KEY IDEAS AND MEMORABILITY IN PROOF

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Both mathematicians and mathematics teachers are well aware that *mathematical explanation* and *mathematical understanding* are terms that lack rigorous definitions, but these terms have proven useful nevertheless in mathematical discourse. Most mathematicians and educators share the view, in fact, that a proof is most valuable when it leads to understanding, and that an explanatory proof is more likely to do so [1]. This view is one facet of the recognition, in mathematics, that a proof is not only a correct syntactical derivation (based only on formal definitions and the manipulation of formulas through permissible logical inferences), but also, and even primarily, a sequence of related mathematical ideas in which these ideas and their relationships are the most valuable features (Manin, 1998; Rav, 1999; Rota, 1997b; Thurston, 1994).

This view makes the assessment of a proof more complex. When the focus on proof is primarily conceptual, the usefulness of a given proof can no longer be assessed by syntactical criteria alone. When choosing which proof of a theorem to present and how to present it, an author or teacher would be helped by having explicit criteria for the degree to which a proof is explanatory and fosters understanding. Unfortunately there are no such well-defined and shared criteria. In their absence, proofs have often been judged on the basis of admittedly ill-defined aesthetic properties such as "clear," "ingenious," "explanatory," "elegant," "deep," "beautiful" or "insightful."

Indeed, aesthetic properties have played an important role in mathematics and in mathematics education, despite having little to do with logical correctness (Aberdein & Inglis, 2013; Rota, 1997a; Sinclair, 2011). Netz (2005), in particular, has gone so far as to argue that though mathematicians tend to focus on the "epistemic values of mathematics activity" (that is, on proof and proving) the aesthetic value of mathematical activity, though non-epistemic, is "*intrinsic* to mathematics" (p. 253). Indeed, for both mathematicians and educators, proofs are "far more than certificates of truth" (Gowers, 2007, p. 37). Gowers notes, too, that "it is remarkable how important a well-developed aesthetic sensibility can be, for purely pragmatic reasons, in mathematical research" (p. 37).

With a view to placing some of these non-syntactical judgements of proof on a firmer basis, Gowers (2007) proposes to examine what makes a proof memorable. When Gowers speaks of a memorable proof or of its memorability, he is using these terms in his own fashion. He has in mind neither historic importance nor rote memory, but rather the ease with which a teacher or student is able to reconstruct the proof without notes in front of a class or in an examination, relying only on a few key ideas. In his view, this ability has broader implications, because there is a "very close connection between memorizing a proof and understanding it" (p. 40). As he writes:

> Instead of remembering the details of a proof, it is much more efficient to remember a few *important ideas* [italics ours] and develop the technical skill to convert them quickly into a formal proof. And it is still better if the ideas themselves are not so much memorized as *understood* so that one feels that they arise naturally. (Gowers, 2007, p. 40)

The idea of memorability leads Gowers to introduce a metric, which he calls the *width* of a proof. This is simply the number of distinct pieces of information, or ideas, one has to keep in mind in order to reconstruct a proof from memory, and thus it would seem to be a metric that is potentially quantifiable. Gowers borrowed this term from computer science, where it refers to the amount of storage needed to execute an algorithm, viewing storage in computers as analogous to memory in humans.

In this article, we delve into the meaning of these concepts, assess their usefulness, and attempt to show how they relate to other aspects of proof discussed in the existing literature on proof and proving. We also explore how the insights behind "width of a proof" and "memorability" might mesh with other teaching perspectives.

Gowers's examples

Gowers maintains that the way to memorability (as he uses the term) is via lower width, and expands on this point through the three following examples, one of which is arithmetical and two of which are proofs. We present them here with some comments.

1. Mental calculations

Gowers discusses how one might best go about performing non-trivial arithmetic operations without writing anything down. He cites familiar arithmetic operations in which one could reduce the width of a calculation (*i.e.*, the number of pieces of information one must keep in mind at one time) by appropriate simplifications. He points out, for instance, that the mental multiplication of 47 × 52 can be greatly reduced in width by bringing into play the difference of two squares. One can represent the product of 47 × 52 as the product (50 – 3) × (50 + 3) less 47, which yields (2500 –9) – 47. The many individual digits that must be kept in mind using

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the customary multiplication algorithm have been eliminated by introducing a single new idea, the difference of two squares. Using Gowers's terms, this insight has transformed the original large-width operation into a lower-width one.

2. Every positive integer is a product of primes

Gowers presents a proof of the factorability of positive integers that he describes as self-generating, meaning that it can readily be reconstructed using familiar mathematical approaches such as reasoning by contradiction, with nothing else to keep in mind.

> If there is a positive integer that cannot be written as a product of primes, then there must be a smallest such integer. Let this smallest integer be n. Then n is not a prime number (or else we would count it as the "product" consisting of the single prime n). It follows that $n = ab$ for two positive integers a and b, both smaller than n. We defined n to be the smallest positive integer that could not be written as the product of primes, so a and b *can* [both] be written as products of primes. But in that case we can combine those two products and we find that n is also a product of primes. This is a contradiction, so the result is established.// (p. 44)

Gowers goes on to ask, "How is this proof remembered? When I wrote it, I did not have to look it up in a textbook; not even in a virtual textbook that I had stored in my brain. Rather I knew that I would have no trouble generating it" (p. 44). He adds that his strategy was a heuristic one he had learned from experience, namely, that it is useful to focus on a minimal counterexample when trying to prove a statement such as "for every natural n" (p. 49).

This proof again shows that, given a certain degree of mathematical experience, one can find and make use of a proof that makes little demand on one's memory, *i.e.*, one of very low width. It may not be elegant, but, as Gowers points out, it virtually generates itself. Even a beginner will quickly realize that assuming only a small degree of mathematical sophistication, it can be generated *ab initio*.

3. The irrationality of √2

This theorem can be proved in various ways, such as by infinite descent, by contradiction, by unique factorization, or by geometrical reasoning. Its proof by contradiction is often used in mathematics teaching, in fact, to illustrate the general technique of proof by contradiction. The following is Gowers's version.

> Assume that $\sqrt{2}$ is a rational number. This would mean that there are positive integers p and q with $q \neq 0$ such that $p/q = \sqrt{2}$. We may assume that the fraction p/q is in its lowest terms, so p and q are as small as possible as a presentation of $\sqrt{2}$; it can be written $p/q = \sqrt{2} = (2 - \sqrt{2})/(\sqrt{2} - 1)$.
>
> Then, substituting p/q for $\sqrt{2}$; $p/q = (2 - p/q)/(p/q - 1) = (2q - p)/(p - q)$
>
> Because $p/q = \sqrt{2}$, it lies between 1 and 2, so we have $q < p < 2q$.
>
> It follows that $2q - p < p$ and that $p - q < q$.
>
> This produces a fraction equal to p/q but with smaller numerator and denominator, and this contradicts the initial assumption that p and q were as small as possible, so the assumption that p/q is in lowest terms ($\sqrt{2}$ is rational) must be false.// (p. 47).

Gowers himself makes the point that this version contains a step that seems to "spring from nowhere," namely, writing $p/q = \sqrt{2}$ as the expression "$p/q = (2 - \sqrt{2})/(\sqrt{2} - 1)$" (p. 47). But it is precisely this key idea of de-rationalizing a surd (*i.e.*, presenting 1 as the ratio of two equal surd expressions, namely "$(\sqrt{2} - 1)/(\sqrt{2} - 1)$" that makes the proof more amenable to reconstruction (or more memorable, to use Gowers's term).

Now, this key idea has two parts: the insight that the fraction can be replaced by another one with smaller numerator and denominator, and the technical skill of knowing how to rationalize (and hence de-rationalize) surds. The clever step makes use of the solution of Pell's equation: proving $\sqrt{d}$ is irrational ($d \neq k^2$) involves solving $x^2 - dy^2 = 1$, which in turn is related to the continued fraction for $\sqrt{d}$. Thus the key idea turns out to require familiarity with continued fractions and related number theory.

Nevertheless, this proof has low width because there is only one key idea that really has to be kept in mind. This key idea does require mathematical sophistication, and also the inspiration to have come up with it in the first place. But anybody who has read the proof once and has grasped this key idea can reconstruct the proof even if he or she is a novice mathematician.

Clearly the clever step in this proof is useful, but where does it come from? A reader might like to know why someone would think of this idea. And an objection in principle would be that no step in a proof should appear as a *deus ex machina.* As Pólya (1954) put it, "it is not enough that a step is appropriate: it should appear so to the reader" (p. 148).

But Gowers is concerned here with memorability (again, as he defines the term). Even if someone just happened to hit upon an obscure and otherwise unproductive idea that made it easier to reconstruct a proof (easier for him and for others, once they understood the idea), that key idea would still make the proof more memorable. From that limited point of view there is nothing wrong with *deus ex machina.* This consideration just shows that being easily reconstructible does not necessarily make a proof good for generating mathematical understanding.

Reflections on Gowers's notions of memorability and width

The context of other proposed measures: Initially, in discussing the notion of the width of a proof, it may be useful to contrast it with two other measures of proof that have gained some currency: *length* and *depth*. The "length" of a proof is the total number of words, symbols and numerals needed to present the proof in a comprehensible form. The "depth" of a proof refers to the richness of its connections to other mathematical ideas and in particular to other mathematical domains, whether that lies in its reliance upon results from those domains or in its implications for them.

The "length" of a proof is perhaps the most quantifiable of these two measures, although it is certainly true that two versions of a proof which are the same from a mathematical point of view may differ in length, depending on the language and symbol set chosen.

The concept of "depth," though it clearly does reflect a quality of proof, would seem to be even less amenable to quantification. One might choose to tally the number of results or sub-proofs upon which a proof relies as stepping stones, but this would raise the question of whether such a result or sub-proof should be counted because it is a noteworthy contributor to the proof from a different domain, or not counted because it is simply a part of the corpus of mathematical knowledge. One is led inexorably to a consideration of the quality or value of each of the mathematical connections displayed by a proof, a consideration that may well be germane to the attractiveness of the proof and its usefulness, particularly in teaching, but hopelessly beyond the reach of quantification. As far as the implications of a proof for other domains goes, it is hard to see, for similar reasons, how this could be quantified—or in fact even known, since such implications may be discovered only later.

How "width" is or might be defined: Width, as defined by Gowers, is a quality of proof clearly different from length or depth, but no more tightly defined. It is unclear, first of all, how the different kinds of information that must be kept in memory (*e.g.*, digits, concepts, rules) are to be counted, and whether they can be given equal weight. In the above example of arithmetic multiplication without the use of paper and pencil, the normal algorithm requires that many interim digits be kept in mind, and thus the procedure has a large width. The width can be reduced, as Gowers states, by recourse to the difference of two squares.

Now, it is true that by using this ingenious procedure—or a key idea, to put it in more general terms—there are fewer digits to memorize. On the other hand, one has to have had in mind the difference of two squares. In Gowers's view, the key idea does not contribute to the width, because it is part of a body of knowledge that any mathematician or mathematics teacher can be assumed to have mastered. This view means that a proof can be more complex, in the sense of using more advanced mathematical ideas, but precisely because of that have less width and so be more memorable (as Gowers uses the term) and more amenable to reconstruction. Of course, evidence for this claim could only come from empirical research.

One wonders, though, if there is not a point of diminishing returns. It is hardly a realistic scenario in education, but if one were to try to make a proof more reconstructible by applying three or even ten key ideas, would not the strategy fail? One could argue that the mental capacity to store a number of sophisticated mathematical notions is greater than that needed to store a few digits, and so one might see here an opening to refine the idea of width by assigning different weights to different types of information.

This leads to the question of how long a piece of information has to be kept in mind. To take the example just discussed, the concept of the difference of two squares must be kept in mind always, so it can be called upon whenever needed, in this calculation or others, while the memory devoted to interim digits can be released as soon as the calculation is finished. In the management of computer storage there is a strong focus on how long a piece of storage is required. The inspiration for the term "width" came from computer storage, so the concept might be refined by taking into account the dimension of time.

Dimensions of proof as replacements for subjective qualities: Gowers proposes the concept of width not only as a way of measuring the ease with which a proof can be reconstructed, but also as an attempt to provide an objective measure that might replace other traditionally useful but ill-defined qualities of proof. One must keep in mind that Gowers sees his investigation as a work in progress. Nevertheless, while conceding that the notion of the "width of a proof" is not precise and lacks a formal definition, Gowers does hope that more work will lead him to a definition "more precise than subjective-sounding concepts such as 'transparent', or even 'easily memorable'" (p. 57).

It is tempting to view length, depth, and width as the three dimensions of proof, as the terms suggest (and may have been chosen to suggest). The first thing that must be said is that it is far from clear that these dimensions are orthogonal. Be that as it may, the obvious question in the context of Gowers's project is whether they suffice to capture or replace the many important non-syntactical properties of proof such as "comprehensible", "ingenious", "explanatory", "elegant", "deep", "beautiful", and "insightful".

The first difficulty is that none of these traditional properties is well defined, and so their relationships to each other cannot be entirely clear. They are certainly not synonyms. Very concise proofs are often considered elegant or beautiful, for example, but they are less likely to be comprehensible or explanatory. Another difficulty is that the proposed measures of length, depth, and width are not clearly defined either, as discussed above. They cannot be seen as objective measures. And even if they were in the course of time to be refined and turned into something more measurable, it would seem far-fetched to think that they would suffice to explicate or construct the many traditional non-syntactical properties of proof.

It can be interesting, however, to reflect upon the relationships that might obtain between width and traditional non-syntactical properties. An elegant proof, for example, could well be one of very low width that relies upon a single key idea of broader mathematical relevance. Or one might take the view attributed to Pólya that "the elegance of a mathematical theorem is directly proportional to the number of independent ideas one can see in the theorem and inversely proportional to the effort it takes to see them."

How such a multiplicity of ideas translates into width is not clear. One can imagine a proof designed expressly to be explanatory, in which the many ideas brought to bear would make it of great depth but of indeterminate width. As discussed earlier, it would seem that at some point the strategy of using key ideas to reduce width would break down when the number of key ideas used became more of a burden than a help—even if they were all familiar from the corpus of mathematical knowledge.

The value of objective measures: We have discussed to some extent the questionable feasibility of quantifiable measures for various aspects of proof. A more basic question is whether there is any point at all in seeking objective measures of non-syntactical properties. Mathematicians know an elegant proof or an ingenious proof when they see one, just as mathematics educators, informed by classroom experience, are quite capable of deciding what kinds of proof are going to be most useful in conveying to their students important mathematical ideas and their connections.

Pedagogical considerations

To explore whether and how the concepts of "width of a proof" and "memorability" might make a contribution to mathematics teaching, we propose to address two questions:

- To what degree does the width of a proof, as Gowers uses the term, represent a new idea in mathematics education, and how does it relate to other insights into the nature of proof?
- How does memorability, as Gowers uses the term, related to understanding, and how could the concept be of benefit to mathematics education?

In putting forward the concept of width, Gowers was not addressing, at least not directly, the broader questions of comprehension or conveying understanding or teachability. He was posing, first of all, the rather more limited question of how to make it possible for a teacher or student to reconstruct a proof from memory with the least effort. His answer, as we have seen, is to base the proof on a key mathematical idea and thus reduce the amount of less important information that must be held in memory.

Now, the idea that a proof can hinge on a key insight is not something Gowers invented. But Gowers has emphasized, in his discussion and his examples, the down-to-earth advantages of such a proof. He shows that a proof that enjoys low width because it hinges on a key idea can make it much easier for a teacher to reconstruct it at the blackboard, for example, or for a student to remember it at examination time. And these advantages are not at the expense of understanding, an objection that would justifiably be raised against rote memory.

In suggesting the concept of width, Gowers also had in mind a contribution to eventually putting at least some of the non-syntactic qualities of proofs on a more rigorous footing. Many practitioners and researchers had been quite aware of the importance of these qualities for some time, and a wealth of studies had discussed their implications for mathematical practice and for teaching. This recognition was Gowers's point of departure, but he hoped to go further by creating a more objective measure that might be useful in defining or explicating these "aesthetic" qualities.

Gowers's thinking is clearly consistent with, though not the same as, the thinking of others who have examined the importance of key ideas quite extensively. An example is the point of view presented by Raman (2003) in her paper "Key ideas: what are they and how can they help us understand how people view proof?". Whereas for Gowers, a key idea is simply a main idea that makes it easier to reconstruct a proof, for Raman, the concept of key idea is related to explanatory proofs and brings together a private aspect (engendering understanding) and a public aspect (containing sufficient rigour).

Sometimes key ideas have been discussed under a different name, as by Mejia-Ramos *et al.* (2012), who, recognizing the significance to comprehension of the non-syntactical aspects of proof, looked at the way in which a proof "is not only understood in terms of the meaning, logical status, and logical chaining of its statements but also in terms of the proof's high-level ideas, its main components or modules, the methods it employs, and how it relates to specific examples" (p. 3). Hemmi (2008) and Malek and Movshovitz-Hadar (2011) have focussed on helping students engage with the main ideas of a proof, showing the benefit of pedagogical strategies such as "proof transparency" and generic examples.

Also closely related to the concept of width is the concept of "structured proofs", which are thought to allow students to more easily identify the major ideas of a proof, as opposed to focussing on its logical details (Leron, 1983, 1985). In Leron's view, an emphasis on overall structure and major ideas makes it much easier for students to grasp the point of a proof as a whole, and thus to reproduce it without undue effort or rote memorization.

The vast research literature on proof and argumentation often touches upon the valuable properties of proof that Gowers sought to capture with his idea of width. For example, Durand-Guerrier *et al.* (2012) and Knipping (2008), when examining the role of argumentation in teaching proof and justification, pointed out the value of an emphasis on the main ideas of a proof when they discussed introducing topic-related ideas (as opposed to topic-free logical inferences) into the process of proving [2].

In addressing our second question, on the connection between memorability and understanding, one must point out, first of all, that memorability, as Gowers uses the term, has nothing to do with rote learning. In education, rote learning is seen as the antithesis of explanation and understanding, and indeed it is quite possible to replicate a proof without understanding the reasoning involved, the range of application of the proof method, or even the result itself. As Fried (2010) has maintained, this does not mean there are no benefits to memorization:

> memory and memorization are held up a little too often as antitheses of all things good in learning and teaching [...] Yet, it may well be that in so easily and frequently opposing memory we have forgotten its educational virtues. (p. 257)

Gowers does see great virtue in what he calls memorability. It must be said that Gowers does not claim to have done research on memory and memorizing. In any case, those who have reviewed such research have concluded that we really do not know much about the role of memory in mathematics learning (Raghubar *et al.*, 2009). Speaking as a mathematician concerned with proof, however, Gowers sees memorization as closely aligned with understanding, where by "understanding" he seems to mean that the ideas central to a proof come to mind in the proper sequence, as if without effort. Put another way, the more coherent a narrative one has for a proof, the more likely it is that it can be reconstructed:

> The fact that memory and understanding are closely linked provides some encouragement for the idea that a study of memory could lie at the heart of an explication of the looser kind described earlier. It is not easy to

say precisely what it means to understand a proof (as opposed, say, to being able to follow it line-by-line and see that every step is valid), but easier to say what it means to remember one. Although understanding a proof is not the same as being able to remember it easily, it may be that if we have a good theory of what makes a proof memorable, then this will shed enough light on what it is to understand it that the difference between the two will be relatively unimportant. (Gowers, 2007, p. 41)

In teaching proving, mathematics educators avoid rote learning. Furthermore, they know that if students are to achieve mathematical understanding, it is not enough for them to learn how to build valid sequences of logical steps (though that too is important). They must also have a broad grasp of mathematics, so that they can draw upon a reservoir of mathematical ideas and apply them to the argument they are constructing.

Gowers takes this a step further in making the explicit suggestion that a proof can foster understanding more successfully if it is constructed on the basis of a key mathematical idea that is well understood and internalized. In his view, the key idea can reduce the amount of less important information, or clutter, that must be kept in mind. Thus a key idea reduces the width of the proof, increases its memorability, and makes it more amenable to reconstruction. At the same time, the intellectual focus, so to speak, is always on the key idea, impressing it on the student's mind and contributing to their mathematical understanding. One might well ask, however, whether a narrow-width and memorable proof does in fact contribute to mathematical understanding when the key idea on which it hinges is an ad-hoc device or technical skill, rather than a mathematical result that offers true mathematical insight. This is a consideration that educators will have to keep in mind.

There are other open questions on the topic of memorability and width, as we have seen, but they would seem to present avenues for productive exploration, particularly as they relate to mathematics teaching.

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Notes

[1] See, for example, Balacheff, 2010; de Villiers, 2010; Hanna, 1990, 2000; Harel & Sowder, 2007; Mariotti, 2006; Mejia-Ramos *et al*., 2012; Tall *et al*., 2012; Weber, 2010.

[2] See also the research reported in the recent ICMI study on proof and proving in mathematics education, Hanna and de Villiers (2012), Reid and Knipping (2010), and research on thinking mathematically (Mason, Burton & Stacey, 1982).

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