# $F(X) = G(X)$?: an Approach to Modelling with Algebra

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The teaching of high school algebra in the United States is frequently criticized as overly focused on manipulation skills with no regard for conceptual understanding [e.g., Heid and Kunkle, 1988]. For example, the terse directions for the symbolic manipulations in algebra (Factor, Simplify, Solve, Multiply and simplify, ....) indicate the non-conceptual nature of these problems. These directions give students little indication of the goal of the problem and do not indicate how one can test the appropriateness of an answer.

Similarly, the relationships between the central objects of study in algebra is not conceptually clear. Equations, defined imprecisely as mathematical sentences with an "=" sign, are the central object of study during the first two years of algebra. In contrast, functions, defined formally as special kinds of relations between sets, are introduced with comparatively little emphasis.[1] Thus in algebra texts, $x^2 + y^2 = 5$ is an equation which expresses a relation between the sets represented by the $x$ and $y$ variables. Some equations, like $y = 4x + 2$, indicate a functional relationship. From this perspective, functions seem to be special kinds of equations. Yet, at the same time, the correspondence rules for functions, like $f(x) = x + 4$, are called the equations of functions. From this contrasting perspective, functions seem to be a higher level concept than equations, since functions "have" equations, tables of values, and graphs.

In order to teach for a greater understanding of algebra, a necessary, but not sufficient, step is that we think about the content of algebra differently. One oft-mentioned suggestion for revamping algebra is to have functions replace equations as the fundamental objects of algebra [Computer-Intensive Curricula in Elementary Algebra, 1991; Fey, 1989; Schwartz and Yerushalmy, 1992; Thorpe, 1988; Yerushalmy and Schwartz, in press]. Thus, rather than appearing at the end of Algebra One texts, functions would come, instead of expressions, right at the beginning. As Thorpe [1988] makes clear, the proponents of this suggestion do not intend to teach the set theoretic definition of function. Instead, like Fey [1989], they propose a definition which emphasizes that functions are relationships where output variables depend unambiguously on input variables.

Such an approach to algebra needs to specify its view of equations and the relationship between functions and equations. This paper is an examination of one proposal for such a conceptualization of equation based on the notion of function.[2] By examining the use of this conceptualization in modelling situations, I argue that it has the potential for helping to make the process of algebraic modelling a conceptually coherent activity for students

## Reconceptualizing the relationship between functions and equations

As mentioned above, the proposed reconceptualization substitutes a more careful definition of equation and explicitly relates the two kinds of objects. To understand this definition of equation, let us start with the examination of an equation out of situational context. One can think of the solutions to an equation as the initially unknown values in a shred domain for which two functions have equal outputs. In other words an equation is a particular kind of comparison of two functions. $3x - 4 = x + 17$ is really a question. It asks, for what values of the shared domain (the default in this case is the real numbers) do the functions whose rules are $f(x) = 3x - 4$ and $g(x) = x + 17$ produce the same outputs? By this definition, $f(x) = x + 2$ is not an equation, because it is not a comparison of two functions. It is an algebraic rule which expresses the correspondence between the domain and range sets of a single function.

This formulation of equations has at least four merits. First, it helps students approach the solution of equations in diverse ways and suggests that students should not be limited to the traditional symbolic manipulations for solving equations. The solutions to equations can be found to different degrees of accuracy in three ways. Traditional algebraic solutions are obtained by applying operations which preserve the solution set to create equivalent equations. We are finished when we have written an equivalent equation whose solution set is understandable by inspection (e.g. $x = 10.5$).[3] Another method is to use guessing and testing to create a table of values and to narrow down efficiently the range in which the solution is to be found. Finally, the solution can also be obtained graphically by graphing the functions on each side of the equation and projecting the intersection points of these two graphs down onto the common domain (see Figure 1 for a picture of a solution to an equation in one variable). Each of these methods of solution can be used to argue that there is only one solution to this equation. This is one way to unify a host of pedagogical suggestions and practices from Dreyfus and Eisenberg's application of graphical solutions to inequalities in one variable [1985], graphical solutions to equations like $\sin(x) = x$, and numerical or tabular guess and test strategies for solving equations proposed in the Hawaii Algebra Learning Project materials [1992].



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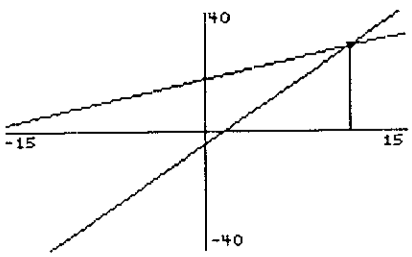


Figure 1
Graphical solution of 3x - 4 = x + 17
At 10.5, 27.5 = 27.5

Second, this conceptualization of equation can be used to clarify the basic terminology of the algebra curriculum. Equations, identities, inequalities, and what are now called relations[4] can all be treated as comparisons of two functions and solved with the same three solution methods. These comparisons can then be grouped by the same three solution methods. These comparisons can then be grouped by the number of variables and the types of functions in the comparison (e.g. a linear equation in one variable $3x - 4 = x + 5$ or a quadratic equation in two variables $x^2 + y^2 = 1$).

Third, this conceptualization of equation helps clarify the utility of algebra. Functions which have explicit correspondence rules, like the polynomial functions, have that peculiar real world utility which many theoretical mathematical constructs share. In this case, because there is only one output for a particular input, they can be used to predict outcomes prior to experience. If we have a model which suggests a rule for the relationship between inputs and outputs in an experiment, then we can investigate behavior (outputs) for inputs without running the experiment. Equations which compare two functions share this characteristic. If we can create two functions which describe the situation and equate them, then we can predict a value for which the two functions will have the same outputs without doing empirical work. It is this characteristic of this type of function which makes the modeling of situations with algebra a useful pursuit.

Finally, I believe that the dynamic dependence relationships captured by functions are accessible aspects of real world situations. Thus, an approach which views equations as built up from functions which vary, instead of expressions which represent an unknown, may make algebraic modeling more natural and coherent for students. It may help students be able to "write equations." I will argue this point by analyzing examples of modelling situations. Each of the examples is a problem that I have used in my teaching of high school algebra over the last three years.[5]

*Modelling problems*

The following situation is adapted from materials provided by the Michigan Department of Commerce in training sessions for citizens contemplating starting small businesses.

**A bike-manufacturing example**

The people who invested in starting this company did not know how to calculate break-even points. They began their business manufacturing bikes to order. They knew that with the shop that they had set up the average cost to make each bike was $60, including labor, materials, and everything else (except for rent and salary for the boss). The rent for the shop was $500 a month. The head of the company was paid $1000 a month as salary. They decided to set the average price for their bikes at $100 dollars a bike figuring that they would make about $40 on each bike. During the first month, they sold 10 bikes.

To solve this problem traditionally, we must write an equation, but it is not a typical type of problem (e.g., distance/rate/time problem) for which a technique for writing equations is readily applicable. One way to think about it is to write an equation for all of the money coming in and out of the business. Thus, we would write $100x - 60x - 500 - 1000$. The last three terms are negative because they represent money going out of the business. In order for the company to break even we want to solve $100x - 60x - 500 - 1000 = 0$.

On the other hand, this sort of business situation lends itself nicely to the $f(x) = g(x)$? Conceptualization. There are two quantities in the situation which we would like to equate. The domain of both functions is identical and the units in which the outputs are measured is the same. Taking that approach, to find the break-even point, one compares the revenue and costs functions for a particular level of sales. The revenue function depends on the number of bikes sold,

$$revenue\ (x) = 100x.$$

The costs have a fixed component, but also a component which depends on the number of bikes sold,

$$costs\ (x) = 1000 + 500 + 60x.$$

Solving the equation,

$$revenue\ (x) = costs\ (x)$$

will produce the level of sales necessary to break even.

Even though we had not studied methods for solving equations symbolically, my students [6] were able to use this view of revenue and costs to show that for 10 bikes the company was 1100 dollars in debt. Furthermore, while some found the break-even point by guessing and testing, many were able to do the equivalent of the algebraic manipulation. They argued that the difference between the revenue and the cost per bike (the margin) was $40 per bike. Therefore, to begin to show a profit, the company would need to sell enough bikes for this margin to cover the fixed costs. Some used repeated subtraction, while others divided and then rounded up to find the break-even point.

Notice that, as in the traditional solution, one could also model the situation with a single function, the profit function:

$$profit\ (x) = revenue\ (x) - costs\ (x).$$

However, the dynamic nature of functions allows us to answer many questions about the situation. For example, to see the result of selling 10 bikes, we find *profit* (10). To find the break-even point, we work backwards; we must find $x$ such that:

*profit* $(x) = 0$

Since in this case profit is a linear function, we have a single answer.

In the Shell Centre's *Language of functions and graphs* [1985, p. 150], the following, more complicated problem appears:[7]

> The point of no return
>
> Imagine that you are the pilot of the light aircraft in the picture (*not included*), which is capable of cruising at a steady speed of 300km/h in still air. You have enough fuel on board to last four hours
>
> You take off from the airfield and, on the outward journey, are helped along by a 50 km/hr wind which increases your cruising speed relative to the ground to 350 km/h.
>
> Suddenly you realise that on your return journey you will be flying *into* the wind and will therefore slow down to 250 km/hr.
>
> What is the maximum distance that you can travel from the airfield, and still be sure that you have enough fuel left to make a safe return journey?

A traditional approach to solving such a problem would be to set up a distance/rate/time chart to create an equation. The equation would then be created by equating the two expressions for distance

| Distance | Rate | Time |
|---|---|---|
| $350t$ | 350 km/hr | $t$ |
| $250(4 - t)$ | 250 | $4 - t$ |

Notice that this technique is a particular strategy which works for a particular type of problem. Students must recognize that this type of problem calls for this strategy and recall the correct technique.

Alternatively, using a similar strategy to the one used in the problem above, we try to create an equation out of the outbound distance and the inbound distance. One can create a function which describes the outbound distance as a function of time and another function which describes the inbound distance as a function of time. In this case, one would write:

*outbound distance traveled*(outbound time) = 350(out bound time)

and

*inbound distance traveled*(inbound time) = 250(inbound time).[8]

We cannot immediately make an equation from these two functions. Though the outputs of these two functions are in compatible units of distance, the inputs are not identical. Both functions are functions of time, but of different times. We need to express both distances as functions of either inbound or outbound time. From the problem, we note that the inbound time is a function of the outbound time and the total time for the roundtrip:[9]

*inbound time*(outbound time) = 4 - (outbound time).

To get a symbolic rule for inbound distance traveled as a function of outbound time, we compose *inbound distance traveled*(inbound time) with *inbound time*(outbound time). This results in the function:

*inbound distance traveled*(outbound time) = 250(4 - outbound time).

Now that we have identical inputs, using $t$ to stand for outbound time, we can write the equation:

$250t = 250(4 - t)$

The solution to the equation is a time for the outbound trip. The solution tells us the maximum time at which the pilot must turn back in order to arrive safely. However, the problem is to find the maximum distance that we can travel and not the time at which we must turn back. The outbound distance is a function of the outbound time, so using the outbound distance traveled function, the maximum time will give us the required maximum distance.

Comparing the two approaches, I believe that the $f(x) = g(x)$? approach allows me to keep the meaning of the situation in mind as I solve the problem. Instead of introducing particular techniques for particular types of problems, an understanding of equation and of the objectives in the problem guide the solution. It also seems natural to write down functional relationships which indicate what depends on what.

As the previous example indicates, this view of equation can handle complicated situations. However, one might object that such an approach cannot deal with the modeling situations currently used in the curriculum, particularly those modeled with relations. I will use a problem to illustrate how relations can be reconceptualized and solved as soon as functions or more than one variable are introduced

Linear relation problems typically involve linear combinations.[10]

> A chemist mixes $x$ ounces of a 120% alcohol solution with $y$ ounces of a 30% alcohol solution. The final mixture contains 9 ounces of alcohol.
>
> a. Write an equation relating $x$, $y$ and the total number of ounces of alcohol.
>
> b. How many ounces of 30% alcohol solution must be added to 2.7 ounces of the 20% alcohol solution to get 9 ounces of alcohol?

A traditional Algebra One solution suggests that the problem be thought of as a relation (not a function) of the form $Ax + By = C$. This relation expresses a relationship between two variables which represent the volumes of the two alcohol solutions which are being mixed. In order to write the relation, we must think about the volume of alcohol present in the mixture, as well as the alcohol present in the original solutions.

An alternative which explains the presence of three quantities is to suggest that embodied in the problem is a functional relationship. The number of ounces of alcohol in the final mixture is a function of two variables, the number of ounces of alcohol in the 20% solution and the number of ounces of alcohol in the 30% solution. By indicating that there are 9 ounces of alcohol in the final mixture, we are indicating that we are looking for elements in the domain which will produce a particular output, $f(x, y) = a$. If we do not specify one of the two domain variables, then there are infinitely many combinations which will provide 9 ounces of alcohol. To solve our equation in two variables, we can:

—solve the equation algebraically,

—generate particular values which are members of the solution set by assigning values to the two variables and checking,

or

—graph the two functions of two variables that are on either side of the equation in three dimensions and project the intersection points of these two graphs down onto our domain plane. (The result of this graphing procedure will be the familiar line in the *xy* plane.)

Because the function is linear in *y*, as soon as we specify that there are 2.7 ounces of the 20% solution, then there becomes only one possibility for the amount of the 30% solution.

This example points out a general issue which has the potential to cause confusion. When we work backwards from a particular output of a function to the members of the domain required to produce this output ("Find *x* for which $f(x) = a$."), we cross the line from evaluating a function to solving an equation. Though in some ways it does not feel natural, in cases where one of the functions is a constant, for example, when we write

$$0.2x + 0.3y = 9,$$

we are nevertheless modeling a situation in which two quantities are being equated. The function on the left expresses the sum of the amounts of alcohol present in the original solutions. The constant function on the right is the amount of alcohol present in the final mixture. When we write this equation, we are asserting that in our model of the situation all of the alcohol in the final mixture was present in the original solutions.

## Ramifications

The proposal to view equations as the comparison of two functions is at the same time radical and continuous with current practice. While the material to be studied in algebra is changed very little, there are many changes in order, pedagogy, and emphasis. Curricular ramifications of this approach to equations include:

- Functions, their representation in literal symbols, tables, and graphs, and relationships between these representations would be taught before solving equations.
- Evaluating expressions would be replaced by evaluating functions
- Equations and inequalities could be treated together as different types of comparisons of pairs of functions.
- Introductory modeling problems would ask students to create and evaluate single functions before asking them to compare pairs of functions.
- The standard relations taught in algebra would be thought of as equations in two variables and would be taught after equations in one variable.

In my view this approach to equations helps make modelling an activity accessible to a broader range of students. Instead of solving for a static unknown, we would be asking students to capture the dynamics of a situation. If this is true, using such an approach, we can ask students to investigate situations more deeply. Rather than answer the single questions associated with each situation, in each case we can open up the situation for wider exploration. After asking students to write the relevant functions which express the quantitative relations in the situation, we can ask questions like: What will be the profit(loss) for particular levels of bike sales? When will the profit reach certain levels? What kinds of changes could be made in the situation to lower the break-even point? What are the range outbound times compatible with a safe flight? What times accommodate a specific safety margin? How would the times for the outbound trip change as a function of the wind speed? What will be the amount of alcohol and the concentration of the mixture for mixtures made from particular volumes of solution? If you want to make a mixture with a certain concentration and you only have a certain amount of the 20% solution, how much of the 30% solution do you need? We can also ask students to write down quantitative questions that would be reasonable ones to ask in such a situation.

In closing, I add that my teaching experiences of the last three years suggest to me that this proposal deserves careful exploration and study. As a part of a larger conceptualization of algebra, it has the potential to build on current work about students' understandings of functions and change current research questions in the field of students learning in algebra. I feel that it has the potential of helping us all understand better how algebra can be usefully applied to the world around us.

## Notes

[1] For the moment, let us focus our attention on functions which have an explicit correspondence rule.

[2] The proposal is one that was introduced to me by Judah Schwartz and Michal Yerushalmy. It is concretized in software they are writing including: *The function supposer: symbols and graphs* [1992], *The algebraic proposer* [1989], and *Unsolving equations* [1991].

[3] Ironically, though this method is emphasized in school mathematics, it is less general that the other two methods proposed

[4] Relations in *x* and *y*, like $x^2 + y^2 = 1$ can be reconceptualized as comparisons of two functions of two variables, e.g., $f(x,y) = x^2 + y^2$ and $g(x,y) = 1$.

[5] My position at Michigan State University allows me to teach Algebra I daily in a local secondary school

[6] The students in this class are 10th and 11th grade students taking Algebra I. They have not done well in mathematics in school

[7] The students had a hard time understanding why an exact answer would be sought for this problem. Instead, they wanted to talk about what were reasonable margins of safety. Though the problems in the paper are

taken from some of the best materials I know, there are important issues of artificiality and context which cannot be addressed in this paper.

[8] We are trying to equate these two distances, so they should both be positive, even though the travel is in opposite directions. We could combine the two distances to get a function for final position. To write such a function, we would need to subtract the inbound distance from the outbound distance.

[9] If the roundtrip time was not a fixed amount, it might feel more reasonable to write the roundtrip times as a function of two variables, the outbound time and the inbound time.

[10] This example is taken from the third chapter of *Advanced algebra* [1990, p 139]

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To theorise requires tremendous ingenuity; not to theorise requires tremendous honesty.

T.S Eliot