MATHEMATICAL PROBLEM SOLVING AND THE ART OF REPRESENTATION

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Research in mathematics education suggests that learning to solve a problem should involve modelling and visual representation (*e.g.,* Lesh & Zawojewski, 2007). Nunokava (1994) advocates that the usefulness of visual representations "lies, we think, in the fact that it can show relationships between elements in the problem clearly" (p. 34). According to researchers, transforming a mental representation of a situation into a visual representation of mathematical relationships between quantities enhances students' mathematical thinking and contributes to problem-solving ability development. In our pre-service teacher training courses, we implement a relational approach to problem solving (Polotskaia & Savard, 2018). We introduce our students to a system of basic quantitative relationships together with their visual representations. To develop students' mastery of this specific modelling tool, we propose some more complex problems for them to represent visually. One such representation triggered a student's remark, "This is Picasso for me", meaning "This is difficult to understand". We would like to explore the following question: What is the connection between modern visual arts (for example cubism) and the visual representation of a mathematical problem? What makes our understanding of a visual representation or drawing difficult?

Visual representations in mathematics

Since ancient times, mathematicians have valued visualization of different mathematical relationships and used visual representations to support their reasoning and proofs. In his book, Roger Nelsen (1993) provides multiple visual proofs developed by ancient and modern mathematicians from all over the world. These days, mathematicians use computers to create representations of very complex mathematical objects. For example, Figure 1 presents a computer-created fractal structure.

It is also known that some mathematicians use their artistic talent to produce images of very abstract mathematical concepts. A representation of the *Deformation of the Riemann Surface of an Algebraic Function* (see Figure 2) created by a mathematician Anatoly Timofeevich Fomenko, full member of the Russian Academy of Sciences, professor of the Moscow State Lomonosov University, is an example of such artistic interpretation of a mathematical idea. These visual representations certainly help to make better sense of the mathematics at hand.

Visual representations of mathematical relationships

In elementary school, children usually learn to represent objects one by one to count them or visually represent numbers as tens and units to carry out a calculation. In some countries, students use simple schemas to represent operations or elementary quantitative relationships

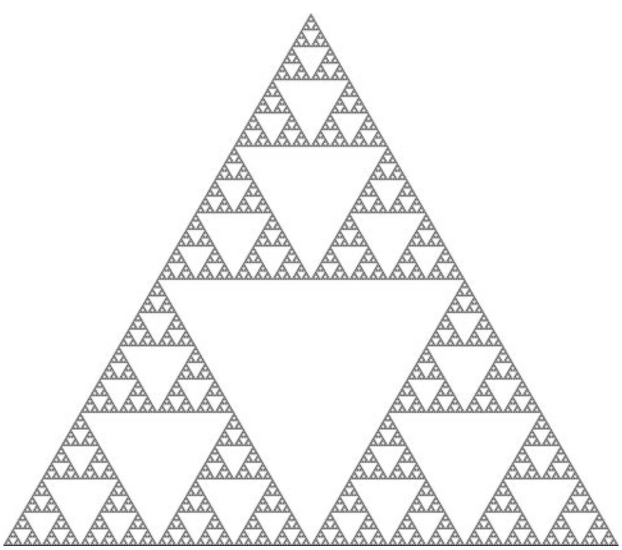

Figure 1. Sierpinski triangle, fractal structure, named after the famous Polish mathematician Waclaw Sierpinski.

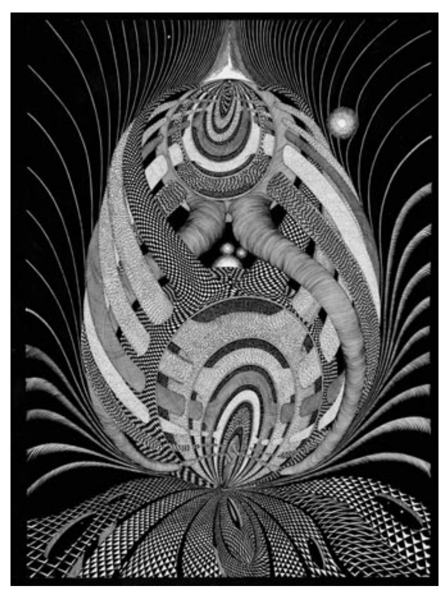

Figure 2. Fomenko, A. T., 1983. Deformation of the Riemann Surface of an Algebraic Function. [1]

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(*e.g.,* Venenciano & Dougherty, 2014; Davydov, 1982). These basic schemas help students learn to solve simple (not complex) arithmetic word problems. Is this basic use of modelling (schematizing) enough for students to be able to solve problems that are more complex? What is a complex problem? Let us analyze the example that triggered the remark about Picasso.

> Problem: I need to pack muffins into boxes. If I put 3 muffins in each box, I would need 8 boxes more than if I put them 5 per box. How many muffins do I have?

We represent this problem in Figure 3.

The left rectangle represents the case when muffins are placed in rows of three, and the right one when the same muffins are placed in rows of five. The upper part of the left rectangle represents '8 boxes more' than in the right case. The shaded right part of the right rectangle represents the two muffins more in each box of 5 (taken all together). It is easy to notice that the unshaded part of the left rectangle and the unshaded part of the right rectangle represent the same number of muffins. At the same time, the left and right rectangles represent the same total number of muffins. Therefore, the shaded rectangles represent the same number of muffins. Thus, we can propose an arithmetic solution to this problem.

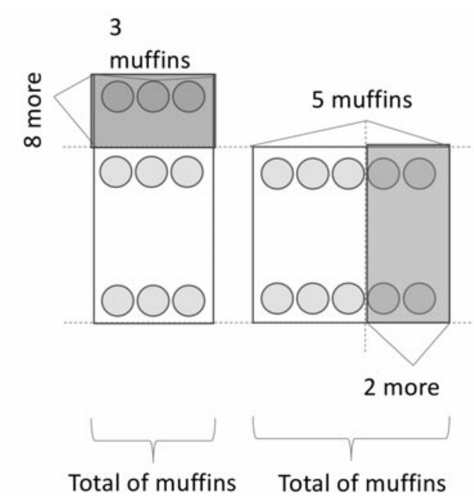

Figure 3 Representation of the Muffins' problem

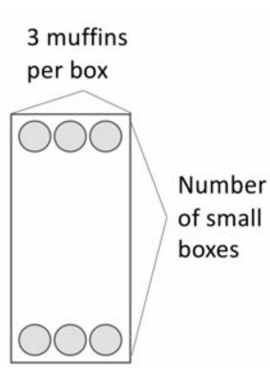

Figure 4. Representation of relationship R1.

> 5 – 3 = 2 (muffins more in each box of five than in a box of three)
>
> 8 × 3 = 24 (muffins in the 8 boxes of three = all extra muffins in the boxes of five)
>
> 24 ÷ 2 = 12 (boxes of five)
>
> 12 × 5 = 60 (total muffins)

In order to evaluate the complexity of this problem, we propose to look at the number of mathematical relationships one needs to analyze and model, as well as their natures.

> *If I put 3 muffins in each box*—This text describes a multiplicative relationship R1. (Figure 4): 3 × NB3 = T (NB3: number of boxes of three; T: a total of muffins)
>
> *I would need 8 boxes more than*—This text describes an additive relationship R2. (Figure 5): NB3 – NB5 = 8 (NB5: number of boxes of five)
>
> *If I put them 5 per box*—This text describes a multiplicative relationship R3. (Figure 6): 5 × NB5 = T

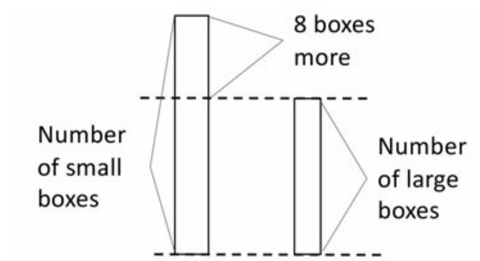

Figure 5. Representation of relationship R2.

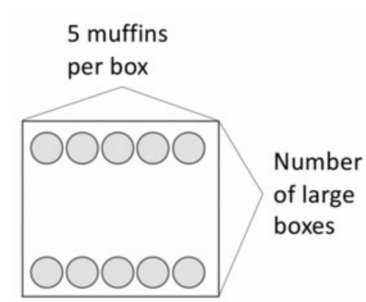

Figure 6. Representation of relationship R3.

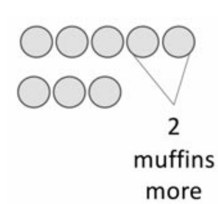

Figure 7. Representation of relationship R5.

Two more relationships are implicit.

> R4: T = T, because in two cases the same number of muffins were used.
>
> R5 (Figure 7): 5 – 3 = 2, therefore, we put two muffins more in each box of five (in comparison to the box of three).

The relationships R1, R2, and R3 involve two unknown quantities each. Therefore, arithmetic reasoning cannot be applied immediately, and our understanding of the problem cannot be simplified gradually by intermediate calculations. Here, we used a visual modelling approach to help visualize and better understand *all mathematical relationships* involved (Figure 3). This way, we can see or find new relationships and finally produce an arithmetic solution. The following relationships can *be found:*

R8: The total muffins in small boxes is composed of two parts: the muffins in 8 small boxes and the rest.

R9: The total of muffins in large boxes is composed of two parts: all muffins in groups of 3 put together and all muffins in groups of 2 put together.

R10: The rest of muffins in small boxes is the same as 'all muffins in groups of 3 put together' in large boxes because these rectangles have the same dimensions: 3 and the number of large boxes.

Another approach would be algebraic modeling. People familiar with algebra would say that it is easy to compose the following equation $5x = 3(x + 8)$. However, the solver composing this equation should be able to analyze the same mathematical relationships altogether (but this time, mentally). Thus, many students will find it difficult to compose such an equation for the problem.

Difficulties with complexity

Let us look again at the analogy launched by our student: "This is Picasso for me". This student's remark suggests that the cognitive obstacle she experienced was of the same nature that some viewers face in front of abstract painting. According to Weltzl-Fairchild's (Weltzl-Fairchild & Emond, 2000) typology of cognitive dissonances, the analogy expressed by the student to manifest her misunderstanding would mainly refer to two types of dissonance related to a difficult aesthetic experience. The first is a dissonance between the expectations of the viewer and the experience of the work (a conflict arising from the artistic object and the notions of beauty and/or communication). The second is a perceived dissonance between the artistic object itself and what it symbolizes (a conflict between certain parts of the artistic object as well as a conflict between the symbolic message and the means of expression). Thus, we will see how the elements of information given by both the algebraic problem and the fragmented composition of the Spanish painter complexify their respective analysis and interpretation. It is a question of studying the similarities between the process of modelling by visual processing proposed above and the process of aesthetic appreciation of a work belonging to analytical cubism, such as *Le Poète* (1911) by Pablo Picasso [2].

The meaning and beauty of abstract art do not come to us directly from the forms and objects themselves. At first glance, Le Poète is composed of rectangular prisms, curves and straight lines. Though, *the interplay* of simple elements, light and dark, lines and angles, their relationships and conversations with each other, can reveal a harmony of rhythms and emotions—a distinct experience of beauty proposed by the artist. We agree with Sinclair (2002) that the "personal and subjective" plays the same role in the aesthetic experience in art as it plays in mathematical problem solving, largely determining the quality of this experience. Thus, for an unprepared observer, multiple relationships (with multiple unknowns) present in an art work or in a mathematical problem make the conversation between those elements difficult to grasp.

Csíkszentmihályi and Robinson (1990) specify that the cognitive dimension of aesthetic experience in art comes in two forms: closed and open-ended. They affirm that, "Certain individuals, for example, employed intellect in the service of achieving a kind of closure, while others used cognitive means to open up works to more varied interpretations" (p. 42). The art of Picasso invites the observer to imagine, to go beyond the directly observable. In our mathematical example, the student was probably trying to interpret the problem and the visual model strictly based on her school experience, where *numbers* are usually represented visually as tens and units. The forms representing *relationships* were difficult for her to imagine. Thus, her relationship to the problem can be interpreted as 'closed'. The role of the university instructor will then be to help students 'open up', to consider new points of view, develop new ways of thinking and modeling, be prepared to better meet the varied needs of their future students. This brings us back to Picasso and the hope that our students adopt his interpretation of learning: "J'essaie toujours de faire ce que je ne sais pas faire, c'est ainsi que j'espère apprendre à le faire" [3].

Conclusion

Surprised by a remark of one of our students, who compared a mathematical representation to an abstract painting, we tried to understand the connections between the complexity of a mathematical problem and that of abstract art. In both cases, an observer or solver should analyze and reconstruct for herself multiple known and unknown elements (visible and implicit objects) and multiple relationships between elements. The goal of this process is to create a harmonious and logically-sound holistic mental representation of the situation: the mathematics behind the word description or graphical representation, or the ideas of the artist behind the lines, forms, and colours.

In his recent publication, Alshwaikh (2018) argues for a better use of diagrams to support school students' learning of mathematics. He writes:

> mathematics is often presented in classrooms and textbooks as abstract, symbolic and devoid of human agency, a view which affects students' access to mathematics. If we understand what is communicated in diagrams—whether they tell a story and include human agency or whether they are conceptual and 'timeless'—we can better design textbooks and better understand what they communicate to learners. (p. 13)

We can add that the ability to embrace the complexity of the world is crucial for mathematics as well as for the work of art. Unfortunately, the development of such ability is regularly ignored by the school curricula. The use of simple drawings, such as the schemas we discussed, could possibly help students access and develop the agency needed to solve more complex problems.

Notes

[1] On-line at http://virtualmathmuseum.org/mathart/ArtGalleryAnatoly/Anatolyindex.html

[2] Pablo Picasso, *Le Poète,* 1911, oil on linen, 131.2 x 89.5 cm, The Peggy Guggenheim Collection, Venice. See https://www.e-venise.com/musees_venise/guggenheim/pablo-picasso-le-poete-peggy-guggenheim-venise.html

[3] "I always try to do what I don't know how to do, this is how I hope to learn to do it" (from https://citations.ouest-france.fr/citations-pablo-picasso-658.html, our translation).

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