PEDAGOGICAL APPLICATIONS FROM REAL ANALYSIS FOR SECONDARY MATHEMATICS TEACHERS

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In this article, we consider the potential influences of the study of proofs in advanced mathematics on secondary mathematics teaching. On the whole, secondary teachers often report that completing advanced courses does not influence their instruction in any perceivable manner (see, *e.g.,* Zazkis & Leikin, 2010). Even when meaningful connections are made in these courses to secondary mathematics, the pedagogical implications for teaching secondary students often remain ambiguous and unstated. Thus far, the literature on advanced courses for future teachers has highlighted the benefits of applying the conclusions of particular proofs to secondary content and of developing a more general sense of disciplinary practices in mathematics in relation to proof. Instead, we consider some implications for secondary teachers that stem from the study of content *within* particular proofs in advanced mathematics. We do so to contribute to the discussion about different ways in which the study of advanced mathematics might serve prospective secondary teachers, and to consider the variety of connections that might be utilized by instructors of advanced mathematics courses with secondary teachers in them.

Connections to secondary teaching

Research literature has explored connections between advanced mathematics and secondary teaching related to both content and disciplinary practices. Attention to content is sensible, especially given that much of the content of secondary mathematics is undergirded by big ideas and concepts studied in advanced mathematics, such as limits and groups. Disciplinary practices are also important as prospective teachers can gain a better understanding of what doing mathematics is really about, on the basis of which these teachers then can help their secondary students develop productive mathematical dispositions and values.

Proof is an example of the second type of connection—a disciplinary practice—that is regularly connected to the study of advanced mathematics. Indeed, one aspect that frequently distinguishes advanced mathematics courses from previous coursework is the emphasis on proof. In a calculus course, for example, one frequently becomes acquainted with particular ideas about limits, derivatives, and integrals; in a real analysis course, the emphasis is on establishing rigorous proof of the ideas used and introduced in calculus. With regard to the product rule for derivatives, in calculus, it is usually sufficient to know how to apply the product rule. In analysis, it is not. Instead, one must also be able to provide proof of the rule. But merely establishing the theorem is often not the main reason for presenting the proof. Proofs are frequently presented to advance one's understanding of the concepts surrounding the theorem. Yet it is also the case that the ideas in the proof of a theorem may involve mathematics not stated in the theorem itself. Indeed, some philosophers and mathematics educators claim that in mathematics, it is the *proofs*, not the theorems, that are the true bearers of mathematical knowledge (Rav, 1999; Hanna & Barbeau, 2008).

Since students spend much of their time in advanced mathematics studying proof, it is interesting to consider how these proofs can be used to inform the teaching of secondary mathematics.

Proofs of the algebraic limit theorems

In a real analysis course, it is typical to study the algebraic limit theorems for sequences—that for two convergent sequences, their sums, products, *etc.,* also converge. These proofs provide the necessary foundations upon which further ideas in real analysis, such as functional limits, derivatives, and integrals, rely. That is, the conclusions from the proofs are used to propel one through the content of real analysis. But how might they be relevant for secondary teaching?

As noted above, one might look for how the content of these ideas about limits and sequences connects to secondary content and secondary teaching. One example that comes to mind is the equivalence classes on the real numbers expressed as infinite decimals—*i.e.,* 0.999… is equal to 1. In fact, not only is the big idea of limit important here, but a fairly common way to prove this fact draws on the algebraic limit theorems for sequences:

$$0.\overline{9} = \lim\left(1 - \frac{1}{10^n}\right) = \lim(1) - \lim\left(\frac{1}{10^n}\right) = 1 - 0 = 1$$

The study of these algebraic limit theorems, then, might be connected to such content in secondary mathematics and, as a result, to the teaching of real-valued decimal representations in secondary school. However, these connections primarily utilize some general notion of limits and some facts about operations on them; that is, the epsilon-delta *proofs* of these statements, which are often the focus in the analysis course, are not particularly pertinent in secondary classrooms—it is the conclusions that the proofs offer that are germane.

Yet, one might also argue that completing these proofs in a real analysis course is not necessarily a waste. As mentioned above, studying and writing proofs in advanced

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mathematics can help teachers further develop their sense of important disciplinary practices in mathematics. So even though the epsilon-delta proofs might not be germane to the secondary content connection described above, they can help develop improved proficiency in proof comprehension and construction, which are important disciplinary practices—ones that might shape a secondary teacher's overarching approach to teaching mathematics. But are there other important implications for teachers from the study of particular proofs in advanced mathematics? Or is it only serving the more general development of disciplinary practices? In this article, we explore another possible implication for the study of proofs in advanced mathematics, related to looking more closely not at particular content, conclusions, or theorems in advanced mathematics, but at the mathematical ideas invoked in the process of proving them.

Connections within proofs

In what follows, we consider this question by elaborating one way that the study of a few particular proofs in a real analysis course might be connected to secondary teachers' future work and practice (and not just useful for developing a better sense of the disciplinary practice of proof in mathematics). Namely, we apply the mathematical ideas and processes contained *within* these proofs (not necessarily the conclusions *of* these proofs) to a common pedagogical situation in teaching. That is, we regard this as different in that it is not about a content connection to secondary mathematics, nor is it just about developing the general disciplinary practices of proof and reasoning, but rather it is an *application* of ideas contained *within* these proofs to a pedagogical situation in teaching.

We begin by elaborating on a pedagogical situation, which deals with a common issue that arises in students' use of rounding. Then, we elaborate on real analysis content—which in this case are the proofs of the algebraic limit theorems for sequences. Finally, we apply some of the mathematics from the real analysis proofs to the teaching situation, elaborating on how the study of proofs in real analysis could be used to enhance the teacher's ability to engage in quality instructional practices.

Secondary mathematics teaching: rounding

Students regularly solve equations in secondary mathematics, and they frequently round values they work with while doing so. (For simplicity, we will consider 'rounding' in this article to mean 'truncating'.) Many times it is because students feel more comfortable representing $4\sqrt{5}$ as 8.94 or sin(59°) as 0.85—decimals feel like numbers whereas expressions involving operations do not (at least for some students). Further, carrying symbolic expressions like sin(59°) through calculations can be cumbersome. Now, rounding itself is not necessarily inappropriate—there are many times when teachers actually *want* students to round. But there is also a (justifiable) sense that when students round, they should round at the end of their solution to a problem, *not* in the middle. And yet using rounded values frequently does not 'hurt' students—their approximate answers are so close to the actual answers that it can feel negligible (and often is negligible, especially on multiple-choice questions). Consider the following pedagogical situation:

> A student sets up and solves the equation
> $$\sin(59°) = \frac{x}{4\sqrt{5}}$$
> by showing the following work: $0.85 = x/8.94$, so $x = 0.85(8.94) = 7.599$. The teacher walks around the room and observes the student's work. The teacher tells the student, "Remember, do not round in the middle of the problem—wait until the end." The student objects to this remark, and says, "Well, my answer is basically the same as Veronica's, and she rounded only at the end. And I finished faster and understand my way better anyway."

The teacher's advice in the scenario is sound—it *is* generally better not to use approximated values in calculations if you can use more precise values. But the student offers two counterarguments: i) the difference compared to the 'actual' answer appears negligible; and ii) the student's solution method was faster for the student and easier to understand. These counterarguments are sensible; there is a need to balance demands for accuracy with other practical issues. But what the student (and teacher) may not recognize is that the student's remark is specific to this problem and is problematic in the more general case. So how does a teacher respond? The teacher could, for example, mandate that students delay rounding until the end of their solution and start taking off points for students who do not do this. However, such rules seem arbitrary, and the classroom consequences, artificial. The teacher could acquiesce to the student—providing her advice but letting the students, in reality, do as they please. But this, too, does not feel instructive. Alternately, the teacher could superimpose a real-world context to make the point that error—even extremely small error—could have negative consequences in the real world. For some students, this analogy may be meaningful; for others, it may fall short. Still, such a response focuses on making the inconsequential error *feel* more 'consequential', which does not illuminate the fact that the student's rounding approach can, in fact, result in very large errors and, as such, only partially responds to the student's first counterargument.

In what follows, we consider a different kind of response from the teacher. In particular, because the student does not buy into the teacher's advice, a powerful instructional response in this situation would be for the teacher to *exemplify* to the student some of the potential mathematical issues going on around the idea of rounding. That is, the teacher could pose a problem that directly pushes back against at least one part of the student's two-pronged argument. In this case, we will look at the work involved in the teacher crafting an example that would counter the student's claim that the use of rounded values does not produce a very different answer from the actual one. Indeed, the key to being able to accomplish this pedagogical objective—of using an example to exemplify the potential issue in a student's approach—lies in understanding how error accumulates when operating on rounded numbers, for which studying proofs of the algebraic limit theorems provides some insight.

Real analysis: algebraic limit theorems

A rounded number is an approximation for a number. One way to think about this is that if we have a theoretical real number, a, then a rounded number, a_{appr}, is an approximation of this real number where, generally, the maximum difference, or error ε, is bounded and relatively small. We can express this with the inequality:

$$|a_{appr} - a| < \varepsilon$$

In a real analysis course, this brings to mind convergent sequences. If we presume that we have two convergent sequences, a_n approaches a and b_n approaches b, then in an analysis course we prove that their limits respect arithmetic operations such as addition and multiplication. Table 1 provides outlines of proofs as they might be stated in a real analysis course of three relevant algebraic limit theorems—in each, one important equation has been included.

For our purposes of secondary teaching, one lens to understand and apply these proofs is in relation to the accumulation of error. In fact, one might argue that such an understanding in relation to error is a productive way to understand these proofs for anyone—they give specific insight into the choice(s) for scaling the given ε value in order to find a sufficient N for the sequence. Specifically, the equation in each proof in Table 1 provides insight into how the accumulation of error can be bounded based on the operation involved—sum, product, and reciprocal/quotient. We elaborate on each.

Suppose that we sum two approximations, $a_{appr} + b_{appr}$ (which is now an approximation for $a + b$), then the first proof essentially says that their errors, at worse, also sum. That is, *when adding two approximated numbers, the error in the new estimate,* $|(a_{appr} + b_{appr}) - (a + b)|$, *is no worse than the sum of the two original errors,* $|a_{appr} - a|$ and $|b_{appr} - b|$. Indeed, if one wanted to consider $a_{appr} = a + \varepsilon_1$, and $b_{appr} = b + \varepsilon_2$, then $(a_{appr} + b_{appr}) = (a + \varepsilon_1 + b + \varepsilon_2) = (a + b) + \varepsilon_1 + \varepsilon_2$, which provides the same conclusion: the errors no worse than sum. So, for example, if we approximate π as 3.1 and ⅓ as 0.33, then 3.1 + 0.33 = 3.43 will be within eleven hundredths (0.1 + 0.01 = 0.11) of π + ⅓ . This conclusion seems reasonably intuitive; however, as the proofs in Table 1 illustrate, the situation for how error accumulates for products and quotients is different and, perhaps, surprising.

From the second proof about the product, the error in the product is no worse than $|a|$ times the error of b_{appr}, and $|b|$ times the error of a_{appr} [1]. We can simplify our understanding of this relationship if we presume our initial errors are the same—that $|a_{appr} - a| = |b_{appr} - b| = \varepsilon$. In this case, the initial error, ε, grows by a factor of $(|a| + |b|)$. That is, the conclusion for the product is fundamentally different from the sum example: the error for the product of two approximations *depends* on the initial two values. So, *when multiplying two approximated numbers (with equal initial errors), the error in the new estimate grows by no larger than a factor of* $(|a| + |b|)$. An important conclusion is that the larger the sum of $|a|$ and $|b|$, the larger the potential error grows in their product. So if we estimate π by 3.1 and ⅓ by 0.3, then 3.1(0.3) = 0.93 must be within 0.1(π + ⅓) ≈ 0.3475 of (⅓)π.

Lastly, with the reciprocal of an approximation, the proof provides a bound on the accumulation of error by no larger than a factor of $2/|b|^2$, so long as the approximation for b_{appr} is closer to b than to 0. (Essentially, this last part indicates an assumption that the original error, $|b_{appr} - b|$, be relatively small in comparison to b.) Again, we see that the error for the reciprocal of an estimate *depends* on the size of the actual value. Although the reciprocal rule may be of limited use at times, combining the product and reciprocal rules gives us a way to grasp the accumulation of error when dividing two estimates: a_{appr}/b_{appr}. Indeed, if we presume both estimates to have the same initial error ε, then *when dividing two approximated numbers (with equal initial errors), the error in the new estimate grows by no larger than a factor of* $(2|a| + |b|)/|b|^2$. So, if we estimate π by 3.1 and ⅓ by 0.3, then 3.1/0.3=10⅓ must be within 0.1((2π + ⅓)/|⅓|²)≈5.9549 of π/(⅓). In this case, we see that even relatively small errors

Table 1. Outlines of the proofs of the sum, product, and reciprocal algebraic limit theorems.

$i. (a_n + b_n) \to (a + b)$	$ii. (a_n b_n) \to (ab)$	$iii. (1/b_n) \to (1/b)$
Suppose we have a given ε. By the triangle inequality, for any particular term in the sum sequence, we have: $|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b|$ Since a_n approaches a and b_n approaches b, we can find terms out in the sequence that make each of the errors on the right-hand side arbitrarily small—in particular less than $\varepsilon/2$, so that, when summed, (a_n+b_n) is still within ε of $(a+b)$.	Suppose we have a given ε. By the triangle inequality, for any particular term in the product sequence, we have: $|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |b_n||a_n - a| + |a||b_n - b|$ Since b_n approaches b, we can find terms out in the sequence that make the second error on the right-hand side arbitrarily small—in particular less than $\varepsilon/(2|a|)$. Since b_n converges, it is also bounded (by a real number, M), and since a_n approaches a, we can find terms out in the sequence that make the first error on the right-hand side arbitrarily small—in particular less than $\varepsilon/(2|M|)$. When summed, $(a_n b_n)$ is still within ε of (ab).	Suppose we have a given ε. Since b_n approaches b, we can find terms out in the sequence that are closer to b than to 0. For any of those terms in the reciprocal sequence, we have: $\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n||b|} \leq \frac{2}{|b|^2}|b_n - b|$ Since b_n approaches b, we can find terms out in the sequence that make the error on the right-hand side arbitrarily small—in particular less than $(\varepsilon|b|^2)/2$, so that $(1/b_n)$ is still within ε of $(1/b)$.

(tenths) can potentially grow to be very large (~5.95) relative to the estimate (~10.33).

Applying real analysis to the pedagogical situation

In the original problem, a student substituted rounded (truncated) values for sin(59°) and $4\sqrt{5}$ that were accurate to hundredths—that is, an error-bound of 0.01. Using these approximations, the student solved the equation, which involved multiplying these two rounded values. The student's answer was 7.599; the actual answer is approximately 7.6667. To the student, this error seemed trivial. For a moment, let us consider some of the real analysis ideas about accumulating error: in the product of estimated values, the error grows by no more than a factor of ($|a| + |b|$). When applied to this situation, the original error in both estimates of 0.01 increases at most by a factor of the sum of the actual numbers, $\sin(59°) + 4\sqrt{5} \approx 9.8$. Thus, the error the student might get in this particular problem is no more than $0.01(9.8) = 0.098$—less than a tenth. (In fact, the actual error as we can see is approximately 0.0677.) The student's argument that this difference appears to be trivial was a challenge to the teacher's advice not to use these rounded values.

Upon considering some of the ways the teacher might respond, we now discuss one based on applying ideas from the real analysis proofs that could help the teacher exemplify to the student a situation in which the rounding would result in a relatively large difference. That is, the teacher could craft an example that would counter the student's claim that the use of rounded values does not produce a very different answer from the actual one. We find the real analysis proofs productive for addressing this pedagogical aptitude of not just stating that the approach *can be* problematic but also pointing to a situation where the student's approach *is* problematic. In the pedagogical situation, the rounded values the student used were ultimately multiplied. In terms of error accumulation, the error in the product of two approximations grows as a function of ($|a| + |b|$)—so, the easiest way to increase the potential error is by increasing this sum. So, for example, by changing the equation to $\sin(59°) = x/360\sqrt{5}$ (and presuming the student still rounds both values to the tenths (*i.e.,* 0.85 and 804.98)), the potential error then grows by a factor of $(804.98 + 0.85) \approx 805.84$, which makes the error no worse than approximately 8.0584. (Alternately, one might change the equation to use tan(79°) since, unlike sine, the tangent function is not bounded.) By changing even just one value, we see the teacher can exemplify to the students situations in which there would be greater potential error involved in their approach. In this case, the actual answer (about 690.006) would be off from the estimate, of $804.98 (0.85) \approx 684.233$, by about 5.77 units—a difference most students would likely not regard as trivial.

By understanding the big ideas behind how error accumulates, the teacher can alter the potential error in this fairly common student approach in ways that can help the teacher exemplify the potential limitations to students. That is, the teacher's response to the student could be: "Hey, try your rounding approach on the following problem: $\sin(59°) = x/360\sqrt{5}$. How close was your answer this time?" The primary point being that by having the student engage in another problem, the student would recognize that there are times when the initial error might grow quite a bit larger. Indeed, such exemplification could help convince the student of the limitations in a way that simply stating a maxim—"don't round until the end"—may not, as the modified example is a more direct rebuttal to the student's argument. Lastly, we observe that the insight required for constructing this example did not come from the real analysis theorems, but rather the proofs of these theorems. For instance, consider the product theorem—that the product of two convergent sequences will converge to their product. How error accumulates is not stated explicitly in the theorem. Implicitly, the theorem statement only tells us that accumulated error converges to zero as n increases. The teacher in our previous scenario did not apply this fact. Rather, the teacher used the insight of how error accumulates for a particular term of a sequence, and this insight is evinced not in the statement of the theorem, but in the proof of the theorem.

Discussion

In this example, we have considered how the study of proofs in advanced mathematics can be productive for secondary teaching. The real analysis proofs in this case are being used to resolve not necessarily a 'mathematical' issue (not an issue about the secondary content of solving equations, nor a disciplinary issue about precision), but primarily a 'pedagogical' one—of being able to exemplify to a student when an approach might fail. That is, the specific proofs studied in real analysis (not just the theorems themselves) are being applied to an issue that arises in the teaching of secondary mathematics. We elaborate on a few pertinent points for discussion.

First, while the example above is singular, it represents a broad swath of situations that arise in secondary teaching. Rather than trigonometric equations, the equation could be a simpler one with rational coefficients: $\frac{2}{7}x - 60 = \frac{6}{7}$. And some students still may prefer to use approximated decimals, such as $0.28x - 60 = 0.85$. Solving for x now involves the quotient (not the product) of two approximated values. Perhaps somewhat surprisingly, rounding to the hundredths in this equation will lead to an approximate answer of $x \approx 217.3214$, which is more than 4 units away (*i.e.,* 400 times the initial error) from the actual answer, $x = 213$. Indeed, if a student were to use a substitution method to check the value of x in this case, they would be led to think that the solution (and likely solution method) was incorrect. But in fact the solution process was correct—it was simply using the rounded values in this problem that led to a dramatic increase in error. This makes sense when looking at the real analysis theorems, which indicate that the original error in the quotient of two approximated values grows by no more than $(2|a| + |b|)/|b|^2$. That is, a *larger* numerator or *smaller* denominator (or some combination of the two) results in larger potential error when dividing two approximations. The point is that the potential error accrued in early rounding will depend on the operations and the values involved, and can be engineered to some degree based upon ideas within the real analysis proofs.

Second, in this teaching situation we see the teacher applying ideas learned from the process of proving the algebraic limit theorems to respond to a student. We elaborate on two facets of this response. First, the advanced mathematics proofs were relevant for their teaching. That is, the ideas learned about error accumulation from the process of proving the algebraic limit theorems did meaningful work for the teacher in this situation. It afforded the teacher the mathematical insights (*e.g.,* by increasing the sum, ($|a|+|b|$), the error in the product potential increases) that were necessary for generating an example that would have a larger error—which directly responded to the student's first counterargument. Second, however, is that while knowledge about error accumulation was productive for the teacher, the teacher's response was not an exposition of these ideas to the student—the response was to have the student look at another example. That is, the ideas about error accumulation operated in the background, helping the teacher respond to the student, but were not necessarily made explicit to the student. Now, it could be that one might want secondary students to have a sense of how error accumulates when rounding, but the broader point is that even if one does not go into such detail with secondary students the ideas from such proofs can still be valuable to secondary teachers' work.

Third, in teaching advanced courses such as real analysis to secondary teachers, mathematics teacher educators have to consider ways of making the content relevant for their future work in the classroom. This could be by helping students see content connections in real analysis that may arise in secondary mathematics (*e.g.,* 0.999… = 1) or to instill broader disciplinary practices. Both are valuable for secondary teachers. However, as discussed in this article, the ideas within specific proofs in advanced mathematics, and not just the theorems, can also be leveraged to make such courses more relevant to secondary teachers. In this case, we regard making the connection to situations in secondary teaching explicit as being important. That is, in their study of the proofs of the algebraic limit theorems in a real analysis course, secondary teachers should be asked to look at ideas within the proofs through the lens of error accumulation, they should be prompted to consider how these ideas might inform operations on approximated numbers, and they should be given specific teaching situations to which they have to respond. Indeed, in mathematics teacher education, we regard being explicit as an important component in elucidating connections between advanced mathematics and secondary teaching.

Lastly, we point out that there are other examples of this kind of connection. For instance, proving that a group of even order has a non-identity element *g* whose square is the identity, which is typically accomplished by pairing each element with its inverse, can be useful in teaching secondary mathematics situations as well, such as justifying that perfect squares have an odd number of integers. However, such applications appear to be relatively difficult to identify. Indeed, our efforts to locate these kinds of examples in a real analysis course were considerable. Consequently, simply asking mathematics professors to 'make connections' between the proofs in their courses and the teaching of high school mathematics is likely to be ineffectual. We need those who have great expertise both in pedagogical issues in the teaching of secondary mathematics as well as mathematical issues that arise in advanced mathematics. As a mathematics education community, we will need to support this recommendation by doing the difficult work of seeking out a broader range of examples of connections.

Conclusion

From the literature, the potential upshot for secondary teachers in taking advanced mathematics courses has been related to an improved understanding of secondary content, and/or the development of broader disciplinary practices that come from advanced study (*e.g.,* Even, 2011; Heid, Wilson, & Blume, 2015; Wasserman, 2016; Zazkis & Mamolo, 2011). This article similarly discussed how advanced mathematics can be useful for a secondary mathematics teacher. In it, we explored one example for how the study of proofs might be applied to a secondary teaching situation: using proofs of how limits respect arithmetic operations to understand the accumulation of error and to exemplify why rounding decisions matter. Broadly, this represented a type of connection between the study of advanced mathematics and the teaching of secondary mathematics, particularly in relation to the study of proofs in advanced mathematics, which we have not yet observed in the literature. The main argument was that the study of proofs in advanced mathematics can serve purposes for secondary teachers other than just acquiring a better sense of mathematics as a discipline. In particular, ideas within proofs can be applied to situations in the teaching of secondary mathematics.

The aim in teacher education is to help teachers develop not just mathematical knowledge but also particular pedagogical aptitudes. We hope that the one example in this article pushes us to consider other kinds of connections that can be made for teachers in advanced mathematics courses, and that this is only the start of becoming more explicit about the kinds of direct and indirect applications for secondary teachers we seek from their enrollment in advanced mathematics and in utilizing such ideas in teacher education.

Notes

[1] Technically, the error in the product is no worse than $|a|$ times the error of a_{appr}, and $|b_{appr}|$ times the error of a_{appr}; but with truncated, positive, values, the use of $|b|$ instead of $|b_{appr}|$ is not problematic.

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