REVERSAL THEORY TO UNDERSTAND STUDENTS' DIFFICULTIES WITH GRAPHS IN MATHEMATICS

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Betsy, a secondary school mathematics teacher, assigns some homework to her grade-11 class, collects students' answers via email and corrects them during a classroom lesson. In class, when she projects the task shown in Figure 1 on the whiteboard, she comments:

> This task was completely different from the ones we have done together in class, it asked you to apply different concepts at the same time, and to approach them in a new way: namely, to read the graph, to get information from the graph somehow indirectly, and to integrate data coming from the text with data coming from the graph. Did you enjoy it?

The tone of her voice induces us to think that she is expecting that the students answer, "Yes". While checking their homework, she noticed that the majority succeeded in answering it. The first reply, from Naomi, is instead, "No, not at all". Naomi adds that she liked the previous exercise. In that one, there was a complicated formula to apply and a nonlinear equation to solve. However, she felt confident that she could solve it correctly. For this exercise, with the graph, Naomi "did not understand anything". She adds that she was not able to do anything, because she understood nothing. Betsy invites Naomi to do the exercise in class. After a discussion with her classmates, Naomi solves the first two questions. When Betsy writes the formula necessary to answer the last two questions [1], Naomi confidently applies it and answers correctly. Finally, she comments, "It was easy, actually, but at home I hadn't understood the graph and I wasn't able to go on".

Betsy expected her students to find graphs playful and to enjoy them. Why did Naomi prefer tasks with a formula? Why such a difference between formulas and graphs? We notice that Naomi uses the verb 'to (not) understand' more than once in the episode and we wonder what she means by saying that she was blocked by not understanding. Was that truly a cognitive issue, or is it an affective one? There is a research tradition dedicated to finding out the typical mistakes students make when they interact with graphs (see *e.g.*, Ivanjek, Susac, Planinic & Andrasevic, 2016), where drawings and diagrams are taken as indicators of general mathematical development, but what can be missed taking a strictly cognitive approach to students' difficulties with graphs? What is the role of teaching choices that somehow promote classroom preferences for formulas / difficulties with graphs? And what is the impact, if any, of an exercise, or a problem, felt as 'impossible' by a student, when she is alone, at home? Reversal Theory might be useful for helping us answering those questions.

Motivational states and being blocked

Reversal Theory aims at explaining inconsistencies in human behaviour by looking at both changes in the situation(s) and in an individual's interpretation(s) of the situation she is living. The main idea of Reversal Theory is that experience is structured into four fundamental domains, each of which is experienced in two different and opposite ways. We experience one state for each of the four domains, but one (or two) of these four states is prevailing and guiding our actions. Since we reverse frequently between two opposite states, it is interesting to focus not on personal traits, which are considered to be stable and unchanging, but rather on dynamical, unstable and even contrasting states (Lewis, 2013). In this article, we see three domains in action and we recall only those that are relevant for our research. The first domain of motivational states is about the goals and purposes of our actions and is labelled 'means-ends'. We can experience it in a serious way, when we are looking for achievements and results, or in a playful way, when we are more likely to be spontaneous and creative. The second domain is about rules, it can be expressed as the need to fit in and do what is expected or, on the other hand, as the need to challenge the way things are done. As noted in Lewis (2015), "when the need for conformity operates in 'fail' mode, students feel excluded from the community of *people who can do maths*" (p. 32). On the other hand, the rebellious state can be a positive state if students are able to 'challenge' the rules and develop their own methods for solving problems and exercises. The third domain, transactions, is about interactions with people or things. In a mathematical context, students are in the mastery state when they have control of the situation, they feel able 'to do mathematics', and complete a task successfully. The opposite state is sympathy, in which students are highly sensitive and desire to

Table 1: Three motivational states, according to Reversal Theory.

Domain	Opposed States	
Means-ends	Telic/Serious (achievement)	Paratelic/Playful (enjoyment)
Rules	Conformist (fitting in)	Rebellious (freedom)
Transactions	Mastery (power/control)	Sympathy (love)

feel that they are cared for. Table 1 summaries these three motivational domains, to which two motivational states with their core values are associated. We analyse the episode with Betsy and Naomi, at a finer grain, employing Reversal Theory.

To return to Betsy and the class discussion in her final recap of the lessons about financial literacy [2]. Betsy starts the correction of the last exercise (Figure 1) feeling optimistic and *playful*; she thinks that the exercise is less boring than the previous ones and wants to share this feeling with her students. From her words in the following quote, we notice that she sees this graph as an opportunity for her students to find their own way of engaging and managing the exercise, instead of adhering to the somehow fixed rules of given formulas:

Teacher	This task was completely different from the ones we have done together in class, it asked you to apply different concepts at the same time, and to approach them in a new way: namely, to read the graph, to get information from the graph in a rather indirect way, and to integrate data coming from the text with data coming from the graph. It was not an ordinary exercise *[...]*. Did you find it fun? *[addressing Naomi]*
Naomi	No, at all!
Teacher	*[silence]*

Betsy was not expecting this reaction and is left speechless for a few seconds. She saw Naomi talking with her classmates about the exercise and thought her students were sharing their feelings about the challenge of this exercise being different from the previous ones. Unfortunately, Naomi is in a complete different state:

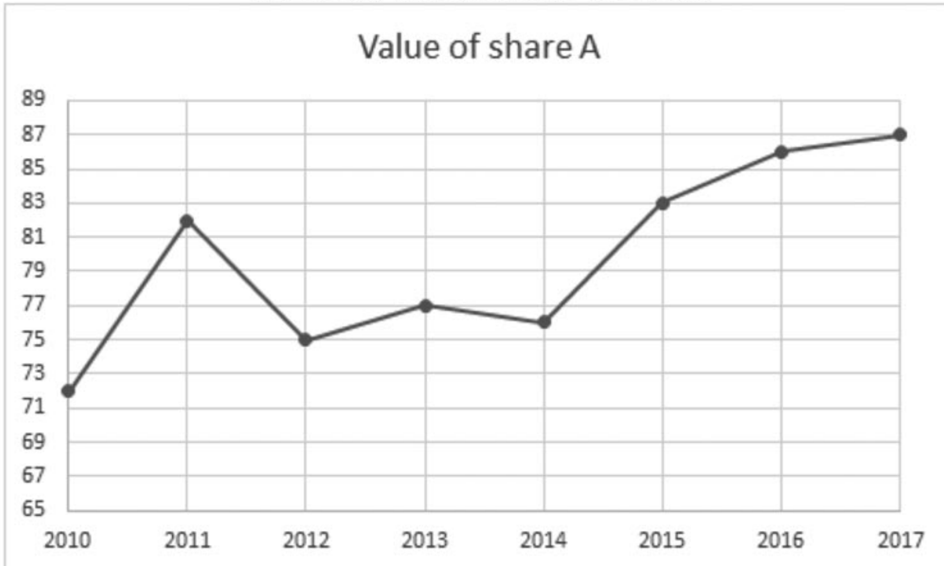
Naomi	I liked more the previous [exercise] than this one
Teacher	What is that you did not like about this one?
Naomi	ehm *[pause]* I was not understanding anything.
Teacher	And when you did this at home? What about it?
Naomi	I didn't do this one, because I did not understand anything.
Teacher	So, this is your time.
Naomi	Yes, indeed, I will listen *[pause]*
Teacher	Read it!
Naomi	Ah.

As we noted above, Naomi was not able to solve this specific exercise at home. She is blocked and seems to be in a rather opposite state with respect to her teacher: namely, to be in a serious/telic, conformist and mastery state. In fact, it seems that Naomi mostly wants to get the job done (telic), instead of being creative and playing with graphs. She says that she likes to adhere to the rules of formulas and to apply them (conformist). She is in a mastery state, since she says that she does not know how to deal with the graph and she desires to feel competent and good at math. The previous exercise, where students had to apply a formula, was better fitting her motivational state ("I liked more the previous one"). It seems that a straightforward application of formulas serves a mastery and conformist motivational purpose more than graphs. Being in a telic/serious state, Naomi is likely to look for a clear and unique approach, she is not ready to find her own strategy to solve the exercise and is expecting a procedure to follow. Actually it seems that Naomi is waiting for the correction of this exercise in order to see what the solution will be, to learn a new procedure to follow, copying what the teacher or her classmates will do to solve the exercise ("I will listen"). There is a mismatch between the state of Naomi and the one of the teacher, who involves Naomi in the correction of the exercise, pushing for the invention of new ways of doing math with graphs (rebellious and paratelic motivational state).

After the previous conversation, Naomi reads the first part of the exercise. When she finishes reading the first question, she stops and asks the teacher "Do I have to answer? I do not know". Again, Naomi is blocked and waits for the teacher to tell her what to do. Betsy's request to read the exercise was not enough to change Naomi's initial state, the student is still waiting for instructions. Naomi wants to solve the exercise (achievement), but she is blocked since she is not ready to try different strategies (moving to a different state), as the teacher suggested at the beginning.

Other students intervene and then there is a class discussion with the teacher about how to answer the first two questions of the exercise. Naomi is not directly involved in the discussion, however, the teacher adds as a general remark to all students: "You are looking for easy and mechanical rules to apply and this completely blocks your thinking". Betsy has in mind Naomi and her need for conventions. The teacher goes on correcting the exercise and reads the third question. Betsy writes the formula [3] to

Exercise: Look at the following graph that displays the value of a share on June 1st from 2010-2017, then answer the questions.



Suppose that three friends, Davide, Elio and Fabio, bought this share: Davide bought it on June 1, 2010, Elio bought the share on June 1, 2011, and Fabio bought it on June 1, 2013. Suppose also that all the transactions described in the following happened on the same day of the year (June 1).

1. If all the three friends sell the share today, who is the one earning the most?
2. In which years Elio should have sold his share, in order to make some money?
3. Suppose Fabio sold his share in 2016 and there was not a dividend distribution, what is the percent return of this investment?
4. Suppose that there was a dividend in 2012, Elio sold his share in 2015 and the percent return of his investment was 10%. Find the value of the dividend.

Figure 1. The exercise assigned by Betsy [2].

compute the percent return of Fabio's investment, and then again addresses Naomi to involve her in the discussion:

Teacher This is the formula, isn't it Naomi?

Naomi *[pause]* this was extremely easy, actually *[pause]*

Teacher Would you like to solve it?

Naomi Yes, so *[pause]* The selling price is 86. Then, I have to find the dividend [pause] Do I have to compute them?

Teacher Read it again.

Naomi Ok, I have to compute the dividends.

Teacher Can you read the question again?

Naomi *[reading] [...]* there was not a dividend distribution. ah

Teacher So?

Naomi There are no dividends, so it is *[pause]* zero?

Teacher Zero *[pause]*

Naomi Oh, come on! So, 0. Then the selling price is 77

Naomi now feels that she has a goal (*i.e.,* to find the numbers to be put in the formula, such as the dividend) and this makes her feel confident while solving the exercise. She is puzzled about the value of the dividend distribution, probably because she can not find an explicit number neither in the text nor in the graph referring to that. When she realises that there was not a dividend distribution, thus the number she is looking for is 0, she comments with "Oh, come on!" as if this were something outside the rules and she was not expecting it. After that, she finds all the numbers, she uses the calculator and approximates the result correctly, as asked by the teacher. The state of Naomi does not change: she is still in a telic and mastery state, but now she is able to solve the exercise that meets her needs. Naomi is still looking for achievement, but she can reach it, with the help of the teacher and of her classmates, and successfully solves the exercise.

Teacher Naomi, you keep solving the last part.

Naomi Yes! *[reads point 4]*

Teacher The formula is already written.

Naomi Yes *[pause]* I know the formula *[pause]* So, now, I *[pause]* I take a shot, eh *[pause]* but I think *[pause]* Since I do not know the inverse formula *[pause]*

Teacher But is it necessary to know the inverse formula?

Naomi No, no, indeed *[pause]* I would do *[pause]* one sec that I need a number *[pause]*

[Naomi uses a calculator to find 0.1 for 10%]

Naomi So *[pause]* I will put in the formula the 10%, that is 0.1 equal to *[pause]* the selling price that is 83, ok, plus x, I would put x, minus the purchase price, that is 82, divided by 82.

Teacher Ok.

It seems like Naomi's motivational state has changed: it is (almost) playful, as the teacher was prompting her at the beginning. First of all, Naomi is ready when the teacher calls her and immediately starts to read the question using a more incisive tone. Naomi accepts the challenge of not knowing exactly what to do and to have a 'guess and try one' strategy in order to solve the exercise ("I take a shot"). This change of state allows her to acquire more confidence in the solution process ("I would put x"), until she successfully solves the exercise. Naomi now speaks and acts in first person, without waiting for the teacher's instructions.

In the sequel of the episode, there is a class discussion about how to solve the equation and finally the result x=7.2 is written on the blackboard. At the end of the exercise, Naomi intervenes again:

Naomi I say, it was not difficult, but I did not understand the table and then I *[pause]*

Teacher The table, you mean the graph?

Naomi Yes, I lose myself in reading, I mean, reading this stuff *[pause]*

Teacher You know that this is an important skill, right?

Naomi Yes, I know, but these are things that I know how to do, but *[pause]*

Teacher You needed some time?

Naomi Exactly.

Teacher You do not feel confident in reading a graph, but this is really an important skill you should have.

During this dialogue we notice Naomi changing state again: she goes back to a telic state. Actually, in going back in time to describe how she felt at home, she is again in that motivational state.

Discussion and implications

Ivanjeck, Susac, Planning and Andrasevic (2016) report that students tend to prefer the use of formulas with respect to graphs. They further argue that formulas may hinder the use of other strategies that may reveal to be more productive to carry on a task. Even if some features of formulas make them challenging, once a student has become acquainted with the rules of symbols in mathematical formulas, as Radford (2010) observes, algebraic transformations within the symbolic domain can be relaxing and provide a sense of comfort. Such a mechanical dealing with formulas seems not to be exploited by students when dealing with graphs: a graph usually contains information that is spatially spread out. Graphs are structured in holistic blocks that require global reading and are not subsequent to unpackaging. In addition, graphs have an iconic component that is absent in formulas (see Andrà, Lindstrom, Arzarello, Holmqvist, Robutti & Sabena, 2015, and references therein). These difficulties in reading and interpreting graphs seemed to emerge also from Naomi's words. By resorting to Reversal Theory, we proposed to interpret Naomi's 'feeling blocked' with the graph as 'not finding a purpose', namely as being in a telic motivational state that does not find an object for action. This results in a shift of focus from cognitive difficulties with graphs to the way(s) graphs and formulas are learned. Naomi shows to be able to perform single actions related to graph interpretation (*e.g.,* to read the value of the share at a certain time point), namely to satisfy her telic calls for purpose in her actions, but she seems blocked when she has to put all these actions together to deal with the task, using her creativity (like in a playful state).

We conjecture that playfulness, on the side of the teacher, took on an interesting role in dealing with her own surprise when Naomi reacted negatively to her question about whether they liked the exercise. Instead of narrowing her perspective on Naomi's difficulties (which could have been at stake in a telic motivational state that is concerned more about reaching goals and thus difficulties might be seen as obstacles), Betsy opens the opportunity to engage Naomi in a conversation about the exercise. By looking at how the episode unfolds, we question how much richer classroom discussions could be, when teachers' telic concerns are put aside for a while. This does not give us a way to solve the puzzle posed in the opening of our article, but leaves open a general question why graphs are associated with playful motivational states and formulas with telic ones.

Another possible interpretation for Naomi's block derives from Sinclair, Moss, Hawes and Stephenson's (2019) warning that students should be engaged in meaningful activities in order to become able to deal with mathematical graphs.

We can interpret Naomi's lack of purpose in her telic state as related to the request of rather passively interpret a graph. Reversal Theory provides us with a means to capture this state in the process of dealing with a graph, connecting the value of active and purposeful production of meaning to affective states that shape this process. According to Roth and McGinn (1997), at school "students make graphs for the purpose of making graphs" (p. 93), whilst it would be more effective to engage students in classroom activities where to draw is essential in order to support one's own reasoning (see *e.g.* Sinclair *et al.,* 2019) and to find one's thoughts in what she has produced, such as in graphs (Roth, 2015). In Naomi's case, the graph does not emerge out of a dynamic activity, but it represents an already crystallised situation: time has passed, data has been collected, and the student has to draw conclusions without having participated in the process of both collecting the data and reporting it in a diagram. This seems to hinder not only her sense-making, but also her will to engage in the homework. Naomi's first block seems, thus, to confirm the importance of movement for thinking: lack of active production mirrors lack of purpose. A motivational layer can, thus, be added to studies like Roth's (2015) and Sinclair's *et al.* (2019) to support the plea for teaching and learning activities in mathematics that involve the body of learners in all its facets.

Lewis (2015) further suggests to take into account also the classroom climate for examining a student's motivational state(s). The students seem not to accept Betsy's playful and rebellious state, they are not following her in solving the exercise using different strategies and approaches. Betsy starts to move to another state, so that she can tune in with her students. She moves to a mastery and conforming state ("We need the formula", "the formula is written") to help Naomi achieve the result and acquire more confidence. After this moment, Betsy moves Naomi to a playful motivational state. We note that attunement between Betsy and Naomi takes place at both cognitive and motivational levels, as if one cannot exclude the other. This dynamic seems to be relevant for both the students' understanding of the task and the teachers' effective teaching in that particular situation; students and teachers need to align their goals and motivations in mathematical activities so as to better understand each other, their difficulties and the mathematical content at stake. In this respect, de Freitas, Ferrari and Ferrara (2019) elaborate on the concept of sympathy as a way to understand affect not only as 'internal' to an individual, but as a social phenomenon: the authors elaborate that sympathetic relationships do not result in the erasure, or complete identification or passive alignment of either a student or the teacher. Sympathy is much more a matter of modulating related movements, a resonance of two different sounds. The focus is, thus, shifted on how the collective, *i.e.* the classroom including the students, the teacher and the task itself, reaches consensus and understanding of the concepts. Not only do the students and the teacher interact, but also they affect and are affected by the task. This comprehensive view of affect, carried forward by de Freitas, Ferrari and Ferrara, seems to apply also in the context of our research and Reversal Theory provides a specific focus on motivational states that link actions to emotions. With respect to previous studies, we can add a new layer for our comprehension of the role of sympathy in teaching and learning mathematics.

Finally, the last part of the conversation reported in this study deserves some attention and connects back to the role of homework for deep learning. To recall, Betsy expresses her satisfaction in having solved the exercise with Naomi, who in turn recalls that, at home, she was not understanding and that she was blocked. In the student's words, we perceive that she is going back to her initial (negative) motivational state, as if the memory from homework activity already crystallised in a trait-like attitude that is hard to change. The (positive) state-like motivation Naomi got from solving the exercise in class leaves space to a resisting and opposite state. A role of homework surfaces, namely: not only at cognitive level ideas might fix, but also affect-related states might emerge and consolidate, to a point that (whatever the in-class activity) students do not change them. This result partly contrasts with other findings in affect-related research, such as Liljedahl (2018), which showcase the positive and long-lasting effect of in-class experiences with mathematical discovery. Interestingly, Liljedahl (2018) proposes to understand affect as a system, rather than a collection of single and almost juxtaposed variables, and (as a system) subject to transformations that can take place at its core and be permanent. Reversal Theory seems to be a good candidate in allowing the researchers to capture the tension between trait-like understanding of affect as a system, and state-like analysis of in-the-moment positive experiences, which can be wiped out in the long run.

Further examination on the role played by homework activities in blended learning formats (*i.e.,* when students are left alone with their doubts and their negative emotions) needs to be conducted, in order to explore under which circumstances the emotional and motivational burden lived at home might impede learning and students' flourishing in mathematics, instead of promoting it. More specifically, how do attitudinal shifts from homework compare to potential cognitive benefits of homework? If it holds true that individual work at home, belonging to the private dimension of learning, has the benefit of letting students feel free to make mistakes, to express their doubts and even to be creative (that is, to be in a playful state), and hence has the potential of enhancing reflection, internalisation, visualisation, and the (re)creation of mathematical meanings (Fried & Amit, 2003), this article highlights that the potential benefits of homework might be hindered by the risk of damaging motivation not only towards a specific task, but generally with respect to the overarching topic. The episode under study here suggests that, in order to avoid the risk of damage without discarding the unquestionable benefits of homework to understanding, fluency and reflection, teachers should make sure to attune to students' feelings and motivations in approaching homework. It seems necessary to embed affective competences in teachers' professional development and education programs, so that they can exploit the potential of effective attunement with their students during their lessons and also in distance learning.

Acknowledgements

We thank the teacher, anonymised as Betsy, and the students, for welcoming us in their class during the lessons. The research has been partly funded by the Italian National Project Lauree Scientifiche 2019 and partly by Fondi di Ateneo per la Ricerca (FAR 2017) at the University of Eastern Piedmont (Italy).

Notes

[1] The formula to compute the percentage return r is the following: $r = (S + D - B)/B$, where S denotes the selling price, D the dividends and B the buying price of a share. This formula was introduced in a previous lesson.

[2] In OECD (2013), two of the cognitive processes within Financial Literacy are: "to identify financial information" and "to analyse information in financial context" (p. 152). Both processes focus on students' ability to search, recognise, and interpret information from different representations. In one of the items (see Example 6 in OECD, 2013, p. 152), there is a problem similar to the exercise Betsy assigned to her students. There is a line graph describing the price of a share over a 12-month period and two true-or-false questions about when is the best time to buy the share and what is the increase of the price over the year. OECD (2013) reports that only one-half of respondents correctly answered both questions.

[3] To recall, the formula emerged from previous class activities in previous lessons.

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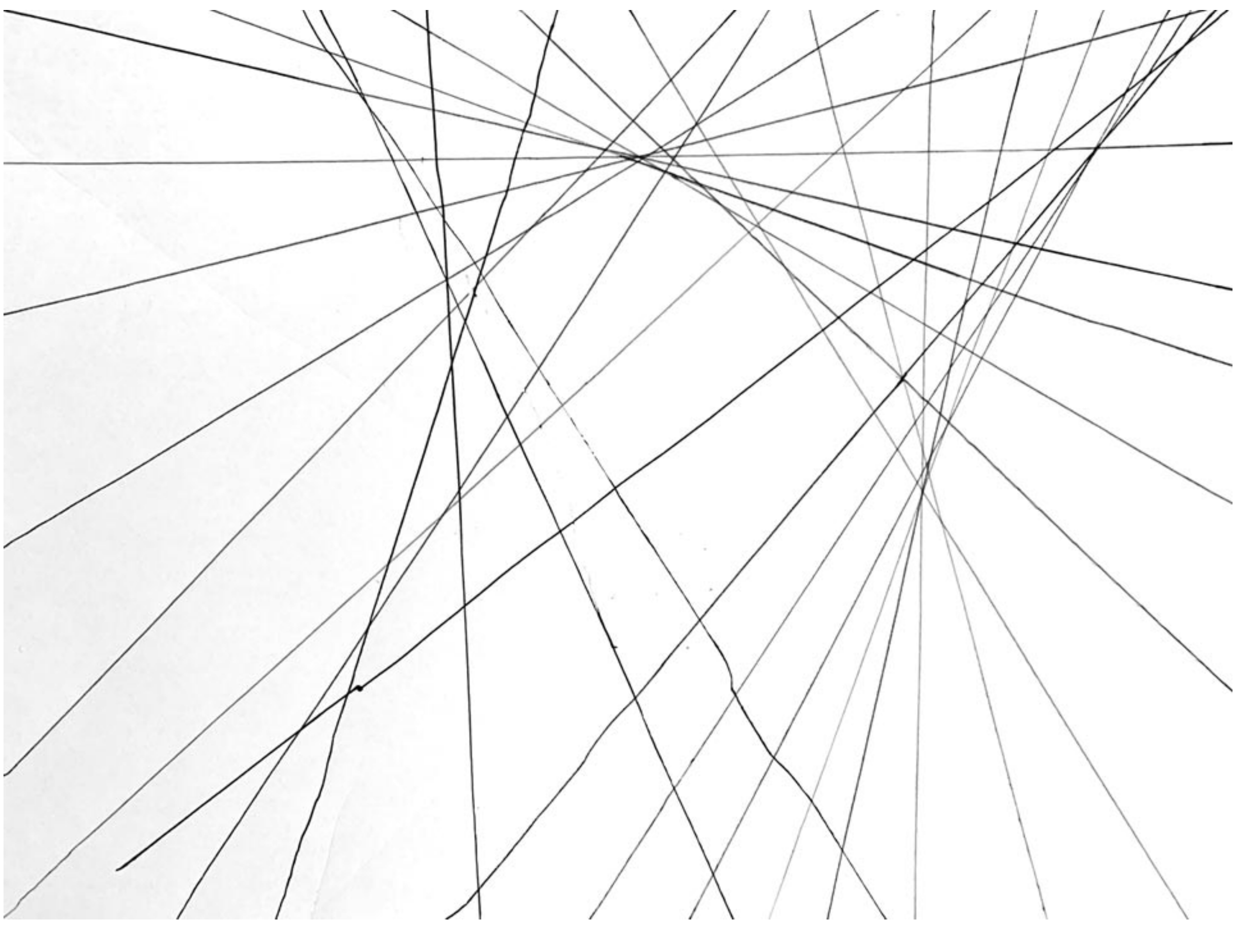

Drawn by Iris, age 12, on being asked to "Draw something mathematical."