Communications

Does Practice Make Perfect?

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熟能生巧, corresponding to the English proverb *Practice makes perfect*, is an ancient Chinese idiom Many teachers in China as well as in East Asia believe it and consider it to be a general principle for all kinds of learning: through imitation and practice, again and again, people will become highly skilled. Is practice an effective way of teaching and learning, especially for mathematics education? In fact, I cannot answer simply 'yes' or 'no' to this question

On the one hand, from the experience that excellent students are diligent in their practice and the fact that students from East Asia often top the list in several international assessments of mathematics education and mathematics competitions, these achievements could be attributed to this traditional way: a large amount of routine practice, problem solving and frequent tests. On the other, it is an obvious shortcoming that students are burdened with too much manipulative practice Mathematics teaching sometimes appears dull and boring, so that some students have no interest in mathematics learning I acknowledge that learning mathematics cannot be equated with learning handicraft. *Perfect* in mathematics learning should mean understanding: what remains to be explained appropriately is whether or not training in manipulative practice promotes understanding

Most mathematics educators in the West think that understanding is the most important goal. They adopt an attitude of negating drill and practice and regard imitative practice as purely behavioural manipulation, resulting in mechanical memorising through repetition This can be seen as Western scholars' basic point of view The issue of practice first versus understanding first is nearly the same dilemma as the question of which came first, the chicken or the egg Amongst the varied literature on this topic, I note what Bruner (1978) wrote in his book *The Process of Education*:

> It has been the experience of members of the School Mathematics Study Group that computational practice may be a necessary step toward understanding conceptual ideas in mathematics (p 29)

It is meaningful to reflect on this traditional method of mathematics teaching and explore what possibilities it offers for developing understanding.

Manipulative activity and reflection: the form of learning

It is now widely accepted that teaching mathematics means teaching through mathematical activities. Activity or action, in other words, involves thinking, doing and physical or mental manipulation. Why is activity necessary? Because the form of activity has profound implications and performs particular functions for cognition.

First of all, mathematics learning involves (quasi-) empirical actions (Lakatos, 1976) Manipulative practice is a fundamental action for mathematics. No matter how students are to learn, whether in the traditional way with pencil and paper, or, in a more modern way, with the aid of a computer, mathematics is not learned by a wild flight of fancy. Although people sometimes may have a sudden inspiration in problem solving, it is still dependent on cumulating experience. Students, the same as mathematicians, will do mathematics themselves: knowledge is acquired through practical activities. In some aspects, the practicality of mathematics is not completely the same as that of experimental sciences such as physics, chemistry, etc. However, behavioral or mental operation is still needed in mathematical activity.

Most mathematical actions are manipulations of mathematical objects. Look at the origin of number. Where do numbers come from? They stem from a kind of behavioral action - counting a set of marbles or sweets. The laws of arithmetic operations are generalized from operations in practical computational activities or real life, which implies that a mathematical concept would be a tree without roots or a river without a source if there were no actual or mental manipulation

Nevertheless, mathematical concepts do not come directly from fact. They are not abstracted from actual things, but are a product of coherent actions on things. So a mathematical object is actually a particular object, a so-called mental object For instance, the concept of addition is not produced from more marbles but from the process of adding or combining. The limit of a sequence does not arise from numbers themselves, but from a special developing trend underlying these terms. For this reason, the concepts of addition and limit are concepts built through manipulative processes. These concepts were created by mathematicians in earlier times. It is a re-creative activity for students when they repeat mathematicians' procedures in similar situations: the empirical processes are necessary. As the constructivist von Glasersfeld (1989, p 162) claimed: "Knowledge is not passively received but actively built by the cognizing subject."

This means that mathematical concepts are built through human activity. For example, 3 plus 6 may involve a counting procedure from 1 to 3, then going forward from 4 to 9 Later on the process from 1 to 3 can be omitted Counting need begin only from 4 to 9. When these procedures are repeated several times, they could become condensed and the concept of addition is formed (cf Gray and Tall, 1994).

What psychological mechanism is needed by students in this course of concept formation? The basis of the organizing concept is provided by experience but the concept itself is not provided To construct a concept, a more important thing is a leap in thinking, i e. reflective abstraction (Piaget, 1971). From Piaget's point of view, after doing something people can change to be a spectator, to place what oneself has done as the object of thinking In reflection, the process of counting itself becomes the object of thinking and conservation is found from counting in different orders. The concept of the class of equivalent classes is then formed. From processes such as $3+6 = 6+3$, $1+7 = 7+1$, etc., the commutative law of addition is generalized. There are a great many such mental behaviors although there are some exceptions. The formation of almost every mathematical concept involves reflective abstraction. In this kind of situation, the basis of reflection is manipulative activities.

Without manipulation, the subsequent reflection cannot be put into effect. And if there were not a sufficiently strong background of manipulation, many contexts and properties would only be viewed as accidental phenomena that would not enable students to discover precise conclusions. Therefore students' manipulative practice in their learning will lay a foundation for their reflective abstraction. Moreover, these activities must be of their personal experience. Students should be involved in practical activities, organizing situations, sending messages and constructing their understanding. Even to see someone else's doing, s/he must perform it her/himself and make sense of her/his manipulation. Nothing can take the place of her/his own thinking. One of the functions of routine practice that we stress is to urge students to participate in their activities and let them learn swimming by swimming. In short, practice makes *perfect*; perfect will be formed on the solid basis of practice. If there were no fundamental activities, reflective abstraction would be a castle in the air.

The procedure of concept formation: the content of learning

The form of construction of mathematics is empirical activity and reflective abstraction. However, compared with the content of what is done, the form is, after all, only secondary. Explicitly, the substance of understanding lies in the content of the manipulation. Understanding the nature of understanding should focus our attention mainly on the developmental procedure of concepts. Is manipulation beneficial to concept formation? Does practice facilitate comprehension? The course of concept formation will be probed, based on Sfard's (1991) theoretical framework about the duality in the majority of mathematical concepts.

As Sfard and Linchevski (1994) have pointed out, in mathematics, especially in algebra, a lot of concepts are both operational processes and structural objects. So a concept is of dual nature, different sides of the same coin. In application, students need to change their sights flexibly. A concept is sometimes viewed as a set of operational steps, and at others considered as a whole object depending on the context. For instance, the trigonometric function $\cos(A)$ can be regarded as the ratio of the adjacent side to the hypotenuse x/r of an acute angle A in a right triangle. It may also be seen as the result of the computation of the division or a ratio, a fraction. The binomial expression $5(x + a) - 7y$ is both a series of operational processes and a structural object involving 5, x, a, 7, y in certain particular relations.

In the latter case, what is to be emphasized is not the operation, but the result of the operation, a permanent thing that cannot be reduced further. Even the small equals sign, which is seemingly the simplest and is not paid more attention, is also of dual nature (Kieran, 1981). In some settings, it is an instruction that indicates you are to carry out the computation. So when a young pupil is confronted with a problem, s/he may write down a sign of equality following closely to the expression before s/he begin her/his thinking. But in the case of an equation, the sign of equality represents an equal relation between the left- and right-hand sides. Thus, when solving an equation, say, $a(x\text{-}3) = 1$, s/he does the same, writing a sign of equality immediately after the expression. The solution $a(x\text{-}3) = 1 = ax - 3a = 1 = \ldots$ suggests that he/she does not yet treat the equals sign as an object.

In the course of teaching calculus, we stress very much at the very beginning that limit is a process, a developing trend of a sequence $\{a_n\}$ when n tends to infinity. In fact, limit is eventually adopted as a 'thing' and becomes an object to be operated on by other operations. Just consider the law, $\lim (a_n \cdot b_n) = \lim a_n \cdot \lim b_n$. If we still take $\lim a_n$, $\lim b_n$ as approaching processes, how are we to carry out the operation $(\lim a_n) \cdot (\lim b_n)$? Apparently the limits here are simplified to be entities, i.e. objects.

According to the further investigations of Sfard and Linchevski, there is an intimate dependence between process and object. The cognition of a concept usually begins with process, then shifts to object. They co-exist as a whole in the mind and play different roles in appropriate contexts. For the concept of function, students are first given several values of the independent variable to find the corresponding values of function. Later, it is to be an object with three aspects: domain, range and the corresponding rule. In the teaching and learning of the concept of limit, it is usual that students look for δ's from ϵ's according to the expression, and turn to see it as an object afterwards.

It is similar with concepts such as displacement, mapping, homomorphism or isomorphism, etc. A concept in the process phase is a sequence of steps or algorithms that is manipulable, easy to follow and comparatively intuitive. But because of its sequential order, the details in every step and its dynamic nature, it is hard to grasp the essentials. Whenever a concept develops into the object phase, it is represented in a statically structural relation, a permanent object, one that is easy to seize in its substance. Only at this moment has understanding been accomplished entirely.

The duality of mathematical concepts brings with it the duality of mathematical thinking and comprehension. The finding of Sfard that operational processes would precede the structural object is in line with humankind's epistemological pattern. Many examples in the history of mathematical development confirm the pattern, for instance, the three representative definitions of function.

Metaphorically, a mathematical concept is like a large, smooth ball which is difficult to manage. It is needed for students to have some place to touch. The process of concept is a possible gap. From here students could approach the concept. They will know the inherent relation through manipulating process and open the door to their successive learning. Routine practice gives a starting point for concept cognition.

The idea that has been analyzed above is only a cross-section of personal cognition. Broadly speaking, a concept is one of the links of a chain of concepts. Let us look at the notion of function again. The process of function involves manipulating variables. As a structure, it controls the relation between the independent and dependent variables, and becomes an object. At a higher level, function will be operated on by adding and multiplying, be differentiated and integrated. From these operations, some higher level concepts are formed. Therefore the concept as an object will play a pivotal role between a certain level and a higher level It manipulates some objects at a certain level and is manipulated by the processes of other concepts at higher levels

It is worth noticing that there is a cognitive crux between these levels In fact, it is a *strange loop*, or may even be called a *vicious circle* (Sfard, 1991), in the reification of the concept On the one hand, if a concept is not operated on by processes at higher level, it will not seem necessary to be reified and it will not become an object On the other, if what is to be operated on is not an object, these operations will be operations without objects. At this point, thinking would be blocked, since the well-defined rules of operation have lost their meaning. Under the conditions in which students are forced to do these operations, the rules seem as those without reason. It looks as if mathematics is only a symbolic game

However, another possibility may be assumed Students could attempt to explore underlying meanings through their imitative practice and find opportunities for insight into the process, making the object explicit from mechanical manipulation When a pupil in primary school understands the concept of multiplication as repeating addition, s/he will not know what 'x' means in 2.56 x 3.87. In this situation, the computation of decimal multiplication is just a purely formal manipulation for her/him. S/he can follow the rules but does not make sense of it But some chances for comprehension are provided by these operations. It will maintain learning and prevent it from stopping

> If a learning process is to be observed, the moments that count are its discontinuities, the jumps in the learning process (Freudenthal 1978, p 78)

Teaching should give students some ways to bridge the gap and free them from this strange loop. Mathematical algorithms or processes are indeed a really driving force. They encourage students thinking and push their learning forward If we insist that students should not undertake routine practice and solve problems until they have understood the related concepts, they may lose any opportunities to be engaged in learning. In short, the operations at a higher level have a reaction on the formation of a concept at a lower level. It brings a bird's eye view to look at a concept from a higher standpoint so as to promote the reification of the object. The function of the phenomenon of *earlierisen operation*, namely operating beyond one's understanding, may be what we customarily mean Whenever you do not understand a concept immediately, follow the process and practice it, familiarize yourself with it. Then you could achieve better understanding gradually through practice

Concluding remarks

Briefly, the discussion above reveals that the mechanism of routine practice and problem solving that are used as a method of teaching and learning is not simply interpreted as a way in which students only mechanically imitate and memorize rules and skills. Manipulative practice is the genetic place of mathematical thinking and the foundation of concept formation. It provides students with the prerequisites of comprehension In other words, the reasonableness and effectiveness of practice lies in its necessity. The positive role of *Practice makes perfect* is confirmed in this meaning. However, it implies at the same time that it is insufficient for concept formation to have proficient practice *only* Many other mechanisms of thinking are needed for a transition from basic activity to reflective abstraction and from process to object. They are worth being explored

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