Textual Strategies

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The argument

1 Comparing two versions of a mathematical text, an original by Courant and Robbins (text A), and its transformation according to a widely used psychological system of readability (text B), one is tempted to say: text A cannot be read but can only be reread, whereas it is exactly the opposite with text B

2. After a presentation of some examples of visualizing mathematical information in textbooks, the complementary properties of the visual vs the verbal-numerical symbol systems are summarized.

3. Some experiences with graphic devices in mathematics education are discussed.

4 The varied use of the balance metaphor is demonstrated. On the other hand, mathematics instruction, throughout its history, has never got rid of its traditional qualms about the metaphorical The following verdict quoted from the preface of Frend's *Principles of Algebra* (1796) is still quite current " Numbers are there (in McLaurin's *Algebra*) divided into two sorts, positive and negative; and an attempt is made to explain the nature of negative numbers, by allusions to bookdebts and other arts. Now, when a person cannot explain the principles of a science without reference to metaphor, the probability is, that he has never thought accurately upon the subject" (p. X)

5. Choosing a sample of mathematics books – university textbooks as well as school books – and a "random page" from each for inspection, we find a very clear contrast along a "verbal versus visual" dimension between the two types of books The result of this little experiment at first surprised me very much After a while I at last began to understand how the diverse conceptions of mathematics, that is traces, of knowledge about mathematics, could be read off the textual surface (The individual's relation to knowledge is the essential problem)

6. The widespread "over-methodization" of textbooks is a very illuminating symptom. It not only demonstrates a detrimental bias of didactical activity towards method, but also has an insufficient epistemological basis. Every evaluation of the relation between knowledge and its textual description implies a view on the other relation between reality and knowing that reality.

7. Some tentative conclusions are proposed:
- The character of school mathematics is essentially algebraic
- Visual cognition, predominantly of diagrams, ideograms and other graphic devices, plays a dominant role within a good textbook
- Instruction which is too method-oriented will often tend to neglect visual cognition
- Excessive methodization of textbooks is a significant sympton in this direction.

1.

In Germany the most frequently used system for improving the readability of (mathematical) texts if that of Schulz v. Thun and Götz. The authors offer a psychological conception of intelligibility crystallized in the following

1 Simplicity (short sentences, simple syntax ...)
2. Organization – order
3. Terseness – precision
4. Additional stimulation (varied and personal style ...)

The authors assign the greatest importance to criteria 1. and 2. In their manual "Mathematik verständlich erklären" (p 47-49), they present, as an example, a quotation from Courant/Robbins *Was ist Mathematik* together with a version which they themselves produced by applying their system to the text of Courant and Robbins

Text A

Kommensurable Strecken

"Vergleicht man zwei Strecken a und b hinsichtlich ihrer Größe, so kann es vorkommen, daß a in b genau r-mal enthalten ist, wobei r eine ganze Zahl darstellt. In diesem Fall können wir das Maß der Strecke b durch das von a ausdrücken, indem wir sagen, daß die Länge von b das erfache der Länge von a ist. Oder es kann sich zeigen, daß man, wenn auch kein ganzes Vielfaches von a genau gleich b ist, doch a in, sagen wir n gleiche Strecken von der Länge a/n teilen kann, so daß ein ganzes Vielfaches m der Strecke a/n gleich b wird:

(1) $$b = (m/n)\,a$$

Wenn eine Gleichung der Form (1) besteht, sagen wir, daß die beiden Strecken a und b kommensurabel sind. da sie als gemeinsames Maß die Strecke a/n haben, die n-mal in a und m-mal in b aufgeht"

Translation:

"In comparing the magnitudes of two line segments a and b, it may happen that a is contained in b an exact integral number r of times. In this case we can express the measure of the segment b in terms of that of a by saying that the length of b is r times that of a. Or it may turn out that while no integral multiple of a equals b, we can divide a into, say, n equal segments, each of length a/n, such that some integral multiple m of the segment a/n is equal to b:

$$b = \frac{m}{n} a. \quad (1)$$

When an equation of the form (1) holds we say that the two segments a and b are *commensurable*, since they have as a common measure the segment a/n which goes n times into a and m times into b. The totality

Fig. 9. Rational points.

of all segments commensurable with a will be those whose length can be expressed in the form (1) for some choice of integers m and n ($n \neq 0$)."

(Courant/Robbins 1978)

Text B

Kommensurable Strecken

"Man sagt: zwei Strecken sind kommensurabel, wenn sie ein gemeinsames Maß haben. Was bedeutet das: Ein gemeinsames Maß haben?

– Angenommen, eine Strecke ist 3 cm, die andere 9 cm lang. Die beiden Strecken sind kommensurabel: Sie haben als gemeinsames Maß 3 cm. Es paßt in eine Strecke genau 1 mal, in die andere genau 3 mal.

– Angenommen, eine Strecke ist 6 cm, die andere 10 cm. Auch diese sind kommensurabel. Das gemeinsame Maß ist 2 cm: Es steckt 3 mal in der ersten und 5 mal in der zweiten Strecke. Selbst für 2 Strecken von z.B. 1.67 cm und 4,31 cm Länge läßt sich leicht ein gemeinsames Maß finden: 0,01 cm. Es ist 167 mal in der ersten und 431 mal in der zweiten Strecke.

Was sagen uns diese Beispiele? Zwei Strecken sind kommensurabel, wenn die eine Strecke (oder ein Bruchteil von ihr) in der anderen enthalten ist, ohne daß ein Rest bleibt."

At a first glance it appears as if indeed the original presentation has been improved considerably. Courant/Robbins seem too much focused on some very fundamental ideas (concept of rational number) and seem a little bit locked within the perspective of full-fledged mathematicians somewhat ignoring the mathematical capabilities of the intended reader. But of course the fundamental ideas are essential and these are not conveyed into the second text. Literally there seems to be the same in both texts, but a text taken only literally and transformed according to semiotic judgments alone must – as in the case above – miss the essential message. The "improved" version B has lost all relevance because it appears to be "closed".

The authors themselves describe their improvements by saying that text B has the following properties

- simple short sentences
- concrete style instead of 'abstract'
- contains examples
- essential rule emphasized
- no diagrams or formulae

The last point especially seems a very great disadvantage because it takes away the essential generalization (the number concept as well as the concept of rationality) and turns the passage into a text without a problem. As soon as one confines the discussion to decimal numbers and does not use variables as well as line segments to signify quantities or the number concept, one has a given common measure to start with.

To continue our discussion, it might be helpful for the moment to distinguish between the levels of text semiotics vs. mathematical content. Text B can be read but cannot be reread. It is read completely in linear-temporal order and it is, as it is read, understood without difficulty and in a literal sense only. Text B certainly represents a type of text which may have dominated in our literate societies over the centuries in the form of economic or other numerical accounts, legal documents, administrative regulations, instructions for use, etc.

Text B represents a generally accepted form of written communication, and this general acceptability obviously puts certain contraints on the epistemology: for instance, inductive reasoning, concentration on particular examples, identifying ideas with their definitory descriptions and considering concepts as names of objects, clear cut yes/no distinction.

For the majority of people printed texts have very seldom been the instruments of speculative and exploratory thinking.

Text B, therefore, meets very well in a certain sense the social standards of literacy. Both its merits and its deficiencies, the relationship to knowledge and the type of schooling it expresses, these and other questions have all to be evaluated according to the general context indicated above.

Text A, regardless of all its insufficiencies which have partly been taken care of by v. Thun and Götz, represents an endeavour to initiate the reader to the science of mathematics in general and to attune him to a certain type of activity. Eco describes this textual strategy:

"Many texts make evident their Model Readers by implicitly presupposing a specific encyclopedic competence. For instance, the author of *Waverley* opens his story by clearly calling for a very specialized kind of reader, nourished on a whole chapter of intertextual encyclopedia:

(1) *What could my readers have expected from the chivalrous epithets of Howard, Mordaunt, Mortimer or Stanley, or from the softer and more sentimental sounds of Belmore, Belville, Belfield and Belgrave, but pages of inanity, similar to those which have been so christened for half a century past?*

But at the same time text (1) *creates* the competence of its Model Reader. After having read this passage, whoever approaches *Waverley* ... is asked to *assume* that certain epithets are meaning "chivalry" and that there is a whole tradition of chivalric romances displaying certain deprecatory stylistic and narrative properties.

Thus it seems that a well-organized text on the one hand presupposes a model of competence coming, so to speak, from outside the text, but on the other hand works to build up, by merely textual means, such a competence" (Eco 1981, p 7/8)

In mathematics instruction, the fundamental generalizations play an important role for this strategy of 'open' texts, and generalizations are represented by variables. They are, on the one hand, the starting point, the advance organizer or essential context for reading a mathematical text They form the regulative basis for any activity On the other hand, understanding the essential ideas and fundamental concepts is the task and the goal which confronts the reader of the text Theoretical generalizations thus serve at the same time as starting points and problems, as means and ends for the learners' activities

How are these distinct but closely linked roles of the theoretical ideas realized within the text? It is obvious that theoretical concepts, and especially mathematic generalizations cannot be identified with their definitory descriptions ["the theoretician's dilemma", cf. Tuomela 1973, p.3] although "proceeding strictly on the basis of an interpretation accepted beforehand" constitutes a very essential part of mathematical activity (hence the permanent advice given to students when they begin learning mathematics: "take the definitions literally and strictly!")

From a dynamical point of view, theoretical concepts thus show a complementaristic structure as their meaning is both operational and representational, and mathematical texts have to be understood both literally and metaphorically In particular, "variables" and other diagrams are principally linked to metaphorical understanding [cf. Otte 1981 p.22/23]. On the other hand, it has been argued [cf Harnad 1982; Lakoff/Johnson 1980, esp. p.108f] that metaphor, considered as a matter of conceptual structure and not as merely a linguistic phenomenon, can fruitfully be investigated from a perspective of "spatialization" or iconic representation.

When we compare texts A and B once more, we notice that B is, on the level of verbal language, by far the better text Language is very well suited to express units which are precise but rather bare in themselves, their effectiveness resulting from sequential order The phonetic alphabet is a perfect concretization of this fact It has led some people to think of mathematics and technical science as linear systems defined by time.

"Remember that written notations of music, words and mathematics are merely formulae for the memorization of fundamentally auditory systems, and that these formulae do not escape from the systems' linear and temporal character The ear can hear an equation on the telephone; it cannot hear a map" [Bertin 1981, p 17']

On the other hand one could argue as well that mathematical language is entirely visual and that mathematical writing demands the greater freedom of the two dimensions of the plane. Mathematical thinking is very much dependent on visualization, and because of the dominance of (spoken) language as an essentially auditory system in communication, difficulties may arise in doing and communicating mathematics.

The French mathematician R Thom even argues that "the dominant influence of the spoken word in the West has resulted in an alphabetical or syllabic system of writing and the expression (*signifiant*) has violently subjugated the meaning (*signifié*)." [Thom 1975, p 329; for a more elaborated and balanced account, see Leroi-Gourhan 1964 esp. chap. VI] And the great American philosopher and mathematician C S Peirce (1839-1914) held the view that the only way of communicating an idea was to do so by the visual means of icons or diagrams.

"Diagram is a word which will do for any visual skeleton form in which the relations of parts are perspicuously exhibited, and are distinguished by lettering, or otherwise, and which has some signification, or at least some significance A system of equations written under one another so that their relations may be seen at a glance may well be called a diagram Indeed, any algebraical expression is essentially a diagram."[Peirce 1976, p 345]

In fact the most important aspect of text A, establishing its open and creative character, is the formulae and diagrams in it And it seems as if we could make such a statement even without considering the details of the subject matter of the two texts A and B By just looking at those texts and comparing them, we get the idea that the possibilities of mathematical writing as well as of intellectualization depend on visual representation and geometrical interpretation This would imply that certain epistemological questions in mathematics are subject to an "investigation by the human eye"

Comparison of text A and text B has led us to two essential phenomena, to "metaphor" and to "graphics" in mathematical texts If there is a relation between those two at all, it will certainly be a very complex one, and one difficult to investigate

Before pursuing these ideas further in the sections to follow I would like to stress the fact that the type of investigation aimed at in the present paper has to make use of all forms of cultural life Quotations from the field of text semiotics or even literary criticism seem necessary and helpful for a descriptive epistemology. Both in linguistics and in the theory of literary criticism, for instance, there exist theories of "spatial form in narrative", which seem to me highly interesting Some of these theories suggest for instance,

... that the literariness of a narrative work, its specific *artistic* quality, may be defined as the disjunction between story and plot – that is, the manner in which the writer manipulates and distorts causal-chronological sequence ...

And it follows that every narrative work of art necessarily includes elements that may be called spatial, since the relations of significance between such elements must be construed across gaps in the strict causal-chronological order of the text

It is obvious that the closer the structure of a narrative conforms to causal-chronological sequence, the closer it corresponds to the linear-temporal order of language Equally obvious, however, is that such correspondence is contrary to the

nature of narrative as an art form Indeed, all through the history of the novel a tension has existed between the linear-temporal nature of its medium (language) and the spatial elements required by its nature as a work of art."
(Frank, J. 1981 p. 234/35)

Here again we find a connection with the semantics of metaphor: Eco, in an analysis of *Finnegans Wake*, puts forward the thesis that the mechanism of metaphor is essentially based on suppressed and therefore hidden contiguity. [cf. Eco 1981, ch 2]

2.
Diagrams or other graphic devices very rarely occur without verbal explanations in mathematics texts because they usually involve conventions referring them to the "stream of social consciousness" and causal-chronological action. The need for reverbalization or verbal commentary is also linked to the exploratory status of these visualizations. Diagrams often imply a tremendous condensation which sometimes makes their effective handling for operational purposes rather difficult. A textbook example:

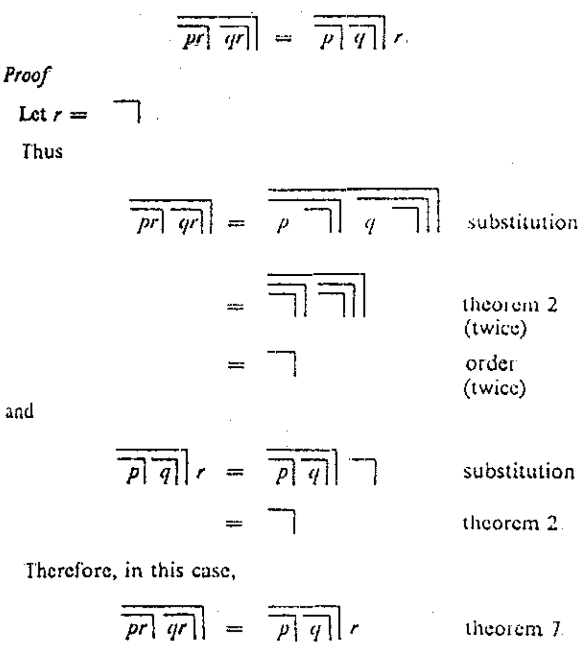


Figure 1
G Spencer-Brown (1979, p 23)

Any formula, for instance the formula giving the area of a triangle, has essentially two sides On the one hand, it is an operational diagram providing a procedure to perform a well-specified task (a calculation). On the other hand, it is an idealized or abstract model which shows relationships and aspects essential for the terms in question Conceived as a model, such a diagram serves explanatory and exploratory purposes [cf. Otte 1980].

To find the area of a triangle, multiply the base and the height Then take one half of the product

$$A = \frac{1}{2} bh$$

$$A = \frac{bh}{2}$$

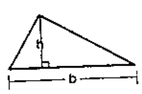


Figure 2
McGraw-Hill (1981, p 172)

Figure 2 demonstrates quite clearly that a presentation in verbal language is very suitable for operational interpretation, but that an idealization of the problem depends on variables and other diagrams Therefore "mixed presentations", following Eco's statement that a well-organized text, on the one hand, presupposes a competence and works to build up this competence on the other, are appropriate sometimes:

The SD is just the r.m.s. size of the deviations from the average. As a formula,

$$SD = \sqrt{\text{average of (deviations from average)}^2}$$

Figure 3
D Freedman (1978, p. 63)

Number is the most important aspect of logical and sequential mathematical language People, who – like statisticians or EDA-experts – have to work a lot with numerical data, may develop a sensitivity for visualization as a necessary complementary facet. An elementary but very effective 'graphical method' in EDA (= Exploratory Data Analysis) is the so-called stem-and-leaf display. The simplest – and most useful – meaningful mark is a digit By taking certain digits as the stem and one more as the leaf, we can stack up the leaves on the stem. Thus, 39, 31, and 33 combine to yield

3/913

which is compact, quickly written, and easily scanned Figure 4 shows an application of this technique (stem-and-leaf display) to a simple example Extensions and modifications to meet a variety of problems are easily possible

Heights of 219 of the world's volcanoes

A) STEM-and-LEAF—unit =100 feet (rounded)

8	0	99766562
18	1	9761009630
40	2*	6998776654442221109850
58	3	876655412099551426
80	4	9998844331929433361107
103	5*	97666666554422210097731
√(18)	6	898665441077761065
98	7	9885543110065210807 3
78	8	653322122937
66	9*	377655421000493
51	10	0984433165212
38	11	4963201631
28	12	45421164
20	13*	47830
15	14	00
13	15	676
10	16	52
8	17*	92
6	18	5
5	19	39730

Figure 4
J.W. Tukey (1977, p. 73)

And replying to the question "Why are graphics useful?" Tukey said:

"Human beings perceive visual patterns more readily than patterns in collections of numbers. This is especially important in exploratory data analysis because pictures dramatically reveal things that we did not expect to find in the data. By contrast, numerical summaries tend to cover up any structure that they were not specifically designed to handle." [p.30]

Tukey's statement implies an essential connection between the "numerical character" of the methodology of classical statistics and its being oriented towards mere hypotheses testing, its being confirmatory rather than exploratory. Graphical methods serve, according to Tukey's view, above all exploratory functions. There is a diverse tradition of ideas of this kind. [Tukey 1977, Macdonald-Ross 1979]

Skemp, in a similar vein, summarizes the "contrasting and largely complementary properties of the two kinds of symbols" in the following way:

Visual	Verbal-numerical
Abstracts spatial properties, such as shape position	Abstracts properties which are independent of spatial configuration, such as number
Harder to communicate	Easier to communicate
May represent more individual thinking.	May represent more socialized thinking
Integrative, showing structure	Analytic, showing detail.
Simultaneous	Sequential
Intuitive	Logical

[Skemp 1971, p.111. We have changed the heading of the second column from "verbal-algebraic" to verbal-numerical"]

Classifications of this type represent simplifications which must be handled with great caution and reservation. A "principle of complementarity" would seem appropriate which takes into account that effective communication and cognition will always depend on an interaction of both types of symbols. With regard to school mathematics, we think that the natural numbers, the discrete and the visual means of plane geometry, are two types of representation of fundamental importance and that a principle of complementary interaction of both types of symbols is essential when mathematical knowledge is presented to pupils.

If one should not want to have the above term of "complementarity" seen as mere wording, one must develop some ideas about the concept of "activity", which is the means of mediating between the subject and the text. The introduction of a dualism as above is neither whimsy nor simply a manipulation of form; but understanding it – that is, understanding the properties of different symbol systems in a purely objectivist manner, – would turn the reader's activity into a mere process framed by the text according to a behavioristic stimulus-response model. Texts of programmed instruction would be the prototypes and paradigms for all textual materials. Hence to the question, "Is the text or the reader the source of meaning?", we should have to reply: "neither or both!" How can we be a little more specific? Both the subjectivist and the objectivist account of understanding consider metaphor as merely a deviant form of normal usage of language or as void. [cf. Otte 1981 p.25, Lakoff/Johnson 1980 chap. 26-28] In the sections to follow, we shall therefore continue to contrast the above described duality of the symbol systems with the complementarity of literal vs metaphorical understanding in an exemplary and experimental way.

3.

The above classification often has epistemological and social connotations. The geometric-graphical symbol system shows something like a "Gestalt" (like wholeness of appearance or pattern of relationships), the numerical system helps to establish order, infinite progression. Sometimes one extrapolates from these aspects, asserting that iconic imagery does not have a voluntary status in behavior comparable to verbal-numerical representation.

Classification, in particular, is a fundamental operation of knowing in school. Arithmetical representation is often seen, in mathematical education, as rather closely linked to formal logic and as especially appropriate for classification or for categorization. This might be a good reason why mathematical instruction in our schools concentrates very much on arithmetic, whereas geometrical representation is seen as serving heuristic and illustrative purposes only. This practice cannot be explained or substantiated on purely epistemological grounds. Gimeno (see Figure 5), for instance, provides plenty of material showing how to use graphics for categorization and classification. [Gimeno 1980]

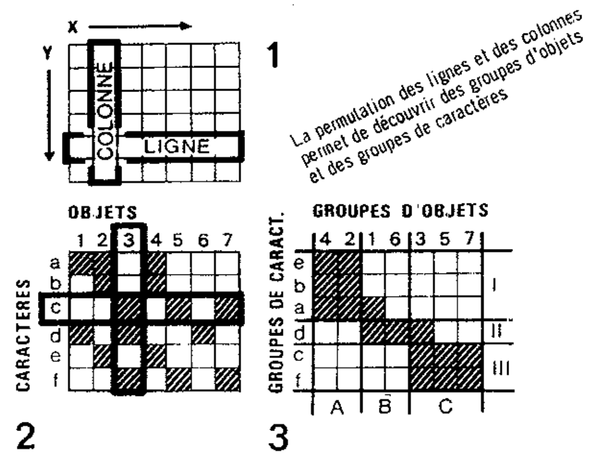


Figure 5
R. Gimeno (1980, p. 14)

The different codes may serve similar tasks, but with very different accentuation and dynamics. If one understands both the "sign-models" and their handling as part and function of the system of cognitive activity, it is easy to see that different types of such models, different "visualizations or embodiments", so to speak, mediate in different ways between consciousness and performance. Thus we get a very rough structural description of the system of cognitive activity: "generalizations of fundamental importance modelling ideas procedures". [Judin 1978, p 323]

S. Papert reports on an experiment in which various proofs of the irrationality of the square root of 2 differing along an "Gestalt vs. atomistic" dimension were presented to nonmathematicians. While the traditional algorithmic, essentially serial proof is experienced as very convincing and powerful "in the way one is captured and carried inexorably through the serial process" [p.202], the more theoretical proof (comparing the number of prime factors on each side of $p^2 = 2q^2$) attracts by its brilliance and immediacy, but depends in its power of conviction on a person having "the right frames of perceiving numbers"

In mathematics education, an empiricist epistemology prevails which usually does not aim at providing the necessary generalized frames. As a consequence of formal pedagogical considerations, relational understanding is sometimes thought inappropriate for learners For an example, let us look at the following description of the "distributive law" in algebra $(2+3)\,2 = (2.2) + (3.2)$ is presented in the following manner showing that numbers 2 and 3 have exemplary substitutional character, i.e 2 and 3 are numbers such that .... etc. ..

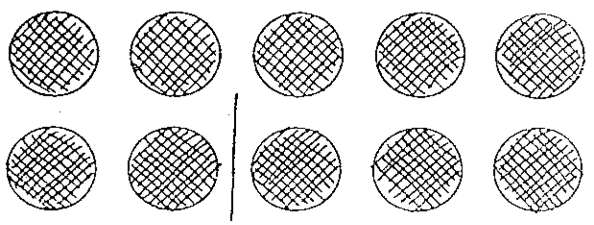


Figure 6

The illustration indicates that the distributive law applies to all triples of natural numbers The only thing missing is the appropriate expression in the symbolic representation

The algebraic formula

$(a+b).c = a.c + b.c$

may be "justified" by means of the following picture

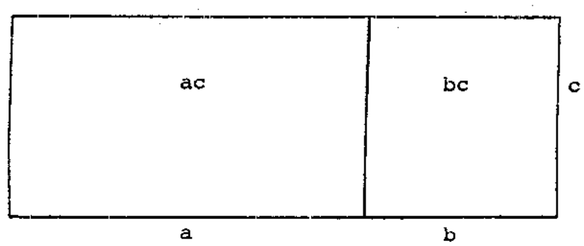


Figure 7

As there is no immediate, direct way of reading off the numerical values for the length and width of the rectangles shown, we cannot but operate with symbolization by means of variables. Indeed, "area" here just means a bilinear function (a tensor), and is not related to scales. At the same time, the distributive law's field of application is extended from natural numbers to positive real numbers.

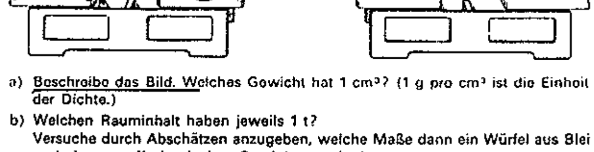
Strangely enough, Figure 7 meets many objections among mathematics educators. Somebody might, for instance, say: "I, and probably Polya, would provide one or more numerical examples and try to get the students to make their own abstractions". Or: "The first exposition gives a valid proof, because it suggests the possible substitution of 2 and 3 by any other numbers, whereas Figure 7 gives only a visual illustration". In fact, these and similar lines of argumentation frequently encountered are exactly those which lead to the transformation of text A into B.

The MTR-Project at the University of Hull reported that, according to their results, Figure 6 and even more Figure 7 were "both unnecessary and confusing appendages" of an "examples-rule" teaching sequence. The generally negative effects of diagrams seems due to the complete negation of the exploratory possibilities offered by diagrams. The MTRP Report itself says that the pay-off of diagrams "seems to be a result of an approach rather than a method " This approach is described as follows:

"When one learns mathematics, verbally, as in any subject, one can only receive the information linearly. If, however, a visual representation is made, it is possible to show two ideas or more in juxtaposition – for example, the area of a triangle and the area of a rectangle If the representation is chosen carefully, and/or is ingenious enough, it will exhibit a relationship between the hitherto unrelated ideas. Thus, by this means one is able to show the pupils that mathematics is not just a series of unrelated facts but possesses many interconnections not only between various branches, but also within one branch." [MTRP Report 1969, p 16]

4.

Before continuing my line of argumentation, I shall present, in Figures 8-15 some randomly chosen, but illustrative, examples of the balance metaphor in mathematics text books. How much thinking may be triggered by viewing a simple task like the following:



a) Beschreibe das Bild. Welches Gewicht hat 1 cm³? (1 g pro cm³ ist die Einheit der Dichte.)

b) Welchen Rauminhalt haben jeweils 1 t?
Versuche durch Abschätzen anzugeben, welche Maße dann ein Würfel aus Blei und einer aus Kork mit dem Gewicht von 1 t hat

Figure 8

H R. Jacobs (1979. p. 207)

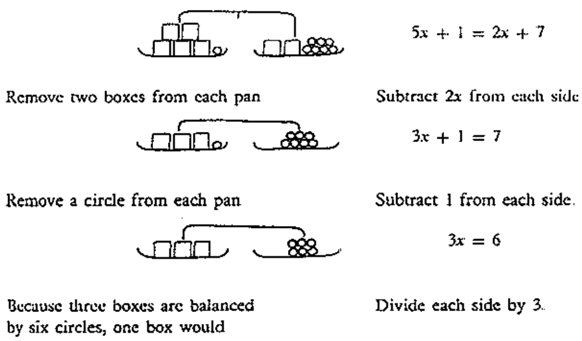


Figure 9
Winter-Ziegler: Neue Mathematik 6 (1976, p 139)

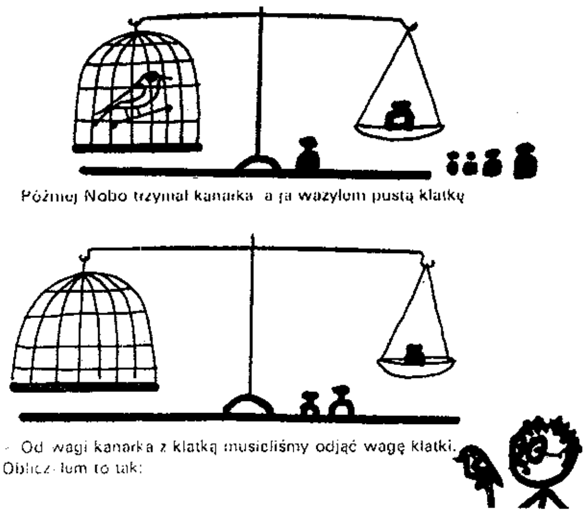


Figure 10
W Zawadowski: matematyka 4 (1979 p 135)

To me these pages show that an algebraic equation can in fact be looked at as an interaction and is not a static relationship This dynamic view is what the metaphor of balance stands for Or should we rather say it is "identity based on interaction" in contrast to tautological identity, which is based on strict, tautological self-reference $a = a$? [This would demand a classification of types of self-reference: "This statement is false" vs "This statement is written in italics" In the first sentence, the meaning of the statement speaks for itself, in the second, we find an expression relating two *different* aspects of the same thing to each other Hence, we have strict self-reference in the first case only, whereas in the second we have interaction cf Levin 1982]

The metaphor "the equation is a balance" ("the night is a velvet blanket") is not an operational diagram showing rules for solving algebraic equations In fact, this metaphor is not at all to be understood from the viewpoint of (mathematical or didactical) *method*! It is addressed to a very general idea, an idea fundamental to cognitive orientation. We may notice the (only seemingly) paradoxical fact, that the general is used to explain the particular and not vice versa

As to the epistemological shift involved here (which in our case is accomplished by an interplay of metaphorical construction and visual imagery), Krajewskij says with regard to the problem of theory reduction:

"We explain an old theory by a new one because the latter is usually more general (explanans must be, of course, not less general than explanandum) True, in the past, when people tried to explain phenomena animistically they did not use general laws but "explained" unknown phenomena by known (or seemingly known) ones The situation was similar with the mechanistic explanation, e g the consideration of an organism as a machine However, in genuine scientific explanation we have an inverse relationship: we explain the better known by the worse known (e g the rainbow by the laws of optics) because we usually know particular phenomena better than general laws" [Krajewskij 1977 30/31]

The terms "general" or "generalization" are not used here as signifying explicit theoretical knowledge, but rather as signifying a level of human experience, where its different forms may meet (personal vs social, practical vs theoretical, aesthetic vs scientific) So I use these terms as parts of a meta-knowledge which is probably implicit and intuitive

One more example:

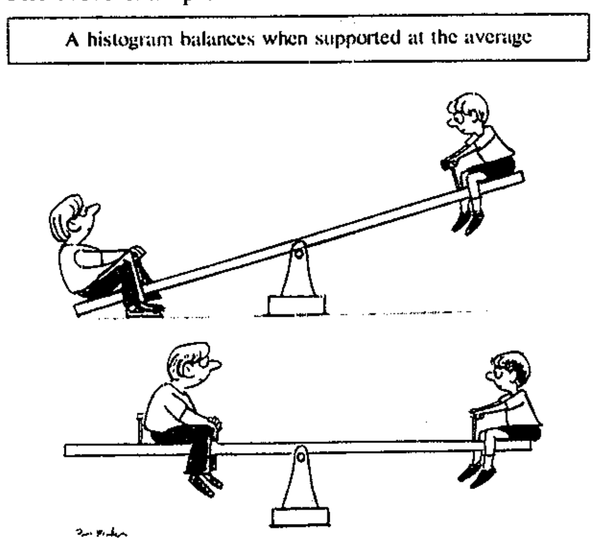


Figure 11
D Freedman (1978, p 54)

And some more:

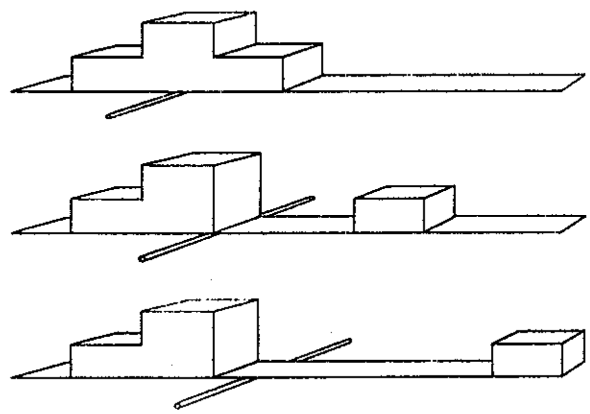


Figure 12
D Freedman (1978, p 55)

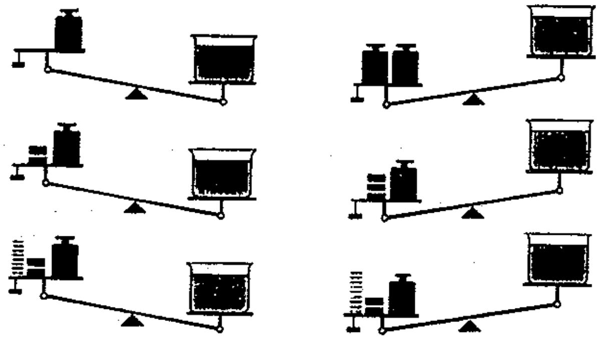


Figure 13
Sorger, Freund, Röhrl: Treffpunkt Mathematik 6 (1980, p 75)

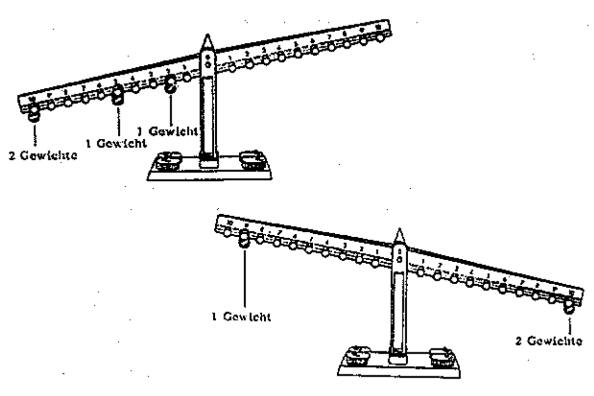


Figure 14
I. Sealey (p 6/7)

Compare Figures 13 and 14 with the following Figure 15, which seems very static and stupid.

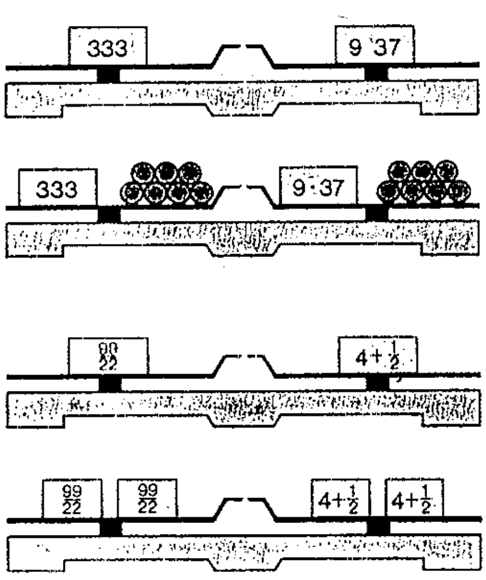


Figure 15
Hahn. O./Dzewas, J.: Mathematik 7 (1979. p. 135)

5.
This simple experiment – searching out visualizations by means of the balance – stimulated me to take a moderate sample of mathematical textbooks from my shelves and choose a "random" page from each for inspection.
At the university level a "typical" page might look like:

To define γ_1, we note first that, for every large prime q_1, there are $\gg q_1$ numbers γ of the form $(p_1/q_1)\alpha_1$ in the interval (1, 2), where the implied constant depends only on α_1, and these have mutual distances $\gg q_1^{-1}$. Further, there are $\ll q_1^{\frac{2}{3}}$ rationals β with $b^3 \leqslant q_1$ and so there are $\ll q_1^{\frac{2}{3}}$ numbers γ satisfying $|\gamma-\beta| \leqslant q_1^{-1}$ for at least one such β. We can therefore select γ_1 so that (iv) holds, and then, by Theorem 7.2, we can choose λ_1 so that the conditions concerning $|\gamma_1-\beta|$ are satisfied. We shall show in a moment that also (ii) holds in the case $j = 1$ if $q_1 \gg 1$.

Now suppose that $\gamma_1, \ldots, \gamma_{j-1}$ have already been defined to satisfy the above conditions; we proceed to construct γ_j. Constants implied by $\ll$ or $\gg$ will depend only on the numbers so far specified, including possibly $\lambda_1, \ldots, \lambda_{j-1}$. First let J'_{j-1} be defined like J_{j-1} but with the additional restriction that the heights of the β in question satisfy $b^{3n} \leqslant q_j$. Clearly the number of β for which the latter inequality holds is $\ll q_j^{\frac{2}{3}}$ and so J'_{j-1} consists of $\ll q_j^{\frac{2}{3}}$ intervals. Further, J'_{j-1} includes J_{j-1} and so, by (ii), we have $|J'_{j-1}| \geqslant \frac{1}{2}|I_{j-1}| \gg 1$. It follows that, for any large prime q_j, there are $\gg q_j$ numbers γ in J'_{j-1} of the form $(p_j/q_j)\alpha_j$, where p_j is an integer $\ll q_j$ with $(p_j, q_j) = 1$. Furthermore, any such γ is in fact in J_{j-1}, for if the height of β satisfies $b^{3n} > q_j$ then $B > q_j^{(3n)^3}$ and thus, on noting that $(q_j/p_j)\beta$ has height $\ll q_j^n b$, we obtain from Theorem 7.2

$$|\gamma-\beta| \gg q_j^{-1}(q_j^n b)^{-3n} > 2B^{-1}$$

Figure 16
A. Baker (1975 p. 93)

A school textbook typically would be much more visual or graphical. One reason for this difference seems to be that mathematicians (as well as university students of mathematics) already have a fixed way of "seeing" mathematics. This again corresponds to the fact that observation of the metaphorical structure of our conceptual system is, on the epistemological level, caused by an interest in understanding, i.e. in the human relations towards knowledge and not in questions of knowledge structures as such.

Only if a page concentrates on conveying strict mathematical knowledge in an effective and economic way does it begin to look similar. Classical mathematics is not very attractive "from the eye's viewpoint":

Um eine Gleichung dieser Form zu erhalten, geht man im allgemeinen in folgenden Schritten vor.

1. Schritt: Man lege, falls nicht vorgegeben, den Lösungsgrundbereich so groß wie möglich fest.
2. Schritt: Falls in der Gleichung Brüche auftreten, multipliziere man die Gleichung mit dem Hauptnenner dieser Brüche.
3. Schritt. Man beseitige vorhandene Klammern durch Anwenden des Distributivgesetzes.
4. Schritt: Man vereinfache die beiden Terme der Gleichung durch Zusammenfassen der Summanden, die die Variable enthalten, bzw. der Summanden, die die Variable nicht enthalten.
5. Schritt: Man addiere einen Term, der die Variable enthält, zu beiden Termen der Gleichung, so daß die Variable in einem Term der Gleichung nicht mehr auftritt.
6. Schritt: Man addiere einen Term, der die Variable nicht enthält, zu beiden Termen der Gleichung, so daß der eine Term der Gleichung die Variable enthält, der andere Term der Gleichung die Variable nicht enthält.
7. Schritt. Man dividiere beide Terme der Gleichung durch den Koeffizienten der Variablen, falls dieser ungleich Null ist.

S Schritt Man gebe die Lösungsmenge bezüglich des vorgegebenen Lösungsgrundbereiches an

BEISPIEL 9 (5.1.2):
Man löse die Gleichung

$$\frac{1}{2}+\frac{2}{x+1}-\frac{1}{2(x+1)}=\frac{1}{x+1}+\frac{5}{8} \quad \text{(der Grundbereich sei } G\text{)}$$

Figure 17
Textbook from the GDR (p. 95)

Sometimes when one is confronted with overcrowded, dispersed and parcelled out pages one would like to have the uninterrupted progression and full explications of verbal language. It may be pleasant in a market place to be all the time distracted by a new optical stimulus, but the excitement lasts only as long as the reacting attention is appropriately rewarded. As soon as stimulus-response becomes something of an end in itself; as soon as an empiricist epistemology leads to mere naming within a world full of objects instead of a genetic development of useful generalizations, the reader becomes both confused and bored:

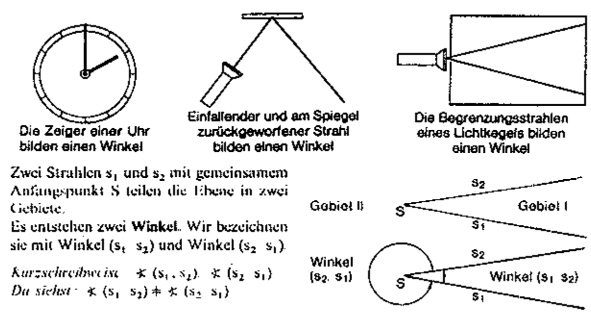


Figure 18
Hahn, O./Dzewas, J. (1978, p. 35)

If we now, for the moment, concentrate on good and instructive examples, we find several graphic means or methods frequently used. Some of those will be described in the following. First of all we may notice 2- (or 3-) dimensional organization as an aspect of this "visual character":

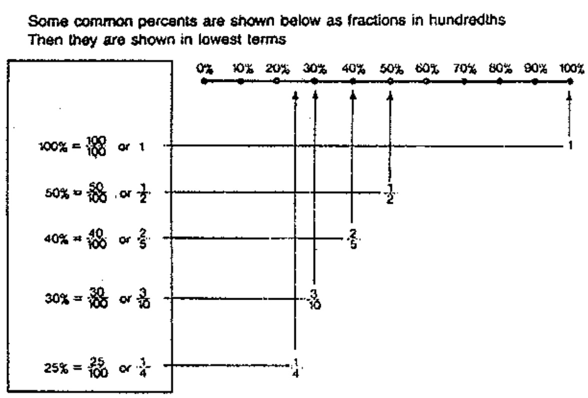


Figure 19
From an American school book

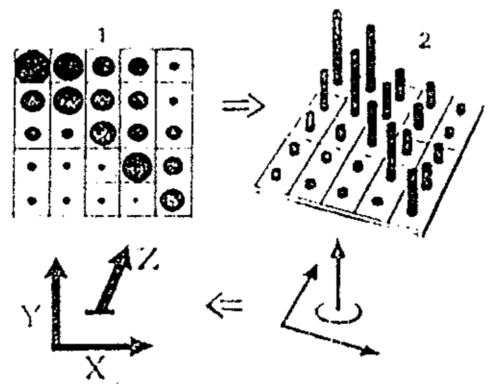


Figure 19a
Bertin (1981, p. 181)

Sometimes we find, in addition to this aspect, ideographic elements, which "speak" through the very arrangement of visual symbols with only a "verbalized" commentary. Unlike a diagram an ideograph does not really need a verbal explanation. On the other hand it is not unequivocal, it has very often to be understood "metaphorically" (remember the balance!). Sometimes it is important to have the correct frame when looking at an ideogram. For instance, the graph of a function describes a relationship within a problem situation and is not a picture of the situation itself.

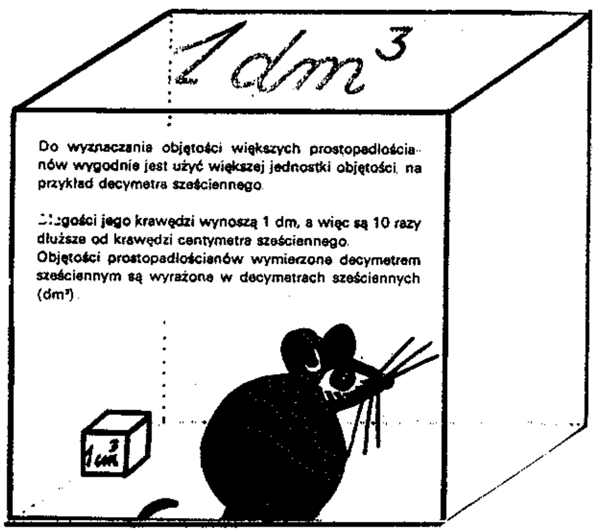


Figure 20
W. Zawadowski (1978, p. 166)

Having the appropriate point of view is sometimes a problem of social psychology. Gombrich has argued that "the visual image is supreme in its capacity for arousal, that its use for expressive purposes is problematic, and that unaided it altogether lacks the possibility of matching the statement function of language." [Gombrich 1974, p. 242/43] This observation cuts several ways with all of its aspects. Therefore I think Cajori has hit on a relevant and difficult problem when he says: "The inspiration of genius has led to a large output of new symbols intended to express all mathematics in purely ideographic form. But this is a phase in the development of our science on which the inspiration of genius is not sufficient; there must be brought to bear upon this problem the wisdom of many minds, and that wisdom discloses itself in the history of the science. New symbols are against the too exclusive use of ideographic signs." [Cajori 1924, p. 941]

In advertisements for tunny-fish or for "natural gas" we find a similar way of communication

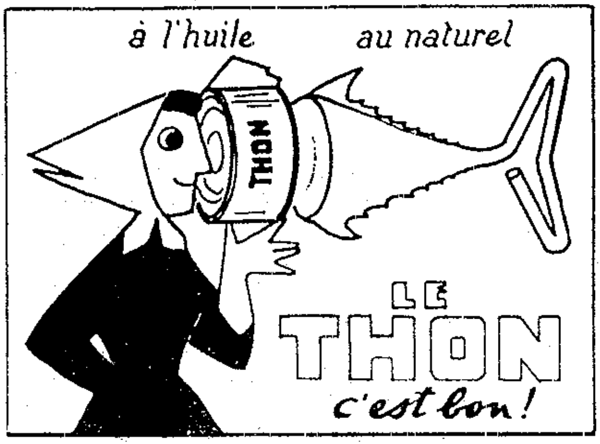
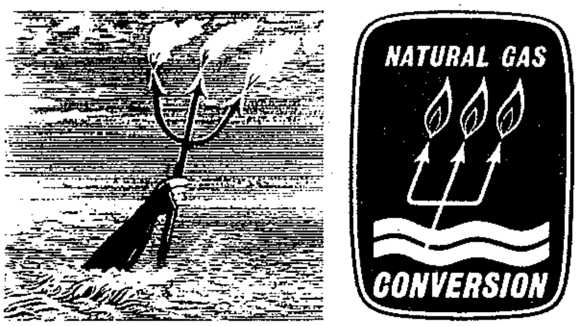


Figure 21
A. Leroi-Gourhan (1964, p. 277)



TRIDENT TRADEMARK adopted for North Sea Gas in Britain combines a symbol of the sea, Neptune's trident, with burning gas. Initially the idea was done realistically (*illustration* at left). Later it was encoded in an abstract form making it easier to remember.

Figure 21a
Gombrich (1974, p. 263)

Literature and advertising are two fields from which something can be learned about textual strategies. A third is propaganda during revolutionary periods in history. An illustrative example of textual propaganda for both new social ideas and political action provide the more than 1600 posters of ROSTA (= the russian telegraph office), which were published between 1919 and 1922 in Moscow, during a historical period of great social agitation and collective ingenuity. This is not the place to study this example of mass communication thoroughly, but we should like to point out one of its aspects which is especially relevant for mathematics education. The ROSTA posters had, on the one hand, to use current phrases and wellknown language, but on the other, had to convey new ideas in an instructive and striking manner to a mass public. To these ends, an interplay of verse and visualization was developed which, by uncovering the essential metaphorical character of the current and well-known verbal phrases made them particularly impressive. The well-known was revealed as something new and unexpected, providing a strong motivation to learn about the new ideas. [C.W. Duwakin 1975, p. 84/85; for other examples of the possibilities offered by the literal illustration of metaphors see Gombrich 1974]

27 ROSTA Nr. 409 Oktober 1920

1. Genossen! Dornig ist der Weg zur Kommun'!
2. Es gilt, ihren Vortrupp gut zu beschuhn.
3. Euer Fuß bricht Dornen und Disteln genug:
4. so geht's zur Kommune ohne Verzug

Figure 22
W. Duwakin (1975, p. 107/111)

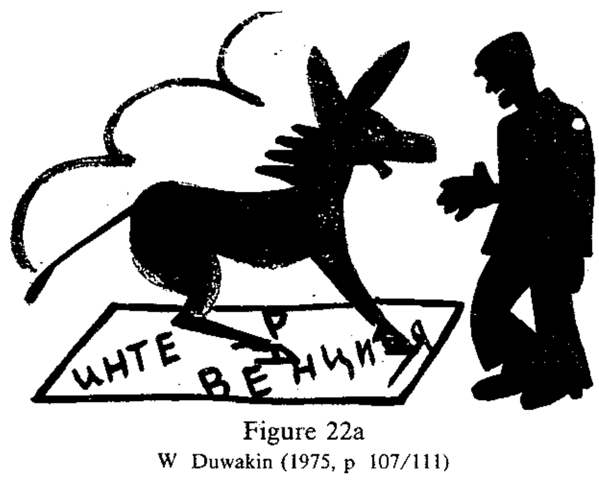


Figure 22a
W. Duwakin (1975, p. 107/111)
Detail from 21 ROSTA Nr. 335

We need metaphors to evoke new insight and it seems as if most of the conceptual system of school mathematics is metaphorically structured The essence of metaphor is sometimes seen as "understanding one kind of thing in terms of another", a statement which connotes that the seemingly concrete and known thus explains the abstract and general. There is a demand that the new notions be exposed in terms of structures which the student has already mastered

But we have seen (remember the balance!) that such a reductionist theory is in no way suitable to explain strategies of textual presentations. In fact, to avoid infinite regression, these theories would be compelled to go so far as to deny the possibility of learning from written instruction

The relation of literal vs. metaphorical meaning signals a circular connection of different levels of understanding which seems to be essential To put it in different terms: to develop an understanding of a piece of theoretical knowledge, one must at the same time, develop some understanding of the very nature of theories and theoretical concepts Knowledge is developed together with "knowledge about knowledge", i e meta-knowledge But the regulatory principles deriving from meta-knowledge are, on the one hand, an indispensable condition of understanding, and the product of the evolution of cognition on the other. The prominent role played by graphic devices and pictorial presentations in mathematics textbooks seems to be an important aspect to understanding this circular connection

"Interaction" of real or allegorical characters is another element of the design of the Rosta posters. Dramatic dialogue of special "characters" (which may be persons, animals or just symbols) is also part of the non-linear, visualized communications of school textbooks I think drama is, from the epistemological as well as from the cognitive viewpoint, the most interesting type of text because it combines in a very outstanding way metaphorical accentuation with an utilization of the contiguity relations established by metaphorical figures in order to achieve direct understanding. It is no accident that the history of science abounds with dialogues of fundamental prominence.

The following page from a Polish text book by B. Chrzan-Feluch and W. Zawadowski shows a varied set of the textual elements described so far

The text reads as follows: "On which weekday will I be finally grown up? I was ten yesterday. Yesterday was Sunday. My 18th birthday should be a Sunday too I should like that. I should not like to come of age on any day during the week Can I figure it out? Starting with last Sunday it will be 8 years exactly How many days? .." And the last paragraph: "417 weeks plus 3 days. Starting with a Sunday 417 weeks leads to a Sunday again and still another 3 days: Monday, Tuesday, Wednesday I'll come of age on a Wednesday A Wednesday after all! Bye, Gapcio. When will you come of age?"

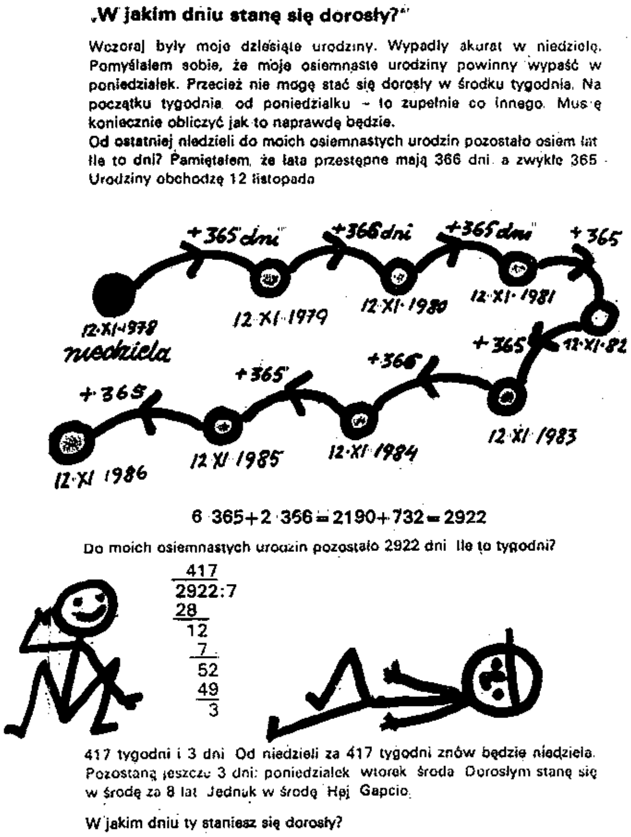


Figure 23
W Zawadowski (1978 p. 33/34)

6.
(School) mathematics is very much confined to "paper and pencil" intuition and a fundamental issue of the pedagogy of mathematics is concerned with the question of how to make written communication ideographic or how to make mathematical thinking visual and active. Although we have many pictures, graphics, and even much colour in the school textbooks of to-day, we usually find too little flexibility. Large parts of mathematics texts are used for mere technical advice, for cosmetic decoration and superficial stimuli. The texts aim at a precise fixation of every single step of the student, developing for that purpose an enormous apparatus of unnecessary terminology which hides the fundamental ideas and causes confusion about the character of theoretical generalizations and about mathematics in particular The text confines the readers' activities to very narrow possibilities Mathematics textbooks are very often – just as text B above – "closed texts" in the sense of the following quotation from Eco: 'Those texts that obsessively aim at arousing a precise response on the part of more or less precise empirical readers (be they children, soap-opera addicts, doctors, law-abiding citizens, swingers, Presbyterians, farmers, middle-

class women, scuba divers, effete snobs, or any other imaginable sociopsychological category) are in fact open to any possible "aberrant" decoding A text so immoderately "open" to every possible interpretation will be called a closed one. Superman comic strips or Sue's and Fleming's novels belong to this category. They apparently aim at pulling the reader along a predetermined path, carefully displaying their effects so as to arouse pity or fear, excitement or depression at the due place and at the right moment. Every step of the "story" elicits just the expectation that its further course will satisfy. They seem, to be structured according to an inflexible project."[p 8]

We have characterized this aspect by the term "overmethodization" (Übermethodisierung). The textbook authors "desire too much" and make too little use of the mathematics itself Obstructive methodization of mathematics textbooks is caused today mainly by a wrongheaded and superficial understanding of the motivation problem Trying to fix the child's activities in an ever narrower way cannot be an answer to this problem Pictures are used to distract the pupil's concentration so as to "hide away the theoretical character of mathematical generalizations" from him Some books even say that they are not designed to teach mathematics better, but to help the child to understand *why* mathematics is important It is easy to sympathize with the humane intention, but this intent as well as the sympathy remain totally abstract and therefore harmful as long as we do not take the child (or the teacher) as an active and capable self and mathematics as an open space for that self to grow "Mathematics for everyday life" or "Mathematics for the real world" are titles of textbooks obedient to the understanding that mathematics is important and motivated because it is useful in everyday business. But first of all this type of motivation does not work When a book suggests to the pupils that the reason for those many hours of arithmetic is to be able to check the change at the supermarket, it is not believed Apart from that the belief in this type of direct learning would lead to a behavioristic drill and routine style of instruction We have seen that the "inflation" of terminology as well as pictures, which have no real connection with the ideas and which are not seriously and genuinely used, not only express a very poor self-organization of the books but are an indication of empiricist epistemology as well as of behavioristic psychology

The terminological inflation, especially at points of some complexity, is a very telling sympton indeed. "Fractions" are such an instance of complexity Fractions, indeed, are the first instance where number shows its *relational* character in an essential way Number x results from the diagram (1) in text A of section 1: $b = x.a$, where b is any quantity and a a standard. The equation is established by measuring (counting, comparing etc.) Number is defined by the relationship between a and b, not by either a or b alone. If b means a set of apples, one apple will be a "natural" unit and it will not be necessary to interpret x as designating a relation. With fractions, the situation is quite different My motivation for knowing fractions arises from the fact that many problems demand the finding and comparing of relevant relations. There is no fixed standard and neither is it settled beforehand which quantities should be compared to which standards This means that fractions are means for solving *algebraic* problems, not arithmetical ones

Now research has documented that the chapters on fractions in German textbooks for grade 6 contain 400 special terms on the average, more than 30 % of which occurred in these chapters only. [Lörcher 1976] Even the simple task of comparing two fractions, as fractions are defined very often simultaneously as "points on the numberline" or "part-whole relations" or "measures" and "quantities" or "ratios" and "quotients" or "operators" and "machines", gives rise to another flood of terms and verbal expressions.

No genetic approach either to knowledge or to instruction is possible without using the flexibility of theoretical ideas and substantial generalizations. Only the complexity of theoretical generalizations which are determined both objectively and socially makes it possible to link individual development to the rich possibilities of the "real world" And we need symbols and texts which allow us to study and use theoretical generalization as such, in "pure form". Or, stated in terms of the theme of this paper: Without the availability of texts, no systematic and profound mediation between the complexities of scientific, cultural and productive progress and the individual's development is conceivable. The complexity of social life demands general organization of learning activities. In order to relate this process to individual conditions, representations by means of texts are indispensable.

This does not mean that texts or other "technologies of the mind" determine the content proper in a direct way, but they influence the way knowledge is conceived within a society, and the individual's relationship to knowledge. Such questions are usually described in terms of *style*, and as schooling establishes above all else both the individual's conscious and unconscious relationships to knowing, matters of style are of great importance in school.

"Indeed, like any social insight, the judgements that teachers make with regard to students, particularly in examination situations, take into account not only knowledge and know-how, but also the intangible nuances of manner and style, which are the imperceptible and yet never unperceived manifestations of the individual's relationship to such knowledge and know-how . " [Bourdieu a o 1974, p. 338]

The sociology of education tends to see matters of style first and foremost as means of social stratification while neglecting or even denying the cognitive potential. With reference to the development of the style of doing geometry and of knowing geometry, Papert writes: "Euclid's is a logical style, Descartes' is an algebraic style Turtle geometry is a computational style of geometry " [Papert 1980, p 55] In our view it is not accidental that these steps in the development of geometrical thinking were parallelled by major changes in the means of knowing and communication such as the shift to alphabetic literacy (Greece), the invention of the printing press and the availability of the computer.

7.

Supposing that school mathematics is still tied to "paper and pencil technology", we may interpret our considerations so far by the following two theses:

- The character of school mathematics is essentially algebraic The variable x and the algebraic equation are the most important of its diagrams.

As texts, being material artefacts, provide proper opportunities for constructive rumination, algebra is reflective as well as exploratory. Its subject-matter is relationships and interactions between objects bound by paper and pencil intuition. Mathematics in schools is modularized and depends in its proceedings very much on juxtaposition by means of spatial forms and graphical devices. Even at points where it cannot affirm its "algebraic epistemology" it is not able to really stand on a different one either (i.e algorithmic or stochastic ....). The most far-reaching and fundamental arguments to substantiate and justify these facts have been provided by C. S. Peirce I'll render two notations from his work which may sound at the same time both familiar and a bit strange:

"Many mathematicians insist that merely talking about a man's income, without saying how much it is, and saying that $4000 being subtracted from it and the remainder divided by two will give the tax, at any rate going on to remark that if the tax be multiplied by a hundred and $4000 be added to the product the amount of the income will be ascertained, – some mathematicians insist that that is algebra, though no letters are used. It is the spirit of algebra, we may grant But as faith without works would be hollow, and confession without heartfelt repentance and amendment would be of little avail; so if you really and truly are attending to operations on the quantity, abstractedly from its value, you certainly will try to express it and them as commodiously as you can " [Peirce 1976, p 322]

"It may seem at first glance that it is an arbitrary classification to call an algebraic expression an icon . but it is not so For a great distinguishing property of the icon is that by the direct observation of it, other truths concerning its object can be discovered than those which suffice to determine its construction Thus, by means of two photographs a map can be drawn, etc. ... This capacity of revealing unexpected truth is precisely that wherein the utility of algebraic formulae consists, so that the iconic character is the prevailing one " [Collected Papers, vol 1, 279]

- Second, although the concept of "meaning" has to be related to realities outside language (be it oral or written), textual strategies nevertheless reflect the logic of extra-linguistic experience of the world

Particularly the way a text refers to itself mirrors the idea of knowledge as well as the forms of knowing it wants to convey Every evaluation of the relation between knowledge and its description, between idea and representation implies a view on the other relation between reality and knowing of that reality.

The over-methodized textbooks provide a poor image of mathematics by missing appropriate ways of self-organization. If mathematics education as a field of discourse can, on the one hand, be defined with only little exaggeration as the production and optimization of textual materials and has, on the other, not developed any competence in really understanding this medium, it would seem useful to explicate such a paradoxical situation more systematically Excessive methodization represents, in fact, a set of symptoms, which show the diagnostic role of the medium of "text" for all didactical questions. Over-methodization expresses first of all the reduction of mathematics education to a bunch of mere methodical devices.

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Unlike more superficial forms of expertise, mathematics is a way of saying less and less about more and more. A mathematical text is thus not an end in itself, but a key to a world beyond the compass of ordinary description.

G. Spencer Brown, *Laws of Form*

This book is not written for the Furious of Pace, for it would like to linger with the things themselves.

H. Keller-von Asten, *Encounters with the Infinite*
