Visualisation in High School Mathematics

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Galton wrote in 1880, "An over-readiness to perceive clear mental pictures is antagonistic to the acquirement of habits of highly generalised and abstract thought and if the faculty of producing them was ever possessed by men who think hard, it is very apt to be lost by disuse. The highest minds are probably those in which it is not lost, but subordinated, and is ready for use on suitable occasions." [Bishop, 1980, p. 1]

Galton's statement is highly relevant to the learning of mathematics, which requires generalised and abstract thought [Krutetskii, 1976], and in which one might expect the ability "to perceive clear mental pictures" to constitute an advantage: after all, diagrams are a frequent accompaniment to mathematical thinking. Krutetskii's research in the U.S.S.R. made an impact on the thinking of Western researchers in mathematics education when his findings were published in English [1976] because his research suggested that, at school level, the strength of the verbal-logical component of thinking determines the level of mathematical abilities, while the visual-spatial component determines their type. As the writer's research confirmed, a pupil may be highly successful in the learning of school mathematics without needing to resort to visual thinking.

The issue then arises of the learning of mathematics by pupils who *prefer* to think in pictures. Many such pupils would like to choose careers in engineering or architecture; success in school mathematics is a prerequisite. Thus the issue of identifying the strengths and limitations of visual processing in high school mathematics is an important one for mathematics educators. Together with some effects of different teaching styles on the learning of high school mathematics by visualisers, this issue is the focus of this article. But first it is necessary to give a brief description of the research instrument which was used.

Definitions and development of a new research instrument

Selection of teachers and pupils for the fieldwork, which was carried out in Natal high schools, required the development of a research instrument to measure preference for visual methods when solving mathematical problems. In the development of this instrument, and throughout this study, the following definitions were used. [For the rationale and a full description of the research, see Presmeg, 1985.]

Visual image. A visual image is a mental scheme depicting visual or spatial information.

Most definitions of visual imagery in the literature refer to the absence of the perceptual object, as in Hebb's definition of imagery as "the occurrence of mental activity corresponding to the perception of an object, but when the object is not presented to the sense organ." [Suwarsono, 1982, p. 270] The writer agrees with Suwarsono and with Piaget and Inhelder [1971] who felt that it is possible to have an image in the presence of the object, which is why no reference is made to this aspect in the definition.

The writer's definition is deliberately broad enough to include kinds of imagery which depict shape, pattern or form without conforming to the "picture in the mind" notion of imagery [Clements, 1982, p. 36] although, of course, imagery which attains the vividness and clarity of a picture is also included in this definition. The definition also allows for the possibility that verbal, numerical or mathematical symbols may be arranged spatially to form the kind of imagery called by Paivio [1971, p. 482] "number forms."

Visual and nonvisual methods of solution of mathematical problems. A visual method of solution is one which involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are also employed. A nonvisual method of solution is one which involves no visual imagery as an essential part of the method of solution.

Following Krutetskii's [1976] model, the position is taken that all mathematical problems involve reasoning or logic for their solution. Beyond this requirement, the presence or absence of visual imagery as an essential part of the working determines whether the method is visual or nonvisual.

Mathematical visuality (MV). A person's mathematical visuality is the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods.

Visualisers and nonvisualisers. Visualisers are individuals who prefer to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods. Nonvisualisers are individuals who prefer not to use visual methods when attempting such problems.

Visual presentation. A visual presentation is a way of teaching which involves formation and use of visual imagery by the teacher or pupils or both.

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Teaching visuality (IV) A mathematics teacher's teaching visuality is the extent to which that teacher uses visual presentations when teaching mathematics.

Visual and nonvisual teachers A visual teacher is a teacher of high teaching visuality. A nonvisual teacher is a teacher of low teaching visuality.

Since the research instrument, called the mathematical processing instrument (MPI)—following Suwarsono [1982]—was required for selection of both teachers and pupils, it was decided that the instrument should consist of three sections, A, B and C, increasing in difficulty, with sections A and B intended for 16 or 17 year old pupils (who were chosen at the end of their standard 9 year) and sections B and C intended for mathematics teachers In the third (final) version of the instrument, the tests for sections A and C each comprised 6 problems, while there were 12 problems in the test for the common section B. For each section there was also a follow-up questionnaire in which subjects were required to select the problem solution which was nearest to theirs from a list comprising from three to six different possible solutions for each problem. Each problem could be solved by visual and by nonvisual methods, and space was provided for individuals to describe original methods of solution which were not listed. For each visual solution a score of 2 was allocated and for each nonvisual solution a score of zero, irrespective of whether the answer was right or wrong. Problems which were not attempted or methods which were ambiguous were allocated an intermediate score of 1 (but this was seldom required) Since there were 18 problems for each, the total mathematical visuality (MV) score was thus 36 for teachers and for pupils.

Construct validity and reliability were tested and judged to be satisfactory in schools in Cambridge, England, and in Durban, Natal (where 342 standard 9 mathematics pupils were tested).

Sections B and C of the MPI were then used to select 13 mathematics teachers (from a sample of 33) in six schools other than the four schools which participated in the final field test of the instrument These selected teachers had MV scores ranging from 3 (nonvisual) to 26 (visual), with a median MV score of 12, which was also the median MV score of the whole sample of teachers. Subsequently, sections A and B of the MPI were administered to the standard 9 mathematics classes of these 13 teachers, after their end-of-year examinations, and 54 visualisers were chosen whose MV scores ranged from 20 to 33, while the median MV score of the whole sample (N = 277) was 18.

These pupils (all visualisers) were then followed into their final year of school and studied for a period of eight months in interaction with their thirteen teachers. The research methodology included participant observation in the classrooms, and interviews with pupils and teachers. 188 interviews were tape recorded and transcribed and 108 lessons were observed; tape recordings of more than 66 of these lessons were transcribed. This wealth of qualitative data was analysed and emergent patterns were used to cast light on the research questions.

Kinds of visual imagery

Images used by visualisers in this study were classified as follows: the number after each category indicates the number of visualisers (of the 54) who used imagery in that category.

(1) Concrete, pictorial imagery	52
(2) Pattern imagery	18
(3) Memory images of formulae	32
(4) Kinaesthetic imagery	16
(5) Dynamic imagery	2

These categories are illustrated in the following transcript data

1. Concrete imagery (pictures-in-the-mind)

Alison M: (after drawing $\begin{array}{c|c} S & A \\ \hline T & C \end{array}$ to help her to see the positive trig. ratios) All students ... take chemistry.
I: Is that an image that you keep?
Alison M: Yup! .. Then it would be sine. .. second quadrant. So .. 180; you take it from 180, 'cause that's your water level.
I: Oh is that how you think of it? And how does the water level help you to get it?
Alison M: Oh, um .. you've got .. It's like a ship sailing: can't sail that way really (indicating up and down, i.e. y-axis). There, that, sort of .. it can. You work it from there You take most of your .. 'cause that's your 360 . it's there as well.
I: Did you think of that for yourself or did someone tell you about the water level?
Alison M: No, I just thought of it

Both of Alison's images in this extract were concrete images.

2 Pattern imagery (pure relationships depicted in a visual-spatial scheme)

Crispin T: (for the special angles in trigonometry) I do it .. I'll do it big here 'cause it's .. I'll draw the three triangles quickly; this is just the way I remember the angles although I probably would be able to do it anyway: 30, 60, 45 and it goes 2, 2, $\sqrt{2}$; 1, $\sqrt{3}$, 1; $\sqrt{3}$, 1, 1.

Confident and at ease, Crispin used pattern imagery frequently in this interview, e.g., in finding the components of a vector:

Crispin T: The next one is four; four times cos 120 You've got your four quadrants, so cos is, that's going to be in the second quandrant so cos is negative. I'd better just show you how I see that: sine, cos, tan; sine goes + + − − cos goes + − − +, tan goes + − + −
I: So you don't actually need a picture of the quadrants: you just need that pattern?
Crispin T: Again, that's another pattern; actually I've got quite a few of them
I: (summing up, later) The important thing about it was the pattern of it
Crispin T: Yeah, the regularity, which just helps.

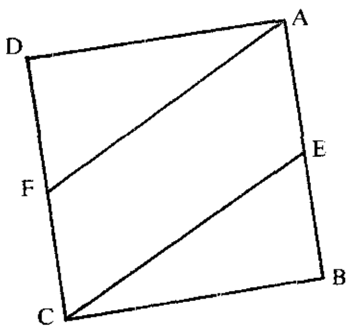
3. Memory images of formulae
Visualisers typically "saw" a formula in their minds, written on a blackboard or in their notebooks. The fact that pupils were "seeing" formulae first came to the interviewer's attention in an interview with Jeni D.

I: Incidentally, for the magnitude, did you need an image? Or is it a formula—to know that you had to square the x and square the y and then add them?
Jeni D: I know it sort of in my mind, the picture. You have the, like the vector and then the two absolute value lines.
I: So you've got a ... Oh I see, it's a picture of the formula, is it? Seeing ... vector, with the absolute value lines.
Jeni D: Yes. And then the square root sign [$|AB| = \sqrt{(x^2 + y^2)}$]

At the end of the interview, Jeni told the interviewer that she had visualised her notebook and thus had seen the formula.

4. Kinaesthetic imagery (imagery involving muscular activity)
Sue B "walked" around imaginary quadrants with her fingers to identify where the tangent of an angle is negative. Alison M "walked" four or five vectors head-to-tail to illustrate her concept of displacement. Many pupils traced out with their fingers an image of the parabola or of the hyperbola (especially when they could not remember the words).

5. Dynamic (moving) imagery



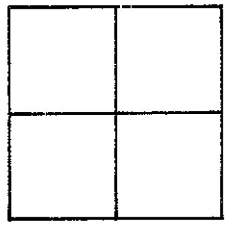
Given that the area of square ABCD is 4 square units, and that E and F are midpoints, Paul R was attempting to find the area of AECF, which had been proved a parallelogram. After some time:

Paul R: Four square units ... (12 seconds) (Then Paul redrew the given diagram.)
I: Think of half your square. (This is a rectangle.)
Paul R: Half my square would be two square units.
I: Dead right. And now look at what they ask you.
Paul R: Parallelogram ... It'll be two!
I: Quite right! Now how did you think?

Paul explained that after seeing that the rectangle would be two, he "slid" the parallelogram up into the rectangle, using a moving image.

After illustrations of some of the ways in which imagery may hinder mathematical thinking, these kinds of imagery will be referred to again with regard to the strengths of visual processing in mathematics.

Difficulties experienced by visualisers
Before Paul R could solve the area problem, as described in the previous section, he had the following image:



In the following extract, the interviewer described to a teacher the difficulty caused by this image.

I: He had an image of a square with a little cross inside: four *square* units, so he had four blocks. And he had difficulty breaking that image and reconciling it with the parallelogram which went across. So his image was hindering his thinking.

Paul's image illustrates several of the following categories of difficulties, which bear out the limitations of imagery described in the psychological literature [e.g. Paivio, 1971].

(1) The one-case concreteness of an image or diagram may tie thought to irrelevant details, or may even introduce false data.

If lines looked parallel in a diagram, they were taken to be given as parallel; a line which looked like a tangent was taken to be a tangent.

(2) An image of a standard figure may induce inflexible thinking which prevents the recognition of a concept in a non-standard diagram.

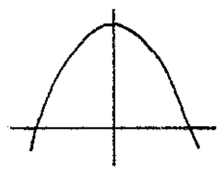
Mara L did not recognise the theorem about the angle at the centre of a circle when it did not conform to her standard image.

(3) An uncontrollable image may persist, thereby preventing the opening up of more fruitful avenues of thought. (This difficulty is particularly acute if the image is vivid.)

After finding the coordinates of the turning point of a parabola, Paul R had difficulty with the intercepts on the axes, as follows:

Paul R: Well the y intercept will be ... (He indicates the turning point.)
I: No. This one will be in the middle of it ... of the shape.
Paul R: Oh? ... The y-axis would be *that?*
I: That is the y value of this maximum point.
Paul R: But that would be on the y-axis, wouldn't it?

Paul apparently had the following false image, which persisted almost until the end of the interview.

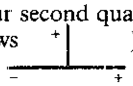


(4) Especially if it is vague, imagery which is not coupled with rigorous analytical thought processes may be unhelpful.

Kathy G had difficulty sketching an angle of 120° in a vector question.

Kathy G: I'm trying to find the angle. So it's that angle there. Oh no that's. yeah, it would be 120.. (12 seconds). 'Cause that whole thing is ..Is 120 half way? I mean halfway between 90 and 180?

Also Shan N persisted in confusing the signs of the components of a vector *OB*, as follows:

I: Just check on your signs.
Shan N: Oh yes it's in the second quadrant; it'll be negative- ... Can it be either one, that can be negative?
I: How can you tell? Think of the second quadrant. Will it be *x* or *y* that's negative?
Shan N: I think the *y*'s negative; *y* will be negative.
I: Are you sure? Just think of your second quadrant.
Shan N: Yes, because it's like this. (Draws ) So it's positive, positive, that's negative.

I: That's not *y*, that's ...
Shan N: Oh, sorry, *x*!
I: Right. So in fact you had the right image but you were just calling it *y* instead of *x*

It was also found that visual methods are often more time-consuming that nonvisual methods. Another aspect which was found to be characteristic of the problem solving of many visualisers in the task-based interviews was a difficulty in communicating the concepts of mathematics. Visualisers stumbled over terminology, could not remember key terms. In these straits they typically resorted to gestures or drew diagrams.

Efficacy of visual processing

The limitations of imagery were considered by Bartlett [1977, p. 220] to be "the price of its peculiar excellences". In learning mathematics, the excellences of imagery were found to reside in aspects illustrated in the following.

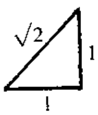
(1) *Vivid imagery of all types has mnemonic advantages.* This advantage was seen, for instance, in vivid memory images of special triangles for 45° or for 60°.

(2) *Concrete imagery is effective in alternation with abstract nonvisual modes such as analysis, logic, or a facile nonvisual use of formulae.*

The imagery used by Enrico B in the task-based interviews was all of the concrete type, but he avoided the pitfalls associated with this kind of processing by combining visual methods with a facile use of formulae, most of which he recalled without imagery, as in the following transcript.

Enrico B: The magnitude is given, and the angle, so .. vector *OA* would equal to . the magnitude of *OA* times the cosine of 45° and the magnitude of *OA* multiplied by the sine of 45°. So, magnitude of *OA* is root two units, multiplied by cos 45 which is, er, cosine is *x* over *r* so it'll be one over root two ..

(quick sketch — took just a second or two while he spoke)



. root two multiplied by sine of 45 which is one over root two . . which equals to root two over root two which is one, and . . one, one, those are the components. So components of *OB* would be . . same formula . . magnitude of *OB* is four units, multiplied by . . that'll be 180 degrees minus 120 which is cos 60 degrees, and 120's in the second quadrant, cosine's negative, so it'll be negative cos 60

(3) *Dynamic imagery is potentially effective.*

This kind of imagery was used very infrequently in this study, but its effectiveness may be seen in the quoted example of Paul R's sliding parallelogram.

(4) *Imagery which serves an abstract function is effective.*

This function was served in two ways in this study, as follows:

(a) Concretising the referent [Werner and Kaplan, in Paivio, 1971]

Concretising the referent refers to the embodiment of an abstract idea in a concrete image. The effectiveness of this process in learning mathematics was illustrated in Alison M's use of her water-level image. Memory images of formulae are also examples of concrete images which may serve an abstract function.

(b) Pattern imagery

The effectiveness of pure relationships depicted in visual-spatial schemes was evident in the thinking of Crispin T, who was the only visualiser in this study to obtain a higher-grade A for mathematics in his final examinations.

This type of imagery is illustrated strikingly in the memory images of chess masters as described by the Dutch psychologist, de Groot [Harris, 1980, pp. 408-9], and by Binet. [Paivio, 1971, pp. 24-5] De Groot showed an unfamiliar chess situation to chess players of various strengths for five seconds. In reconstructing the chessboard from memory, average and even expert players made many errors, while players such as ex-world-champion Max Euwe recalled each situation perfectly. However, when the same pieces were placed randomly on the board, chess masters showed no superiority in recall. Binet found that unlike less experienced players who visualise the board in concrete detail, expert players in blindfolded chess games retained only the geometry of the situation, in images involving patterns of configurations. McKim [1972, p. 105] wrote that "Pillsbury reported a 'sort of formless vision of the positions'; Alekhine said he visualised the pieces as 'lines of force'." Such imagery may be vague or vivid but its essential feature is that it is pattern-like and stripped of concrete details. It seems likely that facilitation of this kind of imagery in the learning of mathematics by visualisers would obviate some of the difficulties associated with the one-case concreteness of imagery.

Effects of teaching styles

Analysis of transcript data revealed that the thirteen mathematics teachers who participated in the field work fell naturally into three groups with respect to the visuality of their teaching. (The Spearman rank-order correlation coefficient between teaching visuality and mathematical visuality — IV and MV — was 0 404 n.s.) With regard to teaching visuality, the groups were as follows:

a visual group, consisting of 5 teachers, all of whose teaching visuality (TV) scores were 9 or 10 (with a possible maximum score of 12),
a middle group, consisting of 4 teachers of intermediate teaching visuality (with TV scores of 6 or 7),
a nonvisual group, consisting of 4 teachers with TV scores of 2, 3 or 4.

It was found that teachers in the nonvisual group were more inclined to adopt a lecturing style (although 12 of the 13 teachers used this style at times), and to teach formally, logically, rigorously, in a manner which could be called convergent. The middle and visual teachers also taught in this manner at times, but in addition to most of the aspects used by the nonvisual teachers, the visual group of teachers — and to some extent the middle group — used many other teaching aspects which are summed up in one characteristic principle, as follows. The visual teachers made connections between the mathematics curriculum and many other areas of pupils' experience, including other subjects, other parts of the syllabus, mathematics learned in past years and, above all, the real world. Visual teachers expressed in their teaching many traits commonly associated with creativity.

In connection with the effect of teaching styles on the learning of mathematics by visualisers, it was found that nonvisual teaching had an inhibiting effect on this learning, but that some visual teaching was also not facilitative of effective learning. Teaching by teachers in the middle group was found to be optimal for these visualisers. Nonvisual teaching had the effect of leading visualisers to believe that success in mathematics depends on rote memorisation of rules and formulae. Teachers in the middle group often used visual methods but stressed abstraction and generalisation, and this aspect aided visualisers in overcoming some of the difficulties associated with the one-case concreteness of an image or diagram. This limitation of visual thinking was one of which very few teachers (even visual teachers) were aware at all. It is clear that visual presentations are not a universal panacea. The visual teachers were unanimously positive in their attitudes towards visual methods, but they were not always able to lead visualisers to overcome the limitations, and to make optimal use of the strengths, of visual processing.

It was felt that if more teachers were aware of these issues, this awareness might facilitate more effective learning of mathematics by visualisers.

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