; above all, nobody can guess in which order they will be used. Furthermore, two different groups of students, starting from this question — I call it *a generating problem* [B_4] — will find some partial answers and several new and more precise questions. Each group will then choose one of these questions as a new problem and will continue the process. At each step they will conjecture and will prove theorems and will find new problems, as always happens in science [B_1].



* Length of curves, area of surfaces, etc

† "Th" means "theorem" and "PB" means "problem"

In [B.W.] many "generating problems" appear. For instance, Chapter I begins with such a problem:

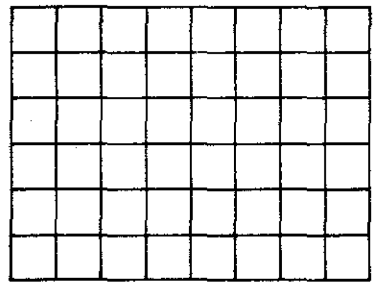
"$x^2 + y^2 = z^2$

After looking at the above equation, respond to the following:

What are some answers?"

Very often, when I give examples of generating problems, for instance:

What is the number of polygons in this figure?



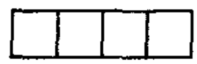
some teachers say: "why do you choose a rectangle"?

But why not?

Many say to me: "This problem is ambiguous: you haven't said precisely what kind of polygons one must use; do you mean squares? rectangles?"

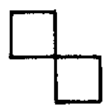
Why should I have to be precise?

If a student decides to find the number of polygons of the following type:



isn't it a nice problem?

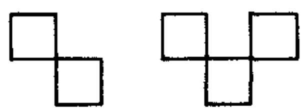
And if another one wants to think about shapes like these:



aren't they polygons?

And if some decide to solve the problem in a 3 by 3 square instead of the given 6 by 8 rectangle, why not?

If some want to find the number of polygons with seven edges, do they have to count these?



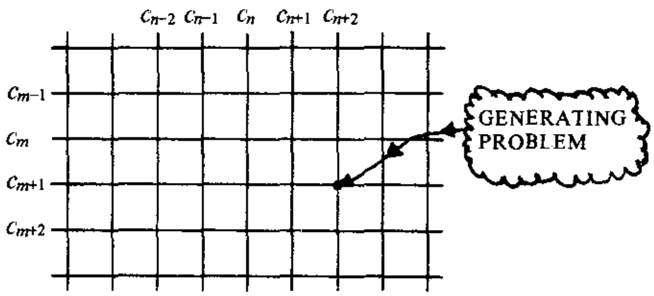
By the way what, for you, is a polygon? Which are the edges and the vertices of the previous one?

Give this kind of problem to several different classes (young students as well as grade 12) and note how many different questions they will ask themselves; how many different problems they will study. How many theorems (partial results) they will get. I am sure it will be very different in each class, but very rich in every one and *every student will find something*.

When the generating problem is well chosen, at each step the number of conjectures and theorems and partial results increases; but *the number of new questions increases much more*.

But if a linear pattern can't be used any more to describe relationships between concepts, it would perhaps be useful to have another pattern. So let me introduce a new kind of pattern, a tool, useful in understanding — or trying to understand — what knowledge is. This idea is due to Michel Develay [D].

You know how a fabric is made from threads following a pattern called a "web" Michel Develay suggests that we think of the different concepts as different threads connected together just as threads are usually connected in a web. This *conceptual web* can be drawn as below, where the c_i are concepts and the crosses are some generating problems, or simply some problems having relationships with the concepts.



If you try to use this kind of pattern to study the relationships in a question like the one about π and 22/7, perhaps you will find it still too poor for this purpose. But it already allows a much deeper study of knowledge than the linear pattern. In fact, the previous pattern is still an oversimplification of Develay's conceptual web. Here, to simplify the idea, I've drawn it in a plane, i.e. in a 2-dimensional space. But now try to generalize this idea and do the same thing in a 3-dimensional space, in a 4-dimensional space, ..., in a 1000-dimension space, ... Here is, actually, Develay's conceptual web. Now, as an exercise, try to draw some conceptual webs (there is not a unique one, but many, many, ...) for some topic you teach: absolute value, Pythagoras' theorem, De Morgan's law, continuous functions, or anything else.

For a teacher who spends time preparing his classes, drawing conceptual webs is not a useless exercise; it is not a replacement of the linear pattern by another which is a little more sophisticated and rather more complicated. With respect to knowledge, with respect to the production of knowledge, and — and so to learning — trying to use conceptual webs, means wanting *to make a break* and change what the teacher is supposed to do: *instead of spilling out many small bits of information, he has to ask good and exciting questions*. In this sentence, "question" must be understood as a *scientific* question or what I call a generating problem, something which creates new questions, new conjectures, mathematical activities. "Question" does not mean the usual problems given on a test! This idea, already shared by many teachers and many educational movements, will become still more obvious for readers — let us hope — after the next section about the learning strategy.

3. Learning strategy

I'll begin with a recent story. But, first, let me give you useful details about mathematical studies in France.

At the end of secondary school, in grade 12, students study calculus in a more theoretical way than used to be the case in North America; for instance, on limits, after grade 11, they are supposed to be able to write some proofs; they learn integration by parts and other methods of integration. At the university, during the first year they study both abstract and linear algebra as well as calculus, including uniform continuity, uniform convergence and the proofs of all the deep results they are supposed to know: the least-upper-bound theorem, the existence of the definite integral, Cauchy's theorem on the limit of a uniformly convergent sequence of continuous functions, etc. But, of course, I do not claim that all the students are actually able to do what their professors suppose of them!

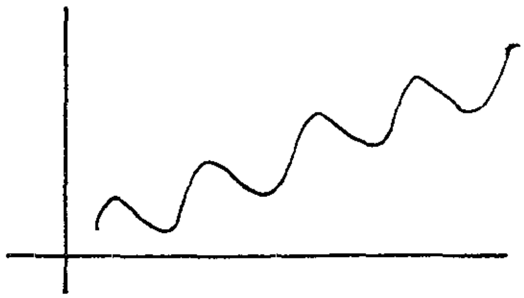
Now, let me come to the story. My university organizes a special class during the first year for a group of foreign students. The agreement with their country asks us to emphasize math teaching. So this group of 15 students study only math and have more hours a week of math classes than the students who also study physics, chemistry and mechanics; they also have more weeks of teaching a year. They have still some more classes after the final exam to prepare them for the second year. This group is taught by a friend of mine and I had asked him to allow me to attend his last class and give the students a kind of quiz I had prepared for another purpose in order to see how these students responded to it. Of course, I didn't really mark such a quiz. Instead, the rules I gave them were: (1) Do the questions by yourself, alone. (2) Then, when I tell you, compare your answers in pairs to try to find the right one. (3) Do the same thing in small groups to give group answers. The quiz contained about fifteen questions on analysis, including the following:

> Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x \to +\infty} f(x) = +\infty$.
>
> Then there exists $a > 0$ such that $f: (a, +\infty) \to \mathbb{R}$ is an increasing function.
>
> TRUE OR FALSE ?

While the students were preparing their answers, I walked around the class looking at what they were writing on their rough copies and on the quiz itself (the questions were printed on individual sheets). I quickly noticed that the students not only seemed to answer TRUE to the above question, but wrote it as if they didn't have any doubts about the answer.

After each step of the exercise was done (individual answers, talks in pairs, small group talks), I asked the whole class to vote about the truthfulness of the statement Except for one or two students who didn't vote, all of them answered "TRUE". None answered "FALSE". I was very unhappy. I didn't want to emphasize a wrong idea about limit in the students minds. Furthermore this was their las class before summer vacation. Moreover, since none of the students had answered "FALSE" I could not provoke a debate among them. What could I do? What would you do? (But perhaps you don't practice this kind of quiz. Anyway, at the end of this story I hope you will be encouraged to do so from time to time.)

Earlier, I had noticed the following picture drawn by a student on his rough copy:



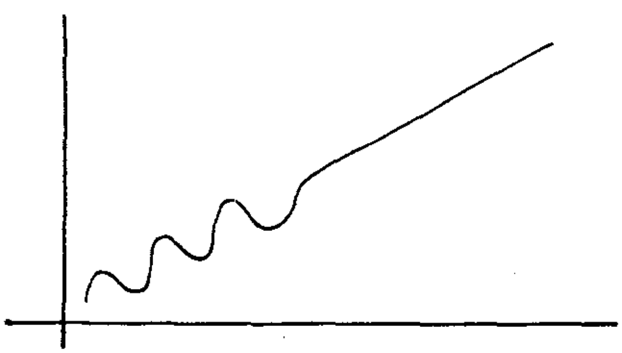
and I remembered seeing him show this picture to his neighbour: for me this picture was really a *counterexample* to the given assertion. Why had this student nevertheless answered "TRUE"? I suggested that he go to the blackboard and, without any further comment, draw the picture I pointed at from his notes. He did it without comment — and nobody in the class said anything! So I insisted:

"What does this picture show?"

Deep silence from the whole group. Not a peep. Then to try to start a scientific debate about the initial question, I asked the student still at the blackboard:

"Does your function satisfy the hypothesis of the statement?"

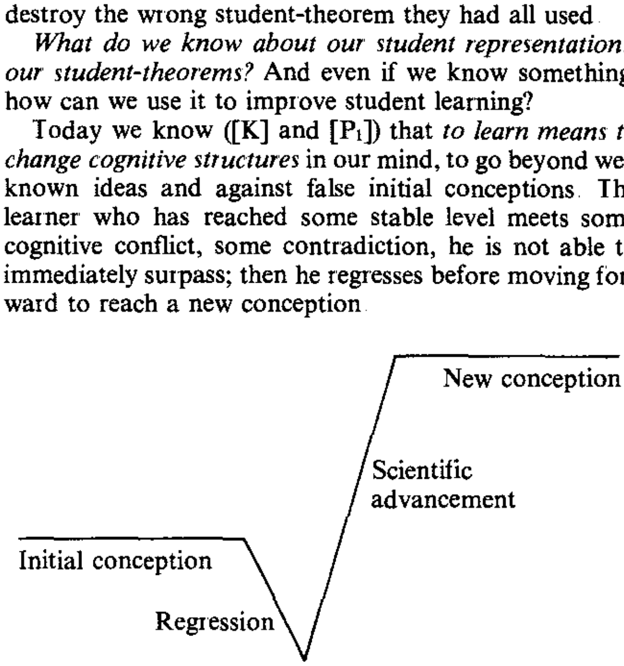
"Yes, indeed; because its graph goes like that," he said, adding a bit to the previous picture to get the one below:



In this situation I could understand how some *models* or *representations*, what I call *student-theorems*, are actually in the cognitive structure of the students and how difficult it can be to destroy them. What could have been used as a scientific counterexample had been falsified so as not to destroy the wrong student-theorem they had all used.

What do we know about our student representations, our student-theorems? And even if we know something, how can we use it to improve student learning?

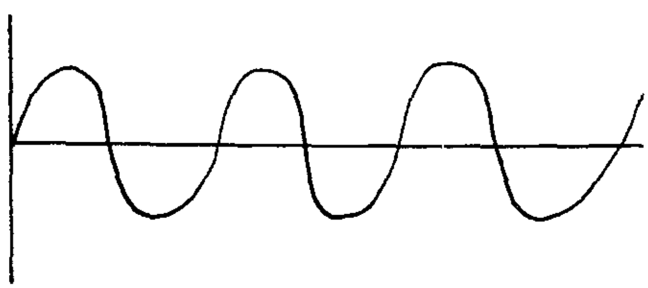
Today we know ([K] and [P_1]) that *to learn means to change cognitive structures* in our mind, to go beyond well known ideas and against false initial conceptions. The learner who has reached some stable level meets some cognitive conflict, some contradiction, he is not able to immediately surpass; then he regresses before moving forward to reach a new conception.



Following Jean Piaget, the teacher's role is fundamentally to furnish some counterexamples to wrong conjectures and create, in that way, new cognitive conflicts. (See [B_3]; new "breaks".) So to come back to my previous story, you would perhaps like to ask, "Why didn't you give to the whole class one or more good counterexamples?" Well, that's a good question! Actually I am not sure why I didn't. But I think I had the feeling it was not the right time to "impose" a counterexample: the group wasn't ready for it Why would they have considered any of *my* counterexamples as counterexamples for them? So I decided instead to get the class to move to the point at which they would ask, first, the following question: "After all, are we so sure the given statement is true? How do we know? How did we check it? Did we try to find some counterexamples? *What could be a counterexample?*"

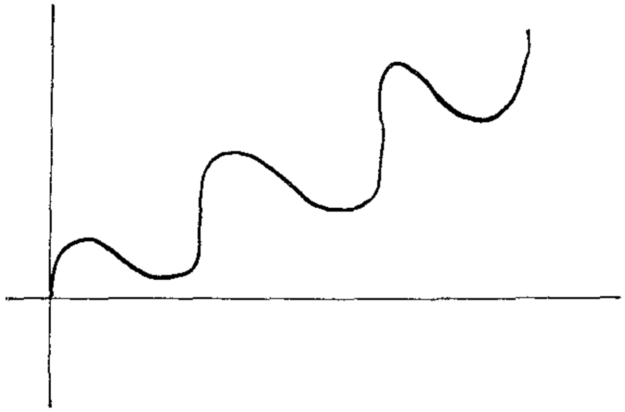
Perhaps you think all these questions have obvious answers for students who have already succeeded in the first year at the university. But wait a minute.

I went to the blackboard and drew the usual sine function:



and asked, "Does this function have a limit when x goes to $+\infty$?" At once the students answered "No" and no one seemed to have any doubt about it So I continued in the following way:

"Suppose, now, I make a small "turn", like this, will the behaviour of the function change?"



"No," answered the students after a few seconds. "But give us an explicit example," they asked me. So, quickly — too quickly, for sure — I answered:

$$\varphi(n) = x + \sin x$$

I didn't add anything else; the students were by then talking two by two or three by three and I guessed they were very close now to considering the given statement as wrong. Suddenly, one student called me to show me his rough copy. He said:

$$\text{"}\lim_{x \to +\infty} \varphi(x) = +\infty.$$

That's O K. On the other hand, $\varphi'(x) = 1 + \cos x \geq 0$, so φ is an increasing function; so the statement is true," and other students agreed with this. No doubt had yet come into their minds; their old — and wrong — conviction was still intact

I made no comments and suggested that they study

$$\psi(x) = x/2 + \sin x$$

instead of $\varphi(x)$.

I gave them a few minutes to think about it. I knew they had come to a crucial scientific step, where they had to admit an "evident and sure result" as definitely wrong. Then, to find out what had happened in their minds, I asked them, again, to vote about the truthfulness of the given statement. To my question: "Do you think the statement is true or false," very few answered "true"; nearly all of them failed to vote and still they didn't yet have a new conception. *How slow the learning process is!*

A half of this two-hour class had already been used. Seeing what had happened the usual teacher of the class decided to change his plans. Instead of teaching what he had prepared to do in the second part he decided to go on with the same topic and to help his students destroy the *obstacle* which had blocked their scientific progress. The wrong knowledge, the "student-theorem" was so obvious, so evident, that they couldn't even consider having a doubt about it. And so the wrong knowledge — *which was knowledge* — interrupted, stopped, all scientific progress and worked against the creation of new scientific knowledge.

This kind of phenomenon was pointed out and studied a lot many years ago, from a historical, philosophical and sociological point of view, by Gaston Bachelard (1884-1967) who called *epistemologie obstacles* [Ba] that wrong knowledge which has interrupted scientific progress for a while: the earth is flat; mass is constant; all continuous functions have derivatives, etc.

> "Great advances in knowledge have taken place by people who have had the courage to look at a cluster of attributes and to ask, "What-if-Not"?" [B.W.].

We know today ([B₂] and [K]) that the learner's process is, sometimes, similar to some parts of past social and historical processes. This fact was ignored fifty years ago But since then cognitive psychologists have proved that *for a learner the main obstacle in learning is not to get new strategies; it's to throw away, but utterly abandon, old, deeply-believed, and less appropriate ones*

Now we may reconsider the teacher's role Isn't it his main goal to improve the students' knowledge? Yes, for sure. So he first has to find mathematical questions that bring contradictions and cognitive conflicts [B_2], making deep "breaks"; but, on the other hand, he has to take care to allow enough time for the whole process to develop, including some regressive steps, into some scientific progress, the creation of new scientific concepts. For a teacher accustomed to observing the learning process of his students at least some of these different steps are known, and it's not too difficult to locate them for a given student The most difficult thing to admit (it's for us another kind of epistemological obstacle) is how slow the speed can be and how different the speeds are for two students faced with the same problem or the same contradictions. Something else is very difficult to admit for a teacher who wants his students to succeed in their studies: the regressive steps can not only be long; sometimes, for some students, the depth of the regressive step is quite incredible.

Let me tell you of an example that happened recently in my first year mathematical class at the university. I team-teach this class with another professor using special booklets written by us for that purpose. The first semester is essentially practical (that means something closer to North American calculus and linear algebra); we use the second semester to study deeply the proofs of the most theoretical results and to study the kind of long and abstract problems students will have to solve for the final exam (twice four hours on two different problems) In this special class* we used to teach three-hour classes to allow the students enough time to go deeply into some mathematical questions. (the same booklet can be studied for 10 or 15 or 20 hours.)

*A paper describing this experience and its evaluation will be published very soon.

At the beginning of the second term one of the booklets deals with the construction of the real numbers. To set the generating problem, at the beginning of the booklet we ask a lot of questions, such as

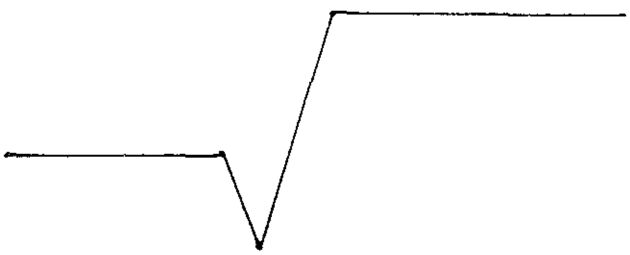
- Is 0.99999 .... equal to 1?
- Let $a = 0.1234567891011121314\ldots$ Multiply it by 8; find a^2.
- Find $(\sqrt{2})^{\pi}$.

One of the main difficulties which appears then is: "What does "write a number" mean?" "Does knowing a number mean knowing how to write it?" "Is there a unique way to write numbers?" "Is a given mathematical formula always a way of writing a precise number?" In other words, the generating problem could be stated as, "What are the relationships between different kinds of numbers and different ways of writing them?"

That day students were studying this kind of question is small groups. Suddenly one student called me; he seemed angry and with an aggressive voice asked me

"After all, why is 1/0 not a rational number?"

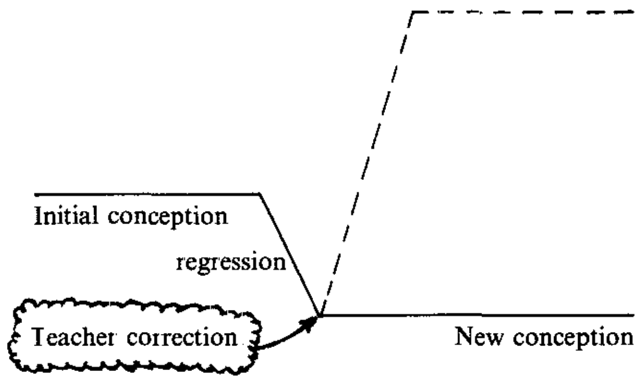
Obviously, inside the small group the discussion had become very hot Surprised by the sudden question, I failed to give a mathematical argument. But — I don't know what stopped me — I only said, "Yes, after all, why is 1/0 not a rational number?" As a matter of fact, my behaviour was very close to the "What-if-not" strategy described in [B.W.] through many examples. Since other groups of students were calling me, I left this one. Later on the same student called me again, but this time gently On his rough copy he had written the sign I had already used several times for them to comment on their learning process:



Pointing to the lowest part of the picture, without any more details, he told me, smiling:

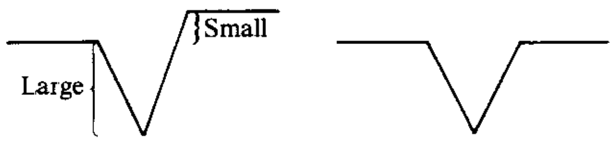
"Just now, I was at the very bottom of the pit."

To end this section, I want to make a few more comments about the consequences that this knowledge about the learning process might have on our teaching style First, we know now that we must find good generating problems and allow students enough time to struggle with difficult questions, to find contradictions and cognitive conflicts; but, on the other hand, we must let them live through the advancing steps to get new and scientific conceptions. Moreover, it seems, if the teacher interrupts the process at the wrong time, he will achieve an effect opposite to the one that he wants. Some researches seem to prove that if the teacher tries to correct a student's thinking when he is at a regressive step, then the student will stay at the current level and will not get a more scientific conception

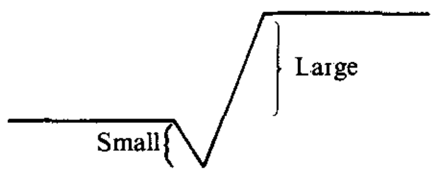


Not only does this kind of correction not improve the student learning process, but it may aggravate it, and this could explain why, after many many corrections by the teacher, some students still make the same mistakes, e.g $(a + b)^2 = a^2 + b^2$ or $\sqrt{(a + b)} = \sqrt{a} + \sqrt{b}$

Certainly, no teacher wants to teach in a way which hampers the student's learning But teachers still have to avoid the following possibilities:



and think, as an ideal goal, of the following strategy:

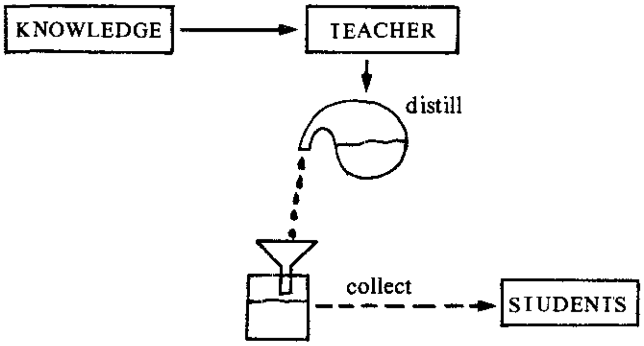


How can one go further?

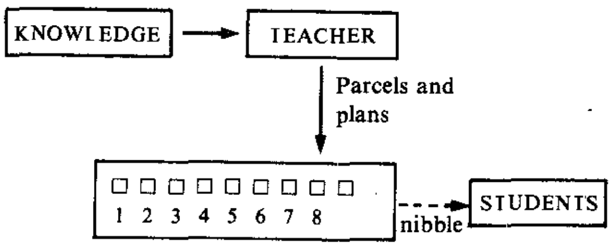
4. Didactic-action strategy

This last section can be viewed as an introduction to [B4] which is a handbook for secondary math teachers who want to use the research-action strategy applied to the "didactics" of math to improve their teaching. In this last sentence "didactics" means the study of learning processes in the classroom: how students produce knowledge. How teachers can increase the knowledge production. What are the relationships between mathematics, the students and the teacher?

To improve our teaching we have, first, to know it, to be able to describe it, to characterize it. Let us begin with an abstract point of view. Teaching has something to do with students, knowledge and teachers. Among several kinds of relationships between these three components, we want to emphasize three. The first can be considered as the most classical pedagogy: the teacher distills parts of his own knowledge while the students try to collect pieces of it:

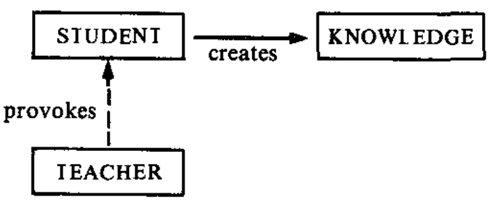


In a pedagogy based on skill teaching, on skill strategy, the teacher plans and organizes the ways students will nibble very well-determined and narrow parts of knowledge



These two strategies are very similar even if, in the second, students play a more active role.

A pedagogy based on learner strategies or learning processes is very different from the two previous ones. Forgetting the non-sensed linear organization of concepts, we saw that the goal of the teacher is to help students to be *knowledge-makers*. Then, the teacher has to create the right conditions to provoke the right learning process:



Compare these three patterns. Of course, you could make other ones. Just as they can't describe all the practical ways of teaching we could find, it's also true that a given teacher wouldn't use only one of them to describe his own practice. Nevertheless the first two, in a sense very similar, are *based on the teacher*. The third one is *based on the learners*. If you wish to think about your own teaching practices, if you want to know whether your teaching is based on you or on your students, answer the following short form and, at the end, compare your answers to the various questions, think about what shows up and ask yourself if you would like to change something.

Main Characteristics of my Teaching Strategy

			1	2	3	4	
1.	Logic used	Discipline logic	x	x	x	x	Learning logic
2	Central subjects	Concepts	x	x	x	x	Generating problems
3	Concepts organization	Through linear patterns	x	x	x	x	Through conceptual webs
4	Learning principle	By accumulation of knowledge	x	x	x	x	By cognitive conflict
5	Attitude with respect to student mistakes	To avoid	x	x	x	x	Considered as part of learning process
6.	Main goals of teacher	Esthetic qualities of the topics. Practice of exercises	x	x	x	x	Student abilities
7	What is evaluated	Bits of knowledge	x	x	x	x	Abilities to solve problems
8.	Time conception	Scheduled in teaching time	x	x	x	x	Scheduled in learning time
9	Use of history of math and epistemology	None	x	x	x	x	Used as part of learning process

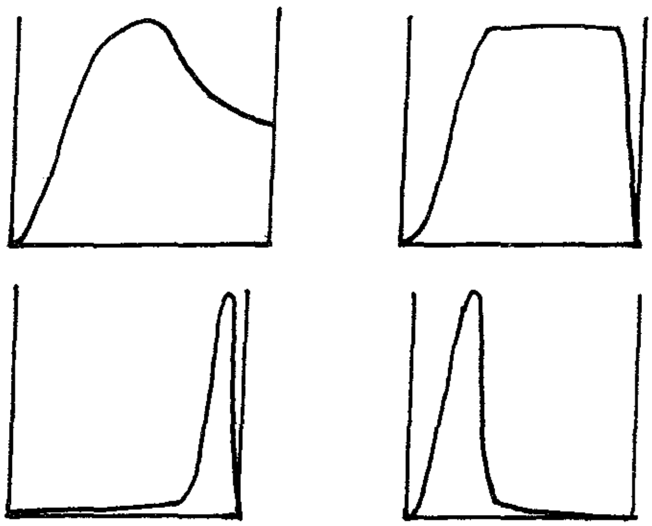
As you have readily noticed, on the left column are teacher-based characters, on the other side are the learner-based ones. For instance, if I apply this form to my own teaching in the first year university class (for some other classes I have to teach the answers wouldn't be the same). I find

3 - 3 - 2 - 4 - 4 - 3 - 4 - 2 - 3

and what I would like to change first is how I think about concept organization and how I deal with time.

Now I know I would like to change my teaching. What can I do? What can help me to do that? I hope the following two stories will help you to find your own answers to these questions

As a child I studied classical music and played the piano. As a teenager I stopped and later on, for my own enjoyment, I began to play some kind of jazz music and popular songs. Last month, for the first time, a friend of mine lent me an electronic synthesizer and I was able to use it for several days. Do you know what it's like? Before then, I didn't. It is something between a piano keyboard and a computer. Around the usual piano keys are many extra keys which can be used to make sounds like a piano, organ, trumpet, flutes (different kinds), etc I could add many sorts of drums, choose rhythms, speeds, levels, ...; give different orders for left and right hands; and many other instructions I learned by myself very slowly. But the most important instruction concerned the qualities of sounds. For each of them, I could determine characteristics, and each of these characteristics was the "sound curve"; on the board, I could see a dozen of curves like the following:



Before using this kind of electronic computer — sorry, electronic "piano" — I couldn't have guessed how rich and how complex it is. I tried empirically many different combinations to produce different kinds of sounds So sometimes what I got was nice, and later on I tried to repeat them. But the sounds and the music depended on so many "command variables" that I didn't always remember all of them and, mainly, *I didn't notice what the most important ones were* Of course, that surprised me: I did something one time; I might not be able to do it again! But my main surprise came later. After using the synthesizer for several hours I was forced to recognize that *I needed to ask myself some questions about sounds which I hadn't asked before* As these questions were very new to me, I wasn't able to answer them and I discovered I had never listened to my piano sounds in that way before And, as a final conclusion, I discovered, with disgust, that despite my love of music and the enjoyment of playing, *I wasn't able to listen and to "analyse" music sounds* (to break them into their smallests pieces, to examine them, critically, part by part). I only had a global impression.

The second story is also very recent As you know, in South European cities, downtown is composed mainly of old buildings (some are several centuries old) and there are very few new ones. When a new one is built, it must be in the same style as the old ones around it. The most beautiful shops are downtown, in narrow streets; and people enjoy living downtown since that is where the theatres, movies, restaurants, night clubs and so on are. Big office buildings are in more modern recently-built suburbs. As it is difficult to drive and to park downtown (many streets now can be used only by pedestrians), as people like to walk in the streets windowshopping — or shopping! — and as the streets are shorter than in North America, Europeans walk much more in the streets than North Americans do.

I was born in downtown Lyon, and I lived there most of my life until recent times; all my family (parents, in-laws, uncle, sister, . . .) still live and work there So I know every street, every building, every shop, and, of course, I know a lot of stories about them.

Last July, I had the opportunity to join a group of people who were going to visit this district, guided by an architect. What an exciting experience it would be to see how people from other cities would react to and appreciate *my* district, with all *my* treasures! And I would see if the architect would be able to explain all the exciting stories I know! And yes, indeed, it was exciting; but not for the reasons I had assumed. Throughout the tour we didn't stop where I thought we would; the architect didn't pay any attention to my treasures; instead, we stopped to look at buildings, shops, and alleys, that I had never seen before. *Sure, I had looked at them a thousand times; but I had never seen them!* The more familiar the places seemed to me, the less I knew about them; or rather, I had always kept the same point of view and nothing had changed in my mind for many years, despite the many changes around in such places.

Let me use these two stories to end with the math classroom and its complexity:

Do I know its "command variables"?
Which ones must I use when I want to reproduce some learning process?
How will I be able to see what I didn't see before now?
How can I prevent familiar situations from becoming obstacles to understanding "didactic" phenomena — in the sense quoted at the beginning of this section?
Furthermore, what I do not know today and will discover tomorrow will become well known very soon and, then, will be a new obstacle to seeing, deeply, the learning process

So there is no eternal, universal, answer valuable for everybody and for all time As mathematics knowledge is something to be discovered and to be built by students, the *didactics of math is a new area of research which must be carried out by math teachers themselves*. To try to organize some classes not only to teach math but also to understand some parts of the learning process is to practice what I call *"Didactic-Action"* [B_3]. Its general philosophy is based on a dialectic attitude between

TO UNDERSTAND ⟷ TO ACT

Can we expect our pupils and students to acquire a research attitude about math if we don't practice research on learning ourselves?

In this paper, I have only tried to explain WHY I think, as a math teacher, I must try to understand and help the learning process in *my* classes. Perhaps, in another paper I will have the opportunity to explain HOW some teachers do that in their classes. Please, if you do it, let me know; I'd be very glad to be in touch with you and to learn from your class experiences.

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