LEARNING MATHEMATICS DOES NOT (NECESSARILY) MEAN CONSTRUCTING THE RIGHT KNOWLEDGE

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Let us eavesdrop on a classroom of eleven-year-olds. The teacher, Mr Lakatos, has the children trying to find the next number in a sequence. They are trying to find a pattern, to come up with a conjecture about how the sequence is created. They will then test their conjectures by predicting what the next number in the sequence would be, and Mr Lakatos will confirm or refute each guess, because he is currently acting as the keeper of the sequence-generating rule. The children all have calculators which they use regularly to do their maths, so the guesses they make about the sequence come in rapid fire succession.

Mr Lakatos: Can anyone guess what the next number in the sequence would be? How am I producing the sequence? 4, 16, 37, ...

Jamie: I think the next number is 1369.

Mr Lakatos: Anyone agree with Jamie's guess? Anyone want to challenge it?

Scott: That can't work, because 16 squared isn't 37.

[It seems that Scott is assuming that Jamie (a) looked at only the first two numbers of the sequence, (b) decided that squaring would get the next number, and hence (c) squared 37 to get the answer 1369. So Scott provides Jamie with a counter-example to what he assumes is Jamie's hypothesis.]

Mr Lakatos: Any reaction, Jamie?

Jamie: Not that I can say out loud!

[Titters of laughter are heard throughout the class.]

Scott: Is the next number 49?

[Scott seems to have focused on the difference between 16 and 4, namely 12, and the difference between 37 and 16, namely 21. He is perhaps guessing that these differences alternate. Hence, 37 + 12 = 49.]

Mr Lakatos: No!

Scott: Then how about 58?

[It seems that Scott is focusing on differences. Hence, 58 would be his next guess because 37 + 21 = 58.]

Mr Lakatos: Yes, that's right, Scott. And what would the number after 58 be?

[The sequence is now 4, 16, 37, 58, ...]

Susie: Well, the gaps now are 12, 21 and 21. Is the next number either, let's see, 70 or 79?

[Susie is not assuming that the difference will now stay at 21. She seems to be covering her bets, as it were, perhaps by reasoning that the pattern of differences might be 12, 21, 21, 12, or that it might be 12, 21, 21, 21.]

Mr Lakatos: No, neither of those is the next number. Should I give you the next number?

Jamie: Yeah, because I agree with Susie. I think it has to be either 70 or 79. What could it be if it isn't one of those?

Mr Lakatos: Well, it could be that the next number is 89, and it is!

[General puzzlement follows. The sequence so far is 4, 16, 37, 58, and 89. Susie, who had made a plausible case for the next number being 70 or 79, has had her pattern refuted.]

Jeff: Does it [the pattern of numbers in the sequence] have anything to do with the squaring of the numbers?

[Jeff, like Susie, is looking for patterns, not just the next correct number. The guessing of an actual number serves only to test the pattern. The pattern is the real guess, the real conjecture.]

Mr Lakatos: Perhaps!

[Mr Lakatos is not too helpful, or is he?]

Susie: Is the next number 120 or 102?

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	[Susie has many conjectures, but she is apparently remaining focused on differences between numbers in the sequence. In the case of 120, she seems to be guessing that the pattern of differences is 12, 21, 21, 31 and 31. In the case of 102, her hypothesizing appears to be 12, 21, 21, 31, 13.]
Mr Lakatos:	No, neither of those is the next number. Actually, the next number is 145. [1]
	[The sequence is now 4, 16, 37, 58, 89, 145, ...]

The children described here are now more than twenty years old. Though Mr Lakatos was not their real teacher, I chose to give the actual teacher that name because the sessions occurred about five years after the writings of Imre Lakatos first appeared (Lakatos, 1963-1964, 1976). It was from ideas contained in Lakatos' articles and book that an alternative way of working in mathematics classrooms developed. The lesson above was an early attempt to teach in this way, the so-called fallibilistic way of teaching, which has gradually gained recognition during the intervening two decades (Dawson, 1969, 1971; Lampert, 1990). The children above made conjectures, that is, reasoned guesses, and then subjected these to the test of (in this case) whether their conjectures accurately predict the next number in the sequence. Lakatos claimed that the creation of mathematics comes about as the result of a process of proofs and refutations, a process in which a conjecture is created, tested and proved, or refuted and modified, or rejected outright. A classroom environment designed to provide opportunities for pupils to operate in a fallibilistic fashion would provide pupils with a problem, a problem about which they could make conjectures as to its solution. The pupils must be allowed to guess the solution and to evaluate their proposed solutions. Opportunities to test and examine critically each conjecture must also be provided. This last provision does raise the issue of how and when guessing is a valid strategy in mathematics classrooms.

Some teachers have worried, for example, that the pupils' "reasoned" guesses may actually be nothing more than thinly veiled, wild guesses, and that to let pupils guess in this way would be disruptive of the classroom situation, thereby making management difficult. However, this need not be the case, if the pupils are provided with the means to test their guesses. If testing procedures are available to the pupils, they are not as likely to guess in a vacuum, as it were, but will be more likely to make guesses appropriate to the testing procedure. Consequently, the testing procedure acts as a control mechanism which does not require the teacher to be the classroom disciplinarian. By the way, it is important to note that a guess is usually called "wild" if it fails; if it succeeds, it is called "daring". Hence, as teachers, we must be cautious in labelling a wrong guess as being "wild". The pupil making the guess may have had strong grounds for making that particular guess, such as Susie had when making the guesses of 70 and 79 above, and it would be a great disservice to the pupil to be criticized unjustly for making what outwardly appeared to be a wild guess, but which actually was not. It is important to elicit reasons, that is, for pupils to offer justifications for their claims.

The fallibilistically-oriented teacher's broad curricular and instructional viewpoint is one which is concerned with pupils learning how to learn rather than with pupils learning specific material. The end result of any mathematics course should not be, according to this view, the acquisition by pupils of the ability to memorize and produce on demand a great number of facts, results, definitions and theorems. Rather, pupils should develop the skills and attitudes for attacking problems in a rational and critical fashion. A teacher who is functioning fallibilistically, like Mr Lakatos was above, establishes a classroom climate in which an atmosphere of guessing and testing prevails, where the guesses are subjected to severe testing on a cognitive rather than an affective level, and where the pupils' goal is to expand their awareness of mathematical relationships and of themselves in a situation where knowledge is treated as being provisional. Because of the provisional nature of knowledge, pupils are encouraged to confront the mathematics, their peer group, and, where appropriate mathematically, even their teacher. In this climate, there are no wrong answers, only refuted conjectures, a subtle but crucial distinction. As we saw with Mr Lakatos, the fallibilistic teacher confirms or refutes guesses, but as Ms Watt will demonstrate below, the teacher can also provide counterexamples to conjectures which pupils feel with great confidence they have already proven!

Ms Watt is working with a group of fourteen-year-old girls who are sitting around a table on which sits one computer. These pupils started using Logo a couple of weeks previously. They have made squares, rectangles and triangles. They know how to use the REPEAT command, so that the instructions of how to draw a triangle, say, can be written in one line. However, the pupils have not yet learned how to write procedures. They are working in immediate mode: when they type an instruction and hit the return key, the turtle performs it immediately, tracing the path it was directed to by the instruction the pupils typed into the computer. The pupils have been exploring the question: when the turtle goes on a trip, how much turning must it do in order to come back to its original starting position? [2]

They have just drawn an equilateral triangle using the command: REPEAT 3 [FORWARD 50 RIGHT 120]. It is at that point we pick up the conversation Ms Watt is having with the girls, who have adopted the group name of the Gang of Five.

Ms Watt:	So, Gang of Five, how much did the turtle turn to get back where it started from?
Janice:	It turned a total of 360°—three times 120°.
Ms Watt:	Anyone want to challenge Janice's conclusion?
Sherry:	No, I agree with it. The turtle turns a full circle, so it has to be 360°. It doesn't matter which way the turtle is pointing when it starts, it still has to turn a full circle to

get back to where it started from.

Ms Watt: Okay! If it turns 120° at each corner, what is the angle inside the triangle? Not how much the turtle turns, but the size of the angle which is made when the turtle turns and makes a new side for the triangle.

Deidre: I don't get it! What are you asking?

Ms Watt: Well, the turtle is heading in one direction for 50 steps, and then it turns 120°, and heads off in this new direction. Type this instruction for the turtle, Deidre: FORWARD 50, RIGHT 120, FORWARD 50. What do you see?

Deidre: Oh, I see. It's an angle.

Ms Watt: And what would the size of that angle be?

Deidre: Let's see. The turtle heads one way, and almost turns completely around in the opposite direction. It turns 120°, so what's left?

Sue: Wouldn't it be 60°?

Deidre: Why is that?

Sue: Because if it turned completely back in the opposite direction, it would have turned 180°. But since it only turned 120°, that must leave 180° – 120°, which is 60°.

Deidre: Right! I get it now.

Ms Watt: So if we added up the three angles of the triangle, what would be the sum?

Jean: That's easy! It's three times 60°. That's 180°.

Ms Watt: Do you think that would be true of all triangles: that the sum of the interior angles of a triangle is 180°?

Janice: It might be true just for the triangles that have all the same angles and the same length of sides.

Sherry: No, it can too be true for all triangles. Look, I can prove it!

Ms Watt: Show us, Sherry.

[Sherry draws a collection of different triangles on the chalkboard. Using a pointer placed along one side of a triangle, she shows how the pointer makes a complete rotation as she moves it around the sides of the triangle. Sherry does this for several of the triangles she has drawn, to the point that the other members of the Gang of Five tell her that enough is enough!]

Janice: So it doesn't matter what size the angles are—so long as they are less than 180°, the sum will still be 180°. Brill!

Ms Watt: Yes, it would seem so, wouldn't it? But imagine this. Suppose the turtle were sitting at the North Pole, and it headed off towards the Equator travelling along a line of longitude. Deidre, get the globe, will you please, and use Sherry's pointer as the turtle.

[Deidre gets the globe, places the pointer at the North Pole directed down one of the meridians. She slides the pointer down to the Equator.]

Ms Watt: Good, now turn the pointer to the right so that it is lying along the Equator. How much of a turn is that?

Sue: It would have to be 90°, wouldn't it? Doesn't the Equator and any line of longitude make a right angle?

Ms Watt: Do you all agree? [Heads nod in agreement.]

Ms Watt: Okay, Deidre, slide the pointer along the Equator, and then turn right back up a different line of longitude.

Deidre: That would be another 90° turn?

Jean: I don't like the look of this!

Ms Watt: Wait a minute, Jean. Now, Deidre, slide the pointer back up to the North Pole. When you get there, turn it right again so that it is pointing back down the meridian you started down before.

Janice: But she has traced out a triangle ...

Jean: Yes, and that triangle has two right angles, two angles of 90°, and another angle of, I don't know, it looks about 30°. So the sum of this triangle's interior angles is, uh, 210°, and not 180°. I don't get this. I thought Sherry's rule worked for all triangles.

Deidre: But the path I traced is not really a triangle, is it? I mean, the globe is curved.

Ms Watt: But didn't the path you traced have three sides, and three angles, and didn't you get back to where you started from?

Deidre: Well, yes, but if that's true, then the total turtle trip must have been greater than 360°. But I thought you said that it was always 360°, so long as the turtle got back to where it started from, that it closed its path?

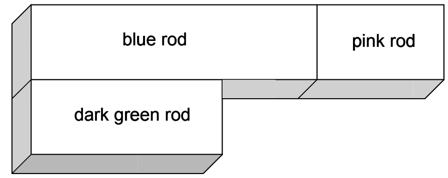
Ms Watt: Oh, yes, well Gang of Five, that is a problem, isn't it.

Central to Lakatos' view of the growth of mathematical knowledge was his contention that there are no immutable truths in mathematics, and that even mathematical proofs can be challenged, and counter-examples found for valid, deductively derived results. Sometimes, these counter-examples arise because of a change of focus, a different realm of application, such as was the case above when Ms Watt introduced a non-Euclidean triangle. She is attempting to expand the pupils' conception of triangles, and acquaint them with spherical geometry. The pupils are excited and animated in response to Ms Watt's ploy and seek to discredit the situation she has put forward as a purported example. On the other hand, they also seem to realize that the path which Deidre followed does, in fact, generate a triangle in some sense, and hence they have a dilemma. In his work on the foundations of mathematics, Lakatos cites many instances where the growth of mathematical knowledge was fostered by mathematicians coming up against just such dilemmas. The behaviour of the Gang of Five, then, is very mathematical in nature, and totally consistent with how mathematicians challenge and expand their own mathematical understandings.

Teaching mathematics in a fallibilistic way is, therefore, derived from the contention that there are no immutable truths in mathematics. Consequently, pupils learn how to investigate unknown terrain in mathematics, not with the goal of finding truth, but with the desire of obtaining an ever-improved map of the terrain. In attempting to chart the unknown terrain, pupils may be guided by their teacher, but they are not given the complete map. Indeed, from a fallibilistic orientation, such a map is not available, not even to the teacher. A teacher working in this Lakatosian tradition allows pupils to create, revise and expand their own mathematical maps. This teacher aids pupils by putting forth conjectures and counter-examples which focus their attention on specific points of the map. This is contrary to what happens all too often in maths lessons where the teacher attempts to give a complete map of the mathematical terrain to all pupils, and then gets upset when the pupils do not "learn it". Each pupil's map of the mathematical terrain is incomplete and basically idiosyncratic. After all, any map is incomplete, and is not and cannot be the terrain! Learning mathematics does not mean constructing the right knowledge, as no one is sure what that knowledge is.

Finally, join me in watching as a visitor, Dr G, to a primary school classroom works with a group of children seated around a table (Gattegno, 1970, 1974). There are three nine-year-old boys (Marty, Arthur and Dick), and two ten-year-old girls (Janet and Cecelia). Dr G has told the classroom teacher that he wants to try something out with the children; namely, he wants to see if they can find the difference between two eighteen-digit numbers, if they can do this entirely as a mental calculation, and if they can write the answer down from left to right and not right to left as is usually done. The lesson begins with the children using Cuisenaire rods to make trains as per instructions given by Dr G. Let us listen in on the ensuing conversation.

Dr G: Make one train composed of a pink rod and a blue rod, and a second train of just a dark green rod, like this.



Dr G: Which rod would fit onto the end of the dark green train, so that the two trains become the same length?

Marty: Well, it sure ain't the white rod! Nor is it the orange rod.

Cecelia: Be serious, Marty! I think it is the brown rod.

Marty: I am being serious! What I said is true, isn't it? Anyway, I don't think the brown rod will work. It's too long—try it!

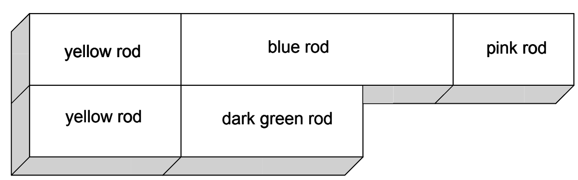
[Cecelia tries the brown rod, and finds that indeed Marty is right, the brown rod is too long.]

Janet: The black rod will fit exactly. It is a white rod shorter than the brown one, and that is how much too long the brown rod was.

[Janet tries it, and the black rod is, in fact, the one which fits.]

Dr G: Now watch what I am going to do.

[He places a yellow rod on the end of each train, so the trains now look like this.]



Dr G: Which rod would fit onto the end of the yellow plus dark green train, so that the two trains become the same length?

Dick: It hasn't changed. It will still be the black rod.

Arthur: That's right. You can keep adding as many rods as you want to the left end of the two trains, so long as they are both the same colour, and it'll still be the black rod which makes the shorter train the same length as the longer train.

Dr G: If that is true of trains and rods, is it also true of numbers?

Janet: I don't understand what you mean?

Dr G: Pick two numbers. Find the difference between your numbers. Keep that in your mind. Have you got it? [There are nods of agreement from the children.] Now, add seven to each of the two numbers you started with, and find the difference between these new numbers. Is this difference the same as the one previously?

Janet: Yes, right! I get it. Do you get it, Marty?

Dr G: What if we were taking coaches off the trains instead of adding them on? Would the difference in the length of the two trains still be the same?

Cecelia: Yes, that wouldn't make any difference to the difference! And the same would be true for numbers.

Janet: Oh, I get it. Look, I can write something like this.

24– subtract 5 gives 19– subtract 7 gives 12–
20 subtract 5 gives 15 subtract 7 gives 8

In all cases, 24 – 20, 19 – 15, 12 – 8, the difference between the two numbers is 4.

Dr G: Look at these differences.

40– → 39 –
26 → 25
14 14

400– → 399–
268 → 267
132 132

4000– → 3999–
2674 → 2673
1326 1326

Marty: So?

Dr G: So, can you subtract 678456234123 from 800000000000 in your head and write the answer down from left to right?

Dick: Eh! That's too hard! You'd have to do all that borrowing, and you have to start on the right, not the left.

Arthur: Wait a minute! I'm not so sure. Let me write it down.

800000000000–
678456234123

If we subtract one from each of these numbers, we can write them as

799999999999–
678456234122

and hence we can write down the answer from left to right.

121543765877

Janet: So, we could do it without writing it down. Just think of it as subtracting one from each number. All the zeros change to nines, except the leftmost digit of the top number. And the rightmost digit of the bottom number decreases by one. Now, we subtract the bottom numbers from all those nines … it's easy!

Dick: Yeah, right, so if we start with something like five followed by a bunch of zeros, and subtract 234345456567678789, we'd get…

Cecelia, Marty: (together) 265654543432321210. Hooray! We couldn't do that on our calculators!

Janet: But what happens if the one number isn't composed of all zeros? Would we still be able to find the difference and write it down in this way?

Dr G: Try it and see, Janet. Can you find a counter-example which would refute the pattern we have here?

Janet is fallibilistic to the end, is she not? Of course, Dr G has led the pupils along a particular path, in a quasi-deductive fashion in order to lead them along to a certain conclusion. In a Lakatosian sense, the result obtained in this example is deductively generated; that is, the pupils made a series of observations and drew conclusions from them with the result that they could eventually generalize the pattern which allowed them to find the difference successfully between the two eighteen-digit numbers. Though one may not wish to call such a derivation a proof, it certainly falls into the classification of convincing argument. (For further discussion of what thirteen-year-old pupils think proofs are, see Balacheff, 1988.) However, Janet is ever ready to challenge this result, and Dr G encourages her to attempt to find a counter-example. This epitomizes fallibilistic teaching writ large!

So I invite you to try a few fallibilistic lessons of your own. The sources listed in the references all give examples of mathematical topics which can be approached fallibilistically. Try being Mr Lakatos, or Ms Watt, or Dr G, and see whether the pupils in your mathematics class become as excited and animated as the children described above.

And, oh yes, do try to find the difference between a couple of twenty-digit numbers tonight before you doze off. Do it in your head, and when you get up in the morning, write the answer down from left to right! Or if you prefer, take a flight on a globe like Deidre's, but start on the Equator. Can you fly around a rectangle? Start by flying north on a line of longitude, but stop short of the North Pole. Make a 90° right turn onto another great circle. Fly on again and then make another 90° right turn onto a third great circle and fly on. Can you make a 90° turn at some point and still be heading home, back to where you started? If so, study the fourth angle you get on returning to your starting-point. Is it also 90°? Or is it less than 90°? Or greater than 90°? And what is the sum of the interior angles of this quadrilateral?

Sweet dreams!

Acknowledgement

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Notes

[1] For those readers who have not already guessed the pattern which creates the sequence, the complete sequence is 4, 16, 37, 58, 89, 145, 42, 20, 4, and then the sequence repeats. Each term is obtained by summing the square of the digits of the previous number; thus 145 produces 42 because 1 + 16 + 25 = 42.

[2] This question gives rise to the "total turtle trip theorem", or the enclosed path theorem, which states that the total amount of turning the turtle does in travelling along a closed path back to its original starting-point is a multiple of 360°.

[3] This imagery is explored further in the chapter entitled "The keeper of the map" by Geoffrey Faux (1991) in the original volume in which Dawson's article appeared.

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Having thrown out the old 'logic' and 'epistemology' sections from the catalogue of our intellectual library, how are we to set about replacing the scattered volumes in a new and more practical arrangement? The full answer would be a very long affair; but some general remarks can be made here about the principles which will govern any re-ordering. Three things especially need remarking on:

(i) the need for a *rapprochement* between logic and epistemology, which will become not two subjects but one only;

(ii) the importance in logic of the comparative method—treating arguments in all fields as of equal interest and propriety, and so comparing and contrasting their structures without any suggestion that arguments in one field are 'superior' to those in another; and

(iii) the reintroduction of historical, empirical and even—in a sense—anthropological considerations into the subject which philosophers had prided themselves on purifying, more than all other branches of philosophy, of any but *a priori* arguments.

Toulmin, S. E. (2003) *The Uses of Argument* (2nd edition), p. 234. Cambridge, UK: Cambridge University Press.
