Historical Stories in the Mathematics Classroom

LUTZ FÜHRER

Some years ago when I was co-editor of the German periodical *Mathematiklehren* we decided to prepare an issue on history of mathematics. To secure the interest of our usual readership, we planned to elaborate a few hints, recommendations, or suggestions on how to include historical items in the ordinary teaching of mathematics in school. It soon became clear, however, that this methodological problem would be the critical point of our project. It turned out, as I had feared from the beginning, that only a very few of my colleagues could be talked into writing about this feature of the subject It was easy to get papers on historical facts or on hypothetical developments, but everyone found it hard to write about practical ways of teaching history in the mathematics classroom.

It can quickly be seen from a survey of the literature that this problem is a general one. You can easily find many papers and books on the history of mathematics, but you will scarcely find so much as a hint on how to integrate the material into your teaching of mathematics at school. Why is this, apparently, so difficult or unheard-of?

I think the main causes are easy to conjecture. There is a widespread uncertainty over historical knowledge. Many people enjoy historical novels, essays or anecdotes, but most of them hesitate to bring that into the open Teachers use their acquaintance with some features of the mathematical past to tell amusing stories by the way, but few such stories meet the general standards of historical science. Maybe that is the main reason why it is hard to find challenging stories in the literature of mathematical history.

And maybe that in turn leads most of our colleagues to abstain from attending to the historical aspects of our subject. From my experience, it is very easy to convince most mathematics teachers that historical insights into mathematics are essential for grasping the human meaning of school mathematics. They would be quite ready to include historical aspects in their lessons, provided someone would help them overcome their feelings of precariousness concerning the uncertainty of history and the methodological problems of teaching it seriously. In fact there seem to be three hindrances to the acceptance of historical approaches in the ordinary mathematics classroom.

— the methodological contrast between mathematics and history;
— the low level of mathematics teachers' historical knowledge, connected with the general uncertainty of historical truth; and
— teachers' memories of their own school history lessons, comprising a boring search for sources, for dark ages, and strange events of no visible actual concern

As far as the methodological contrast is concerned, I think we suffer from the emergence of self-styled scientific didactics and its fight to become established The tone of official discussion about educational matters has become objectivistic; subjective views are out, and the teacher's personality has become an object of research rather than a basis for the art of education. The technological dimension of instruction and educational research has fostered the retreat of mathematics teachers behind the closet doors of true and sure mathematics. As the didactical requirements on the teacher have grown, so opportunities for interdisciplinary teaching have lessened. This seems to provide one reason for the diminution, at least in Germany, of historical literature for the mathematics classroom.

Each of the hindrances I mentioned seems to be closely connected to the fashion of discrediting dilettantism and amateurism among teachers, and the promotion of "professionalism". This is all very well but should not be at the expense of forgetting that full-blooded and stimulating lessons come alive from the uniqueness of the teacher's personality, from his or her spirit and courage to leave the safety zone of professional learning, and from a willingness to discover new aspects of subjects. Personal authority arises much more from the teacher's readiness to learn and adapt by trial and error than from professional scholarship Of course these educational principles are hardly new— they look conservative, and in Germany have been discussed down to the last detail in the first third of our century. But they form an essential background to my demand for a restitution of non-scholastic attitudes by scholastic professionals.

To embark on thinking about the benefits which historical insights can bring into mathematics teaching presupposes that teaching mathematics is more than indicating the truth and training students in the use of algorithms. In fact the most difficult task of the mathematics teacher is to show that mathematics makes, or can make, sense. Further, that it has guided a non-negligible part of the human community to pass behind the curtain of the unknown, and that each mathematical perspective on reality contains several historical strands.

It cannot be the job of the mathematics teacher to idolize the standards of historical science. The teacher must not lie, but should free herself from the heavy burden of exactness. Historical sincerity should concentrate on conceivable, imaginable or at least possible evolutions The mathematics teacher should feel responsibility primarily to the realm *between* human society and the sciences. She is no scientist, and her job is not historical reasoning. The most challenging task is to show that mathematics matters and

For the Learning of Mathematics 11, 2 (June 1991)
FLM Publishing Association. White Rock British Columbia, Canada

display its concerns on the ancient fields of elementary mathematics.

To show this, a good story is much better than the boring truth, although there is no escape from telling small pieces of "the truth" in any case. The greatest sin against historical good taste is to be boring. History is too important to use it to discourage the students. And we cannot leave this important subject to the historians, who seem in general too anxious to be exact at the expense of being interesting. In the mathematics classroom we are not obliged to be historically exact but are allowed to steal from the accumulated stocks of knowledge called history—we cannot show that mathematics concerns everyone without showing how it grew in close connection with the emergence of the spirit of civilization.

It may seem risky to assert that a good (hi)story is better than boring truth. But the risk of perpetuating errors, misunderstandings or prejudices should not be over-estimated in the context of mathematics teaching. In the first place, there is no way to escape from this danger even if telling what we believe to be the complete truth. Secondly, there is little reason to believe that our students remember much in detail from any particular lesson. And finally, good stories have the overwhelmingly beneficial effect of opening the student's mind and attention towards studying mathematical details. The main purpose of historical stories in the mathematics classroom, it is important to emphasize, must be their promotion of a productive attitude towards the student's desire for understanding.

To illustrate what I mean, let me sketch three stories which I have used several times and in different connections in my lessons. They are not for mere amusement, and they should not leave only an air of perplexity at the strange habits of distant times. Their main purpose, like that of every good story, is to use outstanding events as a window on to the meaning and value of mathematics in general. I will point up my view of the significance of these stories, and for the sake of brevity merely hint at the further lessons which readers are invited to construct for themselves.

First story: the ancient history of earth measurement

About 2400 years ago, towards the end of the third century BC, Eratosthenes brought scientific geography into being. A learned, scholarly and ambitious man, librarian of the famous Museum at Alexandria, he is mainly remembered nowadays for his measurement of the circumference of the earth. The result he arrived at was, apparently, 250 000 stades. Several questions arise from his work:

— How did he measure the earth?
— What does his result mean, in terms that make sense to us?
— How precise was his measurement?
— Were there any significant consequences that may concern us?

As to the first question we are well-informed. It is told that Eratosthenes got the basic idea on a boring sea-trip, while watching the shadow of the mast on the rolling deck.

Eratosthenes knew that at Syene (which is called Aswan today, some distance to the south of Alexandria) the dim-day show of a vertical pole vanished at the time of the summer solstice, and a narrow vertical well was said to be illuminated by the mid-day sun to its very bottom. Eratosthenes' idea was to compare this observation with an analogous investigation of the sun's shadow at Alexandria at the same time.

With his *skaphe*, a hemisphere with circles inscribed to indicate the angle of the shadow cast be a vertical rod, he measured the angle of the shadow at Alexandria as the fiftieth part of a whole circle (or 7.2 degrees). To a mathematician of Eratosthenes' generation it was quite clear that the angle difference between the measurements at Syene and Alexandria could only be a consequence of their different latitudes—everyone was convinced by the arguments of Aristotle in the previous century that the earth had the shape of a sphere, and that the sun was sufficiently far away to be thought of as sending parallel rays of sunlight.

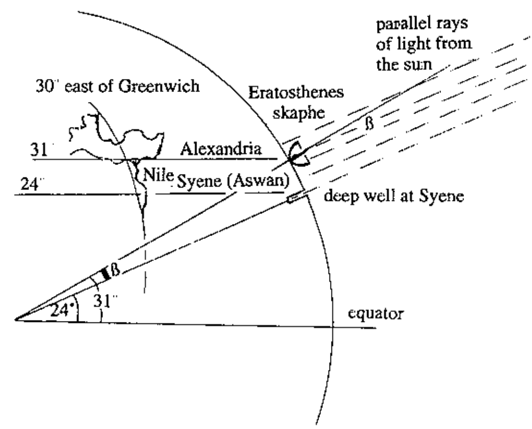

Figure 1

Eratosthenes measured the angle difference ß to be one fiftieth of the earth's circumference.

Thus, by elementary geometry, the measured angle at Alexandria is the altitude difference between the two towns as seen from the earth's centre. In other words, if you travel due south from Alexandria to Aswan's parallel of latitude (which is approximately the Tropic of Cancer), then you traverse one-fiftieth of the earth's circumference. It would be an easy exercise to derive the whole circumference from that observation if you had the correct value of the distance between Alexandria and the Tropic of Cancer. Eratosthenes took this to be 5000 stades, and thus was able to conclude that the earth's circumference is 250 000 stades.

This calculation can give rise to several further explorations, depending on the level of enthusiasm and interest among one's pupils. I mention some here, just as hints for further investigation.

1 The diameter of the sun subtends about half a degree, so there would be no exact shadow-boundary in the skaphe.
2 You can put a small ring on the top of the rod inside the skaphe, watching the evolution of the shadow from

beginning to end. What do you expect to happen, and with what precision?

3 Syene and Alexandria do not in fact lie on the same meridian (i.e., Syene is not due south of Alexandria): the modern coordinate data are that Aswan is 24° 5′ N, 32° 54′ E, and Alexandria is 31° 12′ N, 29° 51′ E. The latitude difference is 7° 7′, while the great circle difference is 7° 39′. How does that affect Eratosthenes' result?
4 Whatever Eratosthenes may have been told, the Tropic of Cancer did not reach Syene in those days (nor does it today), but was at latitude 23° 43′ 24″ N (and is further south, 23° 27′ N, today). In view of his distance value of 5000 stades, Eratosthenes' angle should have been one third of a degree less than 7.2 degrees.
5 How do you estimate the error tolerance of Eratosthenes' measurements with respect to the given data?

Even more instructive than the error analysis proves to be the problem of the meaning of Eratosthenes' result: what was the value of his unit of measurement? The classical definition of the stadion, which does not seem too helpful to us, was the distance a well-trained runner could run without taking new breath. But another source gives a value of 600 feet. This too is less than immediately informative, because several different feet were around. The Pythean foot of 24 cm, or the 29.5 cm foot of Hercules, or the Olympic foot of 31 cm, or the ancient Aeginean foot of 34 cm give rise to values of Eratosthenes' earth varying between 36 000 km and 51 000 km. All of these are at least of roughly the right order of magnitude, which is a tribute to Eratosthenes' skill—or perhaps his luck if several of his errors cancelled each other out.

From Freudenthal [1986] we learn that a dissertation at the Sorbonne sixty years ago showed not only that it seems impossible to determine exactly what value Eratosthenes used for the stadion, but also that the problem is entirely beside the point!—Eratosthenes himself must have rounded his result to 252 000 stades, possibly to obtain a number which is neatly divisible by 60.

The story could close at this stage, but there are two sequels which can be treated in class, depending as always on the pupils and what is to be achieved with them. Both stories take further the important theme of units, and emphasize for pupils through some interesting history the importance of being clear about what units are in use. The first sequel can be found in Resnikoff and Wells [1984], though it seems to be an old story [Fischer, 1975]. Posidonius of Rhodes admired Eratosthenes' work and repeated it. He took the distance between Alexandria and his university at Rhodes to be 5000 stades also, but measured instead of the sun the bright star Canopus, which was 7.5° above the horizon at Alexandria when on the horizon at Rhodes (which itself introduced some error because the refraction of light near the horizon was neglected in these measurements). Multiplying 360/7.5 = 48 by 5000 stades, he reached a value of 10 000 stades less than Eratosthenes, and then transformed this result into its equivalent in the official unit of measurement: 180 000 Persian stades. Later generations, notably the geographers Strabo and Ptolemy, lost sight of what the precise measurement units were. The upshot of several more such adventures of these figures was that a considerable underestimate of the size of the earth was in circulation when Columbus was arguing the case that India could be reached by sailing westwards. As I learned from Resnikoff and Wells, accurate scientific knowledge was not the right way to discover America.

The second sequel pursues the question of units in another direction. Rottländer [1979] and others [see also Press, 1959] have argued that artefacts of Babylonian, Indian, Egyptian and Celtic origin imply a world-wide standard, some five thousand years ago, of a cubit whose value was some 51.86 cm. Exploring the evidence for this, and possible reasons for it and modes of transmission, is a fruitful activity with the right class of pupils, in which much interesting mathematics (and statistical argument, too) can be learned.

In accounting for the transmission problem—how a standard can be accurately maintained and passed on across great distances in the conditions of five thousand years ago—Rottländer describes a hypothetical but challenging theory which is of great attraction to the mathematics teacher. Although the well-known Pythagorean connection between music and mathematics is often described as arising from Pythagoras' observations on the strung monochord, it seems more likely [van der Waerden, 1979] that a wide instrument such as Panpipes, flute, or a precursor of the organ was involved. Given the fact that, for example, a tone of pitch g has from time immemorial corresponded to an organ pipe of length one Roman cubit (which can be related to the earlier world-wide cubit), it seems not impossible that a standard length could have been transmitted across large areas by people with exact pitch establishing a network of pipes the same length.

The truth of such a conjecture is not the point, but the opportunity it offers the teacher: here there is the occasion for telling a story showing the interrelations of knowledge along the lines of imaginable history. Besides the connections with physics and with the story of Eratosthenes, the study of units is valuable for showing the intimate connection of measurements in the past with the human body, as can be seen in many examples from different cultures [see Winter], 1986].

Second story: ideas of π across two thousand years

In the German gymnasium it is usual to explain circle measurement in the tenth school year. Every teacher can choose the appropriate way to do this, but should at least introduce the idea of approximation in order to prepare for the calculus which comes later on. I have found the most inspiring elementary method for measuring the circle to be a mixture of ideas from Archimedes, Viète, and Descartes—namely, the continued bisection of arcs, noticing the corresponding coordinates in a Cartesian system.

The story this time is not one of activity with pupils, but rather how exploring mathematics for teaching can inspire the teacher to investigate his precursors and return refreshed to the classroom with a thrilling sense of the immanence of mathematicians of the past. There are many lighter stories about circle measurement in Beckmann [1977], which I highly commend to the attention of all fel-

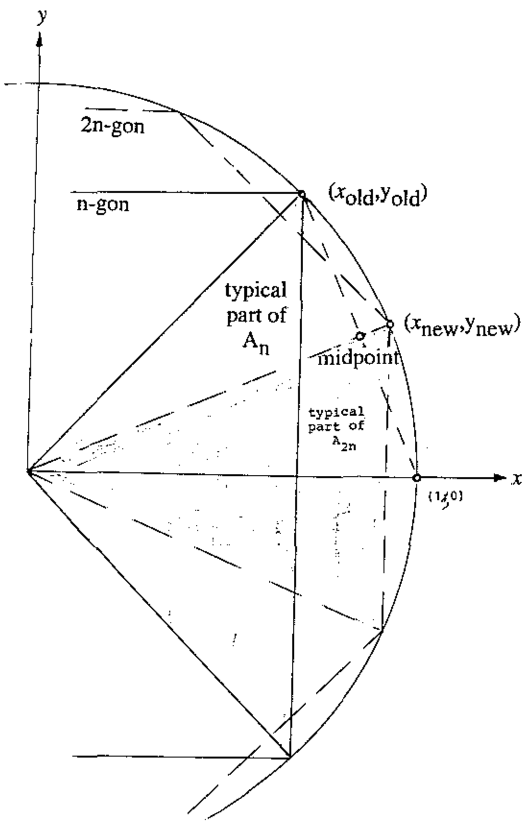

Figure 2

The method of continued bisection within the unit circle (Archimedes and Viète)

low-teachers, and the classroom use of Archimedes' text itself is well described in Bühler [1990]

The common point of Knorr's and our story was the moment we grasped that we had thought along the same lines as Huygens and Archimedes had done before. We have learned from that that good ideas seem to have a touch of eternity, and in our best moments we were pleased to feel like colleagues of an interchronological society of mathematicians. In short, we were happy to meet Archimedes personally!

Third story: the overcoming of mathematicians by complex numbers

The most inspiring result of the mathematical revolution ot the sixteenth century was without doubt the solution of the cubic equation. It is worthwhile taking a closer look at this story because it gives rise to deeper thoughts on the existence of mathematical objects.

In outline the story is well-known, of the great argument between Cardano and Tartaglia over whether Cardano was right to publish a method which Tartaglia thought he had communicated in private (and which was due to Scipione del Ferro, in any case). Let me sketch the strange accumulation of tricks which lead to the formula published by Cardano. Given the general cubic equation

(i) $ay^3 + by^2 + cy + d = 0$,

the first modification is fairly obvious remembering the procedure we use for quadratic equations:

(ii) $(y + b^*)^3 + c^*y + d^* = 0$

can be made equivalent to (i), by dividing the latter through by a and adjusting the coefficients. Let $x = y + b^*$. Then, with suitable coefficients, (i), (ii) and (iii) are equivalent:

(iii) $x^3 + px + q = 0$.

We may call (iii) the reduced normal form of (i)

So far the subject is simple. The Italian mathematicians used to set each other problems on cubic equations. It seems very probable that they experimented with exercises which were constructed from the solution. There are arguments that they appreciated constructions of the following type, for they were proud of being able to multiply complicated terms of numbers and roots. For example,

> From $(x-4)\cdot(x+2+\sqrt{3})\cdot(x+2-\sqrt{3}) = 0$
> you will get the reduced normal form
> $x^3 - 15x - 4 = 0$
> or—in the spirit of the time (without negative numbers)—
> $x^3 = 15x + 4$.
> In generalizing this construction you will get
> $(x-?)\cdot(x+a+\sqrt{b})\cdot(x+a-\sqrt{b}) = 0$
> and with ? : = 2a this will generate a reduced normal form with known solutions: 2a, $-a+\sqrt{b}$ and $-a-\sqrt{b}$ respectively.

Maybe from experiments of this kind del Ferro, Tartaglia and Cardano discovered the same solution of (iii) by using two tricks:

(iv) Assume that x_0 is a solution of (iii), which reads as

$$x_0 = u + v$$

then (iii) gives

(v) $\mathbf{u^3} + 3u^2v + 3uv^2 + \mathbf{v^3} + p\,(u + v) + \mathbf{q} = 0$.

Assume now that the bold terms add up to zero (second trick!) then the rest makes zero too, i.e.

(vi) $3u^2 + 3uv^2 + p\,(u + v) = (3uv + p)\,(u + v) = 0$

which leads to $v = -p/(3u)$ and the sum of the underlined terms now reads as

(vii) $u^3 + (-p/3u)^3 + q = 0$

which is a disguised quadratic equation (multiply by u^3)

Taking u^3 as the unknown to be sought for, you will get

(viii) $u^3 = -\frac{q}{2} \pm \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$

(= v^3 for reasons of symmetry). The only interesting case of (viii) consists of different values for u and v and this leads, after taking the cubic roots, to the Cardanic formula:

$$x_0 = u_1 + v_2 = \sqrt[3]{-\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\ldots}}$$

which solves the reduced normal form (iii) and, by simple transformation, the general cubic equation (i) too.

But there was an ugly snag in this beautiful formula:

In comparing it with the known solutions 4, $-2-\sqrt{3}$ and $-2+\sqrt{3}$ respectively of our given example, the Cardanic formula gives another solution, namely

$$\text{(ix)} \quad \sqrt[3]{2+11\cdot\sqrt{-1}} + \sqrt[3]{2-11\cdot\sqrt{-1}},$$

which Cardano and his contemporaries could not accept. They called this "impossible" situation (when the radical under the square root turns out to be negative) the *casus irreducibilis*

By experiment it soon became clear that the casus irreducibilis is equivalent to the existence of three different real solutions of (iii) (which we can easily prove by calculus). For fifty years they had little success in solving this disappointing riddle, but it should be noticed that Rafael Bombelli did not hesitate to play with the non-existing numbers. In the early seventies of the 16th century he turned the problem upside down by studying $(c \pm d\cdot\sqrt{-1})^3$ instead of $\sqrt[3]{a \pm b\cdot\sqrt{-1}}$ as given by the Cardanic formula. If you take, for example, $(2 \pm \sqrt{-1})^3$ and multiply out unscrupulously, ignoring the non-existence of what you do, then you will get $(2 \pm \sqrt{-1})^3 = (2 \pm 11\cdot\sqrt{-1}$, which shows (ix) to be the known solution: $4 = 2-\sqrt{-1}+2-\sqrt{-1}$. This lent weight to Bombelli's general guess that in the casus irreducibilis the solution has the form

$$\text{(x)} \quad x_0 = \sqrt[3]{a+b\cdot\sqrt{-1}} + \sqrt[3]{a-b\cdot\sqrt{-1}} = c+d\cdot\sqrt{-1}+c-d\cdot\sqrt{-1}$$

where the "bad" numbers cancel out to $x_0 = 2c$. Bombelli was right, but that was of little value because he did not find any way to figure out the real part c.

About twenty years later Francois Viète, who was the greatest of champions in tackling trigonometric equations, met the problem of finding "c" from a different viewpoint. He had studied the ancient problem of trisecting a given angle, by algebraic means which he developed for such problems. The known theorem on the addition of cosines gave

(xi) $4\cos^3(\beta/3) - 3\cos(\beta/3)$ as equal to the given $\cos\beta$.

From the given $\cos\beta$ the unknown x-coordinate $\cos(\beta/3)$ was to be constructed on the unit circle. With $x = \cos(\beta/3)$, and after division by 4, the equation (xi) turned out to be a cubic one in reduced form, namely:

$$\text{(xii)} \quad x^3 - \frac{3}{4}x - \frac{\cos\beta}{4} = 0$$

(written without negative numbers of course) which in most cases was a *casus irreducibilis*.

Viète knew the problems around that subject, but he compared the expected solution $x_0 = \cos(\beta/3)$ with the Cardanic result and with the ideas of Bombelli:

Viete: $x_0 = \cos\frac{\beta}{3}$

Cardano: $x_0 = \sqrt[3]{\frac{\cos\beta}{8} + \sqrt{(\frac{\cos\beta}{8})^2 - (\frac{1}{4})^3}} + \sqrt[3]{\cdot - \sqrt{\cdot}}$

$= \frac{1}{2}\cdot(\sqrt[3]{\cos\beta + \sin\beta\cdot\sqrt{-1}} + \sqrt[3]{\cos\beta - \sin\beta\cdot\sqrt{-1}})$

Bombelli: $x_0 = \frac{1}{2}(\ c + d\cdot\sqrt{-1} + c - d\ \sqrt{-1}\ )$

And now it was easy to guess and to prove what c should be, namely $c = \cos(\beta/3)$ and a fortiori $d = \sin(\beta/3)$.

Having guessed that

$$\text{(xiii)} \quad \sqrt[3]{\cos\beta \pm \sin\beta\ \sqrt{-1}} = \cos\frac{\beta}{3} \pm \sin\frac{\beta}{3}\ \sqrt{-1}$$

it was easy to prove the relation by the method of Bombelli. But through the discovery of Viète it was no longer an intuitive art to solve the casus irreducibilis by finding the "c". To round things off, Viète showed that exactly these cubic equations which are reducible to (xii) by the substitution z = x/const. correspond to the casus irreducibilis in its reduced normal form. In short, Viète discovered that cubic equations with three different solutions are essentially the trisection equation (xii) in a barely disguised form.

After reading this exciting story you should wonder why it took more than two hundred years to see the geometric meaning of (xiii) and of complex numbers in general. But it is one of the most astonishing facts of mathematics history that mathematicians from Viète, Descartes, Leibniz and Newton to the Bernoullis and Euler feared to accept the reality of numbers which proved to work more confidently than they could be said to exist.

I cite this story for many reasons. First of all it seems to be the best example to show that mathematical research and progress is hardly straightforward. In Germany it has become fashionable to focus mathematics teaching on "problem orientation", at least ideologically. Coming from Polya and his followers we have learned a lot of heuristics, and the "Gestalt theory" has made ingenious thinking much more transparent. But I have become suspicious that

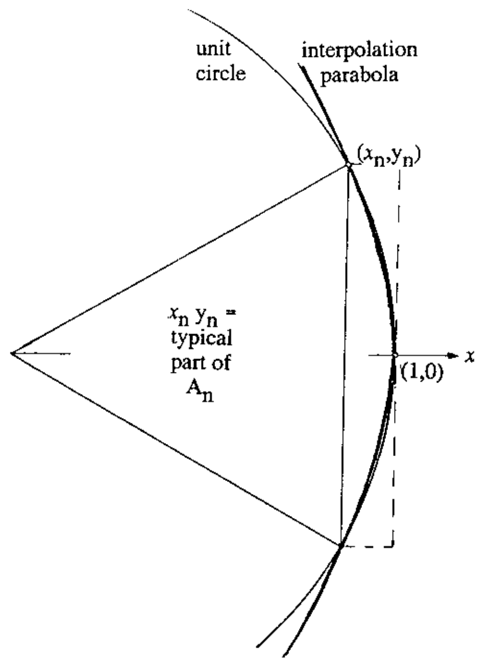

Figure 3

Augmenting the n-gon with n parabolic segments, which make two-thirds of the dotted rectangle, increases the convergence rate significantly.

this ideology tends to simplify things too much, hiding the more difficult items which do not fit the scene. The (hi)story of the cubic equation can serve as critical ground for our understanding of heuristics.

A second aspect is related to the more philosophical problem of the status of existence which mathematical objects possess. I have always found the Platonic view suspect. I think of mathematical items as a product of social being, and have found it more stimulating and rewarding to think of mathematics as invented rather than discovered. My meeting with the spirit of Archimedes, which I reported as the second story, did not contradict my constructivist philosophy, for I could imagine that the software of the human brain has structures which lead all people to think similarly. But the story of the *casus irreducibilis* does make it more credible to think of an eternal commonwealth of ideas as Plato suggested, or at least of mathematical structures in the hardware of our brains as Kant claimed. Unsure as I am today, I have learned that mathematical history can serve to refine our view about mathematical reality [see also Kitcher, 1983].

Lastly, I am convinced that the crucial role which mathematics plays in public education is based on its historical merits, not merely its topical significance in industrial society, as many people claim. Our reputation as teachers of mathematics depends heavily upon the revolutionary role mathematics has played over the past five centuries, a period during which the Copernican Revolution and its aftermath saw a belief in God gradually overtaken by a pseudo-critical admiration for scientific knowledge Mathematics has become the ideology of the objectivists, besides being conservative enough to guard the frontier to the second culture in the sense of C P. Snow. The mathematics we can teach in schools remains essentially on the level of the day before yesterday, and there is little escape from this situation when so much post-Newtonian mathematics is so difficult. But we can and must remember the days when school mathematics was new, and following comprehensible human interests

Postscript

The preceding pages were written before the HIMED 90 conference. Now, after the inspiring discussions I watched and participated in there, I feel moved to emphasize an aspect of general importance which I felt remained too far below the surface despite its significance.

It seems inevitable that in such conferences the professional didacticians and the specialist historians of mathematics control the official discussions, despite the best efforts of the organizing committee to involve non-specialist practising teachers. We need to decide whether the subject of history of mathematics should remain an area for a very few teachers who are sufficiently acquainted with the details and with the general course of history, or whether we have some strong arguments for bringing historical materials within the broad range of educational practice.

If we prefer the second alternative, as I do, then we cannot remain satisfied with such brave arguments as "history is motivating", "historical texts show novel approaches towards mathematical items of the curriculum", "historical alternatives to our methods suggest some reflections upon them", or "after some struggling with this text, most of the students became enthusiastic about historical reasoning". There must be more efforts to encourage our school-centered colleagues to seek their own way to the questions behind all the answers they give in teaching, without having to be asked by their students.

Most parts of our school mathematics are deliveries from the day before yesterday, and in trying to explain and justify its contents entirely in terms of the world of today, the teacher inevitably becomes insincere. The teacher's job is a conservative one by its very nature, for it is based on the knowledge and the rules of the previous generation's adults. There is no escape from this, but we are free to accept those parts of our delivered traditions which are likely to have a constructive potential for our students' imaginable futures. Another conservative claim is that the unconscious sphere of the educational tradition in mathematics deserves respect, for it is a secret storehouse of the human endeavour called "mathematics", the mother-subject of all rational thinking

This claim amounts to a strong argument that *all* school-children ought to learn mathematics with a historical perspective, not merely those fortunate enough to have teachers who are that way inclined. This may be put another way. The curricular traditions which are enshrined in the syllabuses of all European countries, not least in the everyday teaching of elementary mathematics, deserve respect for their creative potential towards the unknown futures of our students—and, in my view, only for that reason. Our students will have to cope with the adult world we deliver to them, equipped with the training we provide for them. Therefore the historical perspective of mathematical processes developed in the past must be shown to all students, not only to those few with the advantage of a historically-minded teacher Elementary mathematics, as the core of school mathematics, is essentially a delivered treasury of potentialities which may become fruitful one day (or never). The teacher's intention is to facilitate the utilization of that wealth, and therefore should never stop at merely clarifying an ancient original or a second-hand text. Her teaching target should always be to bring out the actual relevance of the historical topic to the student's grasp of the world of today and tomorrow, and that cannot be achieved without the teamwork of historians, didacticians, and school teachers.

Finally, I exemplify what I mean by sketching a sequel to my third classroom story

The third story revisited

I do not know the historical truth about Cardano and the Cardanic formula. That is certainly not what I set out to teach my students. But there is an excitingly constructive function of historical reasoning that should be brought into the open, even in the course of mathematics teaching. Original mathematical texts can fulfill an invaluable role in helping to resolve a heuristic impasse of the kind which can be reached with bright students in the classroom. This is what happened to me.

Having explained and discussed the curious development of the Cardanic and the Bombelli-Viète concept, we remained unsatisfied with the unmotivated air of the strange accumulation of tricky transformation steps which constitute the Cardanic argument as I presented it. How could he, or his predecessors, have found that way to the solution?

This observation forced me to think about the subject in more detail. As a matter of fact, there had been no factorized form of algebraic equations in the days of Cardano at all, so my "rational reconstruction" of history was leading me into a quagmire. A rethink was necessary. What was the actual process of Cardano's thought? I was rather reluctant to plunge into the Renaissance prose of Cardano's *Ars magna*, but the students insisted and we began to study the text as it appears in English translation in Struik's 1969 *Sourcebook*.

First we had to realize that our algebraic way of reasoning was not yet invented; Cardano and his contemporaries did not only *speak* in geometric terms, they must have *thought* is that way. The second thing we noticed was that, as is customary in mathematics, Cardano did not say how he found the result, he merely stated it in geometric terms and went on to prove it—and all in terms of one specific example, ***a cube and unknowns equal to a number*** (namely, $x^3 + 6x = 20$). Cardano's strategy was to take difference of two cubes (named u^3, v^3 later on), making the product uv of their edges equal to one-third of the given "6". Then, he claimed the difference of the edges $(u - v)$ would show the unknown.

From Struik's book we learned that this difference trick was announced in a poetic communication from Tartaglia to Cardano in 1539:

> Quando che'l cubo le cose appresso
> Se agguaglia a qualque numero discreto
> Trovan dui altri, differenti in esse....

This provides good practice for any pupil learning Italian (but an English translation may be found in Fauvel and Gray [1987]). It is not too hard to see, through geometric diagrams, what is going on. Taking for granted the idea of a difference of cubes (Figure 4), it is quite easy to find the

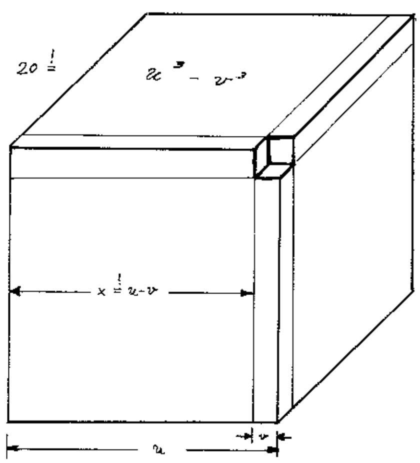

Figure 4

Cardano's reduced cube

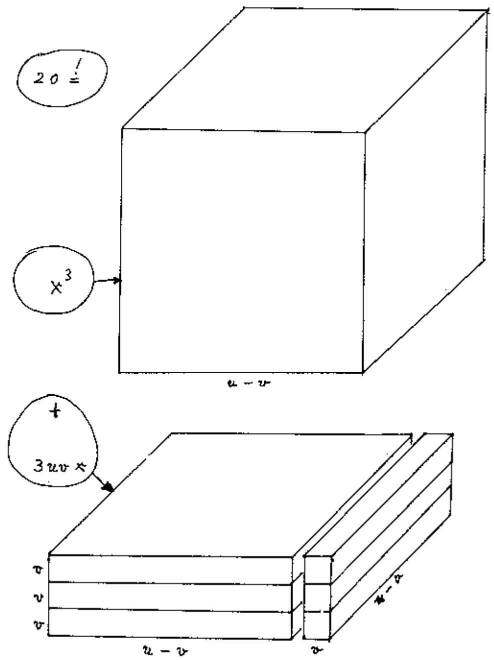

Figure 5

The pieces placed around $x^3 = (u-v)^3$ must make $6x$

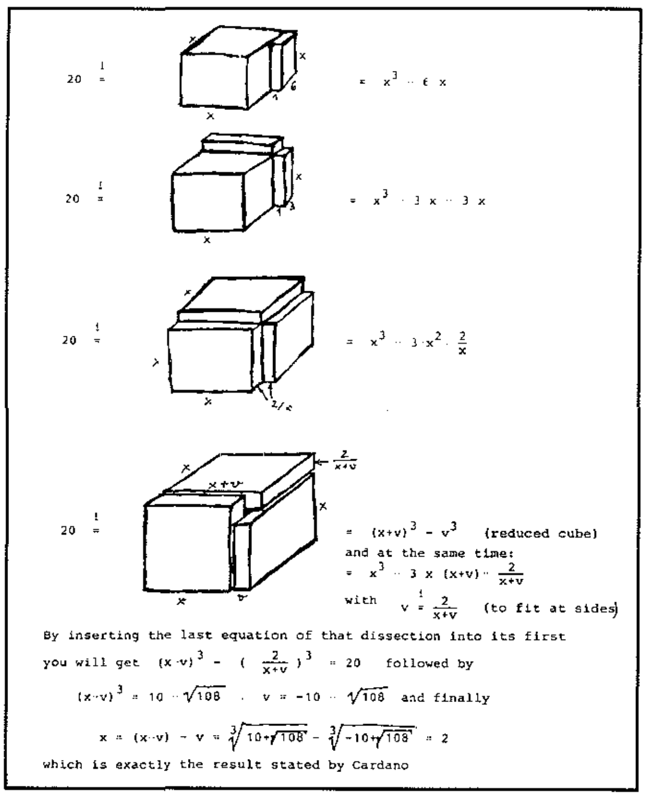

Figure 6

$6x + x^3$

uv = 6/3 requirement (Figure 5). That is clear, but where did Cardano, or whoever, get the difference idea from?

Translating the term "$x^3 + 6x$" into the geometric environment, we realized instantly that the placement of the "6x" offered a splendid occasion to be creative. What is the most convenient way of placing the "6x" around the cube "x^3"?—Look at Figure 6! This enabled us to imagine where the geometric inspiration for the algebraic idea had come from. I do not suppose that we have found out the historical truth about Cardano's processes even yet, through our classroom thought experiment, but we did receive a heuristic approach to an actual problem from looking at reality in another way. Our exploration of past mathematics has proven an enrichment of our heuristic skills and an enrichment of our mathematical insight in general.

Conclusion

To sum up: for many reasons mathematics should be taught as a rational learned behaviour towards reasonable, non-specialist and non-esoteric questions. This enables our students to be prepared as best we can for the possible issues that will arise in their future. This viewpoint can be understood better through being explicit about what it is denying: it seems highly undemocratic to educate pupils while setting as their supreme attainment target a strange collection of conditioned reflexes called "mathematical maturity". From the Renaissance to the French Revolution, mathematics was an emancipatory subject closely connected with the emergence of enlightenment against the dominance of theological and other reactionary forces within society. This is our best tradition: to encourage individuals to think for themselves and to insist on their human right to do that without constraint.

The pedagogical avenue towards redeeming and preserving that right is what Freudenthal has called "to re-invent under guidance". Some acquaintance with mathematical history can help the teacher to design suitable mathematical activities in the classroom, but more significantly provides a changed tone for the framework within which the whole process of mathematics education takes place

Bibliography

Beckmann, P. *A history of pi.* Boulder Co.: Golem Press, 1977

Bialas, V. *Erdgestalt, Kosmologie und Weltanschauung: Die Geschichte der Geodäsie als Teil der Kulturgeschichte* (The history of geodesy as a part of the history of civilization). Stuttgart, 1962

Bühler, M. "Reading Archimedes' Measurement of a Circle', in J. Fauvel (ed), *History in the mathematics classroom: the IREM papers.* Leicester: Mathematical Association, 1990, 43-58

Cantor, M. *Vorlesungen über Geschichte der Mathematik* (Lectures on mathematical history). Leipzig: Teubner, 1913

Fauvel, J. and J. Gray, *The history of mathematics: a reader.* Macmillan 1987

Fischer, I. "Another look at Eratosthenes and Posidonius' determinations of the earth's circumference", *Quarterly journal of the Royal Astronomical Society* 16 [1975] 152-167

Freudenthal, H. "Die Welt wird gemessen" (Measuring the earth). In Führer [1986]

Führer, L. "Zur Entstehung and Bergründung des Analysisunterrichts an allgemeinbildenden Schulen' (Genesis and legitimation of analysis teaching in Germany). *Der Mathematikunterricht* 27, 5 [1981] 81-122

Führer, L. "Die Kreisberechnung als Brennspiegel der Schulmathematik" (Circle measurement as a burning mirror of school mathematics). *Praxis der Mathematik* 23 [1982] 289-298, 323-337

Führer, L. (ed) *Geschichte—Geschichten* (History—(high-)stories). Special issue of *Mathematiklehren* 19 [1986]

Hogben, L. *Science for the citizen.* London: Allen & Unwin, 1938

Kämmerer, W. *Die Quadratur des Kreises im Schulunterricht* (The quadrature of the circle in mathematics teaching). Gießen: Emil Roth, 1930

Kitcher, P. *The nature of mathematical knowledge.* New York: Oxford University Press, 1983

Knorr, W.R. "Archimedes and the measurement of the circle: a new interpretation". *Archive for history of exact sciences* 15 [1976] 115-140

Kommerell, K. *Das Grenzgebiet der elementaren und höheren Mathematik* (At the frontier of elementary and higher mathematics). Leipzig: Koehler, 1936

Prell, H. *Die Vorstellungen des Altertums von der Erdunfangslänge* (Ideas of the earth's circumference in antiquity). Berlin (GDR), 1959

Resnikoff, H.L., Wells, R.O. *Mathematics in civilization.* New York: Dover publications, 1984

Rottländer, R.C.A. *Antike Längenmaße* (Measures of length in antiquity). Braunschweig: Vieweg, 1979

Rudio, F. (ed) *Archimedes. Huygens. Lambert. Legendre: vier Abhandlungen über die Kreismessung* (Four papers on the measurement of the circle). Leipzig:Teubner, 1982

Salamin, E. "Computation of π using arithmetic-geometric mean. *Mathematics of computation,* July [1976] 565-570

Schelin, C.W. "Calculator function approximation". *Mathematics Monthly* 905 [1983] 317-325

Schneider, I. *Archimedes.* Darmstadt: Wissenschaftliche Buchgesellschaft, 1979

Smeur, A.J.E.M. "On the value equivalent to pi in ancient mathematical texts: a new interpretation". *Archive for history of exact sciences* 6 [1969/70] 249-270

Struik, D.J. *A source book in mathematics 1200-1800.* Cambridge: Harvard University Press, 1969

Taylor, E.G.R. *The haven-finding art: a history of navigation from Odysseus to Captain Cook.* London: Hollis & Carter, 1971

Tropfke, J. *Geschichte der Elementarmathematik, Band 1: Arithmetik und Algebra* (History of mathematics, volume 1). New edition by K. Vogel, K. Reich and H. Gericke Berlin: de Gruyter, 1980

van der Waerden, B.L. *Science awakening.* Oxford University Press, 1961

van der Waerden, B.L. *Die Pythagoreer.* Zürich: Artemis, 1979

Winter, H. "Zoll, Fuß und Elle: alte Körpermaße neu zu entdecken" (Inch, foot, and yard: old body measures to be rediscovered). In Führer [1986]