HIDDEN MATHEMATICS CURRICULUM: A POSITIVE LEARNING FRAMEWORK

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The Conference Board of the Mathematical Sciences (an umbrella organization consisting of sixteen professional societies in the US) has released sweeping recommendations for the mathematical preparation of prospective schoolteachers. In the context of elementary teacher preparation, these recommendations challenge the beliefs of those entering the teaching profession about the apparent simplicity of elementary mathematics content, which, on the contrary, is intellectually rich and full of important mathematical ideas. Such beliefs can be confronted through appropriately designed courses in mathematics content for elementary teachers that help them

> make meaning for the mathematical objects under study – meaning that often was not present in their own elementary educations. (Conference Board of the Mathematical Sciences, 2001, p. 17)

In the context of secondary teacher preparation, the recommendations attempt to address the widespread concern that prospective teachers do not have sufficient knowledge to teach high school mathematics and suffer from what Cuoco (2001) called *vertical disconnect* – not seeing the connection between ideas they learn in undergraduate mathematics courses and the school mathematics curriculum. Such a situation, it is said, can be improved by redesigning mathematics courses for teachers to help them

> make insightful connections between the advanced mathematics they are learning and the high school mathematics they will be teaching. (Conference Board of the Mathematical Sciences, 2001, p. 39)

This article is a reflection on courses for elementary and secondary pre-teachers developed in the school of education at SUNY Potsdam in response to the above recommendations. These courses are based on the ideas of connecting different topics in mathematics across the elementary school through college curriculum, exploring traditional curricular content from a deeper perspective, and providing a structure through which it is possible to try to address systematically and incorporate meaningfully the NCTM *Connections* standard and *Technology* principle (National Council of Teachers of Mathematics, 2000).

The assessment of one such course, developed for elementary teachers, involved student portfolios in which they were encouraged, in part, to reflect on the course content and their previous mathematical experiences. In this regard, one pre-teacher's remark is of special importance:

> Certainly the math taught to me in the past helped in my understanding of concepts presented; however, the interconnection of mathematical concepts was something I do not recall being aware of in the past. This may have been a fault […] in the nature of the education system I grew up with but […] it is hoped that I can implement skills learned in this course to […] challenge students to see the relationships.

The pre-teacher goes on to argue indirectly for deep mathematical knowledge by teachers in pointing out that

> […] as students learn [to ask questions] they will begin to challenge the teacher's understanding of the topic.

This last remark can be put in a research context by recalling a phenomenon reported by Bruner (1985) who, referring to work done by Tizard and her colleagues, observed that

> children […] who ask the most searching questions are the ones whose parents are most likely to answer them fully and […] the parents who are most likely to answer are the ones with children most likely to ask! (p. 31)

It appears that Bruner's observation remains true if the word "parents" is replaced by the word 'teachers.' Indeed, the depth of (both elementary and secondary) teachers' knowledge, like that of parents, and their disposition towards discourse has great potential to create favorable conditions in which students can ask more and better questions. However, as noted by Tizard, Hughes, Carmichael and Pinkerton (1983), the absence of such conditions in the classroom ultimately results in the emergence of

> the underlying power relationship between teacher and child: the children seem to learn very quickly that their role at school is to answer, not to ask questions. (p. 279)

Such ill-promoted learning occurs through what is commonly known as the *hidden curriculum:*

> the outcomes or by-products of schools […], particularly those states which are learned. (Martin, 1983, p. 124)

within the unintentionally developed structure of social relations. In other words, hidden curriculum – a term coined by Jackson (1968) – refers to the tacit features that traditionally organize life in schools.

Alternatively, Slater (2003) has referred to this institutional structure as a *hidden contract* between students and teachers; so that when students ask non-routine questions or fail to memorize the material, it is considered to be an attempt to break the contract on their part. It appears, as the above remark by a pre-teacher indicates, that teachers' knowledge is a factor that can significantly alter the terms of

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the hidden contract. When elementary teachers come to change their beliefs in the simplicity of the mathematics they teach and move away from the tradition in which each emerging question is considered to be 'stupid,' they, in fact, cease to perpetuate this ill-promoted learning. The same thing can be said about secondary teachers, who come to view students' success as more than a matter of memorization of rules and procedures.

Hidden mathematics curriculum

This article seeks to challenge the aforementioned counterproductive consequences of schooling by offering a way of increasing teachers' mathematical competence, which in particular includes the appropriate use of technology. To this end, the authors have recently introduced the notion of hidden *mathematics* curriculum, which includes tacit concepts and structures underlying the traditional mathematical content communicated to students through formal schooling (Abramovich and Brouwer, 2003). This notion is based on the observation that many mathematical activities across the school curriculum, seemingly disconnected from a naïve perspective, are, in fact, permeated by a common mathematical concept or structure, traditionally (and, quite possibly, intentionally) hidden from learners because of its complexity. Such complexity may be either procedural or conceptual in nature. It should be noted that while the traditional conception of hidden curriculum (alternatively, hidden contract) has a negative connotation for learning, the notion of hidden mathematics curriculum is proposed as a positive learning framework.

We approach investigating the idea of hidden mathematics curriculum in teacher education by finding, creating, and working with a series of problems across the curriculum that, from a deeper perspective, may be described by a common mathematical concept. Creating problems with, perhaps hidden, mathematical depth is consistent with the recent call for helping teachers to become better teachers by providing rich problems that "convey the distinctive cohesiveness of mathematics" (Steen, 2004, p. 869). Technology has great potential to enhance this approach through appropriate pedagogical mediation.

It should be noted that the notion of hidden mathematics curriculum can be extended to make explicit the connections that exist between secondary school and university mathematics as recommended by the Conference Board of the Mathematical Sciences (2001). This recommendation was in response to observations that secondary mathematics education majors do not see these connections or the relevance of the content of their undergraduate mathematics courses to the material they will eventually teach. Clearly, the failure to see these connections is due to a compartmentalized treatment of mathematical topics learned at all levels. It is such treatment that results in the earlier-mentioned *vertical disconnect* (Cuoco, 2001). Part of the problem is likely to be the lack of collaboration between university mathematicians, mathematics educators, and school faculty. This is unfortunate, because even such basic notions as sorting and counting can be interpreted as rudiments of mathematics studied in probability and statistics, including discrete and continuous random variables. Of course, these ideas are not readily apparent; yet they can be revealed as appropriate through pedagogy that employs a hidden mathematics curriculum framework.

The main argument, for us, is that utilizing this framework to design instructional activities provides a significant opportunity to enhance mathematics teacher education. The premise is that prospective school teachers are given the opportunity to learn traditionally hidden, advanced mathematical ideas in the social context of competent guidance provided by a university faculty. This social context may be supported and enhanced by the use of appropriately deployed technology tools. Embedding advanced ideas in the technology allows for their easier access. Such pedagogic mediation supports the advancement of Freudenthal's (1983) theoretical proposal of the didactical phenomenology of mathematics as "a way to show the teacher the places where the learner might step into the learning process of mankind" (p. ix). Put another way, technology-facilitated learning in the social context of expert-novice collaboration has the potential to uncover the hidden meanings of, what are commonly perceived to be, elementary mathematical concepts.

A focus on pre-teachers' expertly facilitated mathematics learning is consistent with one of the tenets of Vygotskian pedagogy, in which the fundamental educative mechanism is interaction within a broader social context and learning is conceptualized as a transactional process of developing informed entrants into a culture with the assistance of more advanced participants, or, as Bruner (1985) called them, "vicars" (p. 32) of this culture.

Considering mathematical culture, it appears that this notion of hidden mathematics curriculum creates a powerful intellectual bridge between Freudenthal's didactical phenomenology of mathematics and Vygotsky's zone of proximal development (ZPD) that learning by transaction creates. The concept of the ZPD is based on the premise that human learning is, at its core, a social process in the sense that to fully characterize a learner's cognitive development, one must consider what the learner can accomplish with the support of a more expert facilitator. Vygotsky (1978) argued that learning by transaction creates the ZPD and thus "the only 'good learning' is that which is in advance of development" (p. 89).

At this confluence of pedagogical and psychological theories, the combination of Freudenthal's pedagogy of learning mathematics as the advancement of the culture of mankind and Vygotskian theory of learning in a social context provides theoretical underpinning for the pedagogical framework of hidden mathematics curriculum. In particular, teaching pre-teachers within such a framework in the social context of technology-enhanced learning creates the conditions for ZPDs to be developed, which in turn provide the basis for the pre-teachers' deeper understanding of fundamental mathematical concepts. It is through such expertly-scaffolded facilitation at the key points of the zone where assistance is needed (Tharp and Gallimore, 1988) that the pre-teachers develop the capacity and confidence to teach these concepts. Thus, a case can be made that a hidden mathematics curriculum framework enhanced by technology has potential to become a medium for what Vygotsky referred to as "good learning." We offer specific examples of classroom work to support this case.

Partition of integers as hidden curriculum

One profound concept that unites many of the topics found across the school mathematics curriculum, including arithmetic, algebra and geometry, is the partition of integers. As mentioned earlier, it may be due to the complexity of mathematics behind this concept that it has not been explicitly highlighted in the curriculum as such.

Yet, partitioning problems often emerge from simple situations like the one that pre-teachers face on the first day of a graduate education class when they are asked to arrange themselves into working groups. An obvious constraint is that the groups should be as close to the same size as possible. Interestingly enough, whereas some may view this task as a division-with-adjustment problem (divide the number of groups into the number of students to see which interval between two consecutive integers the quotient belongs – these integers determine the sizes of the groups), it can be solved using mathematical reasoning on a kindergarten level by using manipulatives (*e.g.*, square tiles). For example, see Figure 1 for the case of twenty students and six groups representing a result of a hands-on solution.

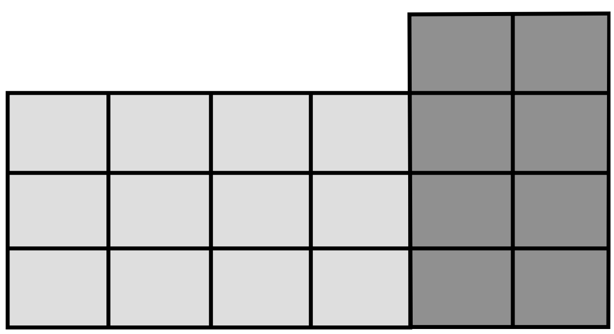


Figure 1: Solution to the student group problem using square tiles.

While kindergarten mathematics was apparently hidden in this upper elementary-level problem situation, secondary-level mathematics can be developed from this situation as well. Firstly, its algebraic representation in the form of the system of two simultaneous equations $3x + 4y = 20$, $x + y = 6$ (coefficients in the first equation result from the inequality $3 < 20/6 < 4$) that could be solved using several methods. Secondly, a graphic representation may be introduced so that the point of intersection of two straight lines on the Cartesian plane can be interpreted as students being split into four groups of three and two groups of four. Finally, on an even more advanced level, yet appropriate in the coursework preparation for secondary teachers, this situation can be understood in terms of partitions of integers; that is, as partitioning of 20 into the summands three and four.

In general, the partition of a positive integer is its representation as the sum of counting numbers without regard to order (Ahlgren and Ono, 2001). Many problem situations include counting partitions of a positive integer n whose summands are taken from the sequence of counting numbers $a_1, a_2, \ldots, a_k$. A formal approach to counting such partitions deals with the method of generating functions that differentiates whether these numbers can enter a partition more than once or at most once. In the first case, this method enables one to find the number of partitions as the coefficient of r^n in the expansion of the following product of geometric series:

$$(1 + r^{a_1} + r^{2a_1} + r^{3a_1} + \ldots)(1 + r^{a_2} + r^{2a_2} + r^{3a_2} + \ldots)\ldots(1 + r^{a_k} + r^{2a_k} + r^{3a_k} + \ldots)$$

Indeed, by presenting the product as a polynomial in powers of r, one can see that the coefficient of r^n coincides with the number of partitions of n into the summands taken from the k-element set $\{a_i\}$.

In particular, for the student-group problem, such a product has the form $(1 + r^3 + r^6 + \ldots)(1 + r^4 + r^8 + \ldots)$. By expanding the product (in which the first and the second series include the terms up to r^{18} and r^{20} respectively) there are two terms, namely, r^{20} and $r^{12}r^8$, that have an exponent of 20. The first term represents the case of five groups of four, the latter term points to four groups of three and two groups of four (six groups in total), thus furnishing the solution sought.

Alternatively, if the students are to be seated at six tables designed for two, four, six, eight, ten, and sixteen people (assuming that the order does not matter and once a table is taken, it is used to full capacity), the method of generating functions leads to the product $(1 + r^2)(1 + r^4)(1 + r^6)(1 + r^8)(1 + r^{10})(1 + r^{16})$, the expansion of which yields the appearance of term r^{20} four times, thus indicating that four possibilities for the students to be seated exist.

In such a way, starting with a seemingly simple division-with-adjustment problem, the hidden curriculum approach enables pre-teachers to see this problem from broader perspectives, both elementary and advanced. Furthermore, as detailed elsewhere (Abramovich and Brouwer, 2004a), the use of a spreadsheet and computer algebra system allows for implicit knowledge about partitions, communicated to the teachers through the basic structure of formal meaning in course activities, to be extracted from a hidden curriculum and become more explicit. Therefore, the use of both manipulative and computing technology makes it possible to elevate pre-teachers' learning of mathematics to "higher ground" (Bruner, 1985, p. 23) so that not only "the new higher concepts in turn transform the meaning of the lower" (Vygotsky, 1986, p. 202), but, in addition, the lower concepts can enhance understanding of the higher.

Uncovering hidden isomorphism between two geometric worlds

Looking at Figure 1, it is possible to observe that two tiles are protruding from the structure, which otherwise would be a rectangle. If the total number of students were 18, then the student group problem could have been solved by dividing six into 18 – division with no remainder. Ironically, the benefit of such simplification is that it opens the door to an even more complex concept than that of partition of integers; namely, partition of unit fractions. To clarify, note that the rectangular structure made of square tiles (see Figure 2) can be interpreted in terms of perimeter and area as these concepts are introduced at the elementary level. Indeed, this rectangle has perimeter 18 and area 18. Surprise! Are there other rectangles that can be made of square tiles (*i.e.*, having integer sides) whose area numerically equals its perimeter?

How can such rectangles be found formally rather than by serendipity?

Van der Waerden's (1961) citation of Plutarch is worth mentioning in connection with this unexpectedly discovered situation:

> The Pythagoreans also have a horror for the number 17. For 17 lies halfway between 16 […] and 18 […] these two being the only two numbers representing areas for which the perimeter (of the rectangle) equals the area. (p. 96)

A significance of this classic remark for this discussion is in the hidden connection (in fact, an isomorphism) between the ancient geometric proposition and a hands-on activity recently observed in an elementary classroom in rural upstate New York. This activity was based on the use of fraction circles – manipulatives representing a unit fraction and shaped as a sector of a whole circle. Young children were asked to cover one half of a circle with two fraction circles in as many ways as they could. Mathematically speaking, the children were partitioning the fraction one-half into the sum of two unit fractions, which, as it turns out, is related to the problem mentioned by Plutarch.

Indeed, if x and y are integers, which measure the sides of such a rectangle (Figure 2), then $xy = 2(x + y)$. Although Babylonians knew an algebraic solution to the last equation (Van der Waerden, 1961) it still may be difficult for some elementary pre-teachers to comprehend. An easier path is to rewrite the expression as $1/2 = 1/x + 1/y$, to interpret this new equation as a partitioning problem, and solve it through the above-mentioned hands-on activity. In doing so, it is possible to discover two – and *prove* there are only two – ways of covering a half of a circle by two fraction circles, or, alternatively, re-inventing the two rectangles known to the Pythagoreans. How can such proofs be developed?

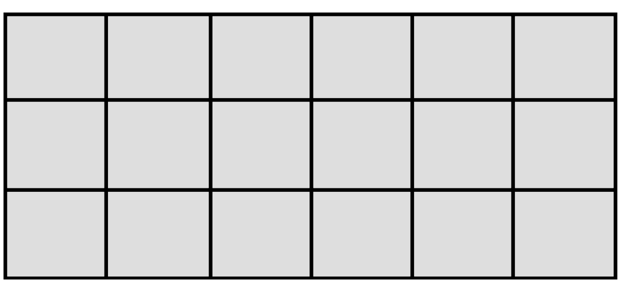


Figure 2: Rectangle known to the Pythagoreans.

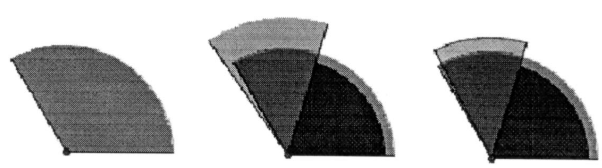
Technology and proof

In turn, the activity with fraction circles can be enhanced by the use of custom tools created within *The Geometer's Sketchpad*. By using these tools, under competent tutelage, it is not only possible to solve the above problem of partitioning the unit fraction one-half, but its natural extensions also, for the equations $xy = x(x + y)$ and $1/x = 1/x + 1/y$ are equivalent in this context. For example, finding all ways of representing the fraction circle one-third as the sum of two like fractions can be used as a strategy for *finding all rectangles (of integer sides) for which the area is three times as much as its semi-perimeter*. The above use of the word "all" implies the need for students to provide formal justification of their results – otherwise known as mathematical proof. In this regard, note that Bell's (1979) claim that the essence of mathematical proof deals with the "public acceptability of the knowledge being discovered" (p. 368) is consistent with the Vygotskian notion of learning as a social activity. Asking pre-teachers to communicate their proof schemata through written speech using computer-generated images as a support system is a form of developing ZPD.

In fact, the need for proof may be motivated by a genuine student question often overheard when Plutarch's remark is discussed: *How do we know that we found all rectangles?* To this end, an elementary pre-teacher attempted to develop a proof with technology in the case of the fraction circle one-third:

> Giving fraction circle $1/x$ a smaller denominator (making the fraction circle larger), will require that the fraction circle $1/y$ increase its denominator (making the fraction circle smaller) to maintain the equation $1/3 = 1/x + 1/y$. The combinations of fractions that are possible then are always finite given the nature of fraction circles. Once the initial fraction circle (1/3) is divided in half, the remaining possibilities for fraction circle $1/y$ are discovered by solving for $1/y$ given that $1/x$ can only be 1/6, 1/5, and 1/4. By making a bigger $1/x$ is 1/3 [*making 1/x bigger results in 1/3*] and […] this will give no value to $1/y$. It only remains to be seen whether all combinations [for $1/x$] are possible.

It is remarkable to read in the teacher's proof schemata, "It only remains to be seen" when what remains of the proof is arithmetic of rational numbers, typically challenging for some elementary pre-teachers. Indeed, the above-cited proof was followed by a combination of geometric representations (Figure 3) and numeric arguments not included here for the sake of brevity. In such a way, experiencing success at a high conceptual level – that is, through doing a proof – expanded pre-teachers' ZPDs, thus empowering them to tackle procedural details with confidence. Finally, note that in the context of technology and proof, the above geometric activities can be extended to include explorations with rectangular prisms, or, alternatively, partitioning a unit fraction into the sum of three like fractions (Abramovich and Brouwer, 2004b).



1/3 does not equal 1/7 + 1/5 (too large) and does not equal 1/8 + 1/5 (too small) – every other fraction smaller than 1/8 will also not help

Figure 3: Geometric representation of proof schemata.

Concluding remarks

The notion of hidden mathematics curriculum advances several agendas within mathematics teacher education.

Firstly, it promotes the interplay between theory and practice within the field by highlighting the deeper meaning in what is commonly perceived as routine school mathematical activity. Through well-chosen and, perhaps, historically significant examples enhanced by technology, pre-teachers can develop an understanding of how these deeper meanings arise from common structures of the explicit curriculum and connect its different ideas and representations. In addition, in a technology-mediated intellectual milieu, achieving control over a concept occurs in a socially created ZPD where intuitive understanding of the concept meets the logic and formalism needed for its representation through a computational medium. The product of this human-computer interaction aided by competent tutelage is a solution, which, once internalized, becomes a part of one's consciousness.

Secondly, as argued in this article, the deeper the mathematical knowledge a teacher possesses, the more likely it is that students will ask more and better questions. The importance of establishing a disposition towards discourse in mathematics classrooms was recognized in the *Professional Standards for Teaching Mathematics* (National Council of Teachers of Mathematics, 1991) by expecting students "to raise questions and challenge ideas generated by other students as well as by the teacher" (p. 104). This ambitious vision of a mathematics classroom feeds back into a remark of an elementary pre-teacher cited in the introduction regarding challenges that a more knowledgeable teacher becomes faced with, as students develop a tradition of asking reflective questions.

In this respect, there are two specific aspects of the notion of hidden mathematics curriculum to consider. One is our belief that it enables pre-teachers' learning of mathematical concepts traditionally considered advanced which, in turn, allows them to communicate mathematics more fully to their students. The other is that the notion of hidden mathematics curriculum, used as a conceptual framework for teaching mathematics, has the potential to create conditions within which the intellectual curiosity of students and their disposition towards reflective inquiry is promoted and encouraged. Therefore, through redefining their traditional role in school as passive receivers of information, students are empowered to take an active role in shaping their own learning trajectory.

Thirdly, the notion of hidden mathematics curriculum has potential to provide pre-teachers with true modeling experience by introducing them to the concept of isomorphism, the basis for mathematical modeling. The latter is essentially a process of exploring properties of objects that belong to one system within another system that is isomorphic to it, so that results obtained in the model can be reformulated in the language of the original system. Indeed, the problem about perimeter and area was not solved by the teachers in the 'world of rectangles' but rather in the 'world of fraction circles.' By using simple algebra, the pre-teachers, in fact, established the isomorphism between the two geometric worlds, so that conclusions derived from work with fraction circles were re-interpreted and applied to rectangles (Figure 4). It should be noted that some pre-teachers expressed uncertainty with this approach remaining skeptical about the validity of such inference. This is not finding fault with these teachers – we rather appreciate their reluctance to take for granted one of the most powerful tools developed within mathematical culture.

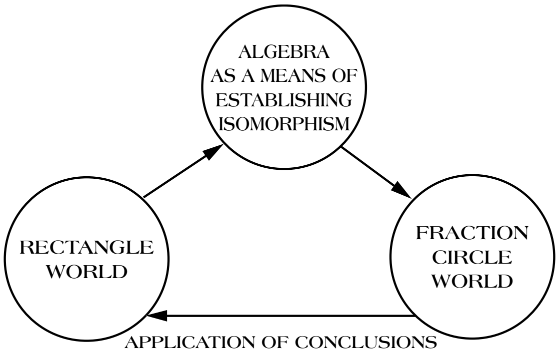


Figure 4: Modeling as isomorphism.

Finally, a hidden curriculum approach that identifies deep concepts and structures of mathematics makes it possible to elevate pre-teachers' learning of mathematics to higher ground, both from elementary and advanced perspectives. Climbing to this new height creates in pre-teachers greater self-confidence in their abilities to teach mathematics. The importance of developing such a confidence was acknowledged implicitly in a comment by an elementary pre-teacher who admitted,

> my biggest fear [...] is that one bright student might figure something out in a remarkable way and I won't be able to recognize it and keep the student interested enough to be excited about the next lesson.

We argue that this approach helps alleviate these kinds of concerns by pre-teachers, especially at the elementary level. Indeed, at this level, traditionally poorly understood topics, like formal arithmetical operations with fractions, when highlighted from a different, sometimes advanced, perspective in which the pre-teachers experience success, leads to a greater understanding of and confidence in those topics.

Using technology, both elementary and secondary pre-teachers can make significant progress in connecting their informal explorations with formal symbolic mathematics. Appropriately used as a pedagogical tool, the notion of hidden mathematics curriculum has the potential to be used as a positive learning framework by significantly broadening pre-teachers' content knowledge at all levels, bringing desirable change to various teaching-related psychological phenomena, and eventually affecting the way that mathematics is learned in schools.

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[The references can be found on page 25. (ed.)]

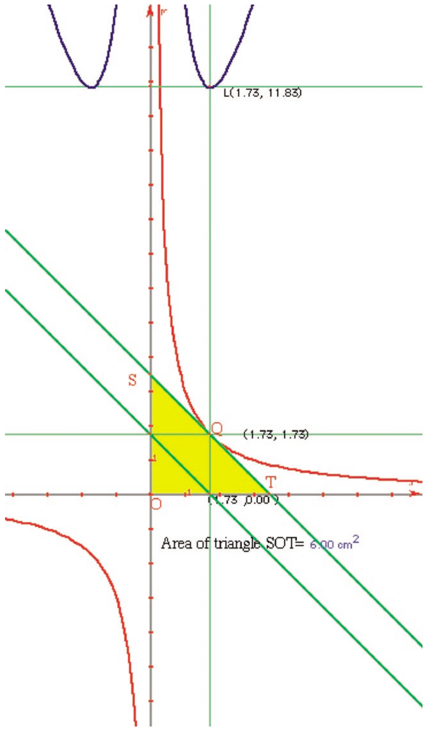

Figure 8: Exploring the behaviour of the area of triangle OTS when f(x) = 3/x).

- to express in writing the representations of the problem
- to explore and communicate properties of mathematical objects that emerge while moving parts or analysing relationships within the dynamic representation.

Here, students often are asked to think of the problem in general terms, that is, beyond the particular case being studied. Again, a proper notation is needed to express and support the validity of those results (Santos-Trigo, 2004). In addition, it seems apparent that with the use of the tool students may consider the dynamic representations of problems as a platform to launch and pursue new and relevant questions that eventually become a source to formulate and generate other problems or results.

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[These references are from the article "Hidden mathematics curriculum: a positive learning framework" that starts on page 12. (ed.)]

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