Beyond Proving and Explaining: Proofs that Justify the Use of Definitions and Axiomatic Structures and Proofs that Illustrate Technique

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Students' difficulties with proofs are well documented: in particular, students typically cannot comprehend proofs, fail to realize their purpose and often consider the process of proof to be pointless (e.g. Harel, 1998). To address these issues, researchers have attempted to identify and categorize the purposes of proof in mathematics and the mathematics classroom (e.g. Hanna, 1990; Hersh, 1993) This article is my attempt to further this categorization

Hanna and Hersh distinguish between proofs that convince, i.e. proofs that an individual can use to verify *that* a theorem is true, and proofs that explain, i.e proofs that one can use to gain an intuitive understanding of *why* the theorem is true. Although proofs that convince dominate the mathematics journals, Hanna and Hersh both argue that these proofs are often not appropriate for the mathematics classroom Instead, they advocate using proofs that explain in the classroom, even at the expense of mathematical rigor

Proofs that convince and proofs that explain both provide *knowledge* about *mathematical truths*, albeit in different ways They are proofs of theorems whose validity is not obvious to the reader By studying the proof, the reader gains insight into the theorem's veracity

While Hanna and Hersh's distinction has been valuable, many of the proofs that are presented to students do not fall into the categories of their taxonomy: consider proofs in axiom-based logic courses that 1 + 1 = 2 or that the sum of two even numbers is even As students never doubted these facts and have probably used them without penalty on their assignments in other courses, one cannot say that the purpose of these proofs is either to convince the student that or explain why these facts are true According to Harel (1998), such rigorous demonstrations of seemingly obvious results frequently lead students to view the process of proof as pedantic and meaningless; some have even advocated removing these proofs from the classroom altogether (e.g. Kline, 1973).

I believe that students' and instructors' difficulties with these proofs stem from the fact that they do not understand the proofs' purpose The proof that 1 + 1 = 2 is not a proof that provides knowledge about this mathematical truth; this proof provides information about the axiomatic system in which one is working and about how one can generate proofs within that system. In this article, I will describe four types of proof. I will review two types of proof that provide knowledge about mathematical truth: proofs that convince and proofs that explain. Then I will describe two other types of proof - proofs that justify the use of a definition or axiomatic structure and proofs that illustrate technique I will conclude with some pedagogical suggestions as a result of this expanded classification.

1. Proofs that provide knowledge about mathematical truth

Over the next two sections, I propose four purposes for presenting proofs in the mathematics classroom I should stress that a decision about which category a proof belongs to is *contextual* and depends on the instructor's intentions and the audience observing the proof.

Proofs that provide knowledge of mathematical truths fall into two overlapping categories: proofs that convince and proofs that explain. As these types of proofs have been discussed previously in the mathematics education literature (as mentioned above), my description of them here will be brief

A proof that convinces begins with an accepted set of definitions and axioms and concludes with a proposition whose validity is unknown An overview of such a proof purpose is given in Figure 1. The intent of this type of proof is to convince one's audience that the proposition in question is valid. By inspecting the logical progression of the proof, the individual should be convinced that the proposition being proved is indeed true.

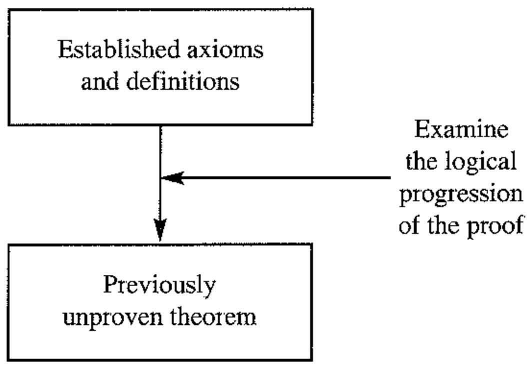

Figure 1: A proof that convinces

A proof that explains also begins with an accepted set of definitions and axioms and concludes with a proposition whose validity is not intuitively obvious, although another proof of this theorem might already be known. In contrast to proofs that convince, proofs that explain need not be totally rigorous. An overview of this proof purpose is given in Figure 2 The intent of this proof is to illustrate intuitively why a theorem is true. By focusing on its general structure, an individual can acquire an intuitive understanding of the proof by grasping its main ideas.

There are proofs that convince but do not explain. Good examples of this are most proofs of the Heine-Borel theorem, which (in its weak form) asserts that every closed

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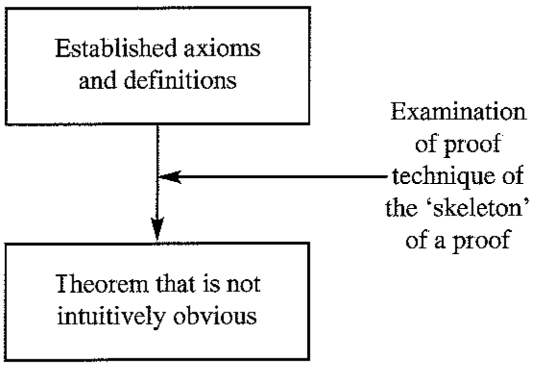

Figure 2: A proof that explains

interval is compact While most undergraduate mathematics majors are expected to be able to follow a logical demonstration of this theorem, most of its proofs (for example, Royden, 1988, p 45) provide undergraduates with little intuitive understanding of why this theorem is true. Other examples of proofs that convince but do not explain include proofs that are condensed into as few lines as possible, thereby hiding the intuition that was used in creating the proof, or proofs that are excessively technical

There are also proofs that explain but do not convince This can occur when a theorem has been established previously, but a new proof proves the theorem in a different or a more intuitive way. Examples of this might include the new proofs of Sylow's theorems in group theory that made use of orbits Although this theorem had been proven for over half a century, the new proofs made the theorem more accessible to undergraduates [1] (For a comparison of the original proofs and the proofs described above, see Gow, 1994.)

Many similar examples can be found in publications such as the *American Mathematical Monthly*, where results that were established by esoteric or technical proofs are reproven in a way that is more intuitive and accessible. Proofs that explain but do not convince can also occur when students are shown an intuitive argument that would not qualify as a rigorous proof Of course, many proofs serve the dual purpose of convincing and explaining. For a more thorough discussion of this subject, the reader is invited to read Hersh (1993) or Hanna (1990).

2. Proofs that justify the use of terminology and proofs that illustrate technique

To convince and explain are the more common purposes of proofs and most proofs that appear in mathematical journals are of this type. Indeed, I suspect that most students and perhaps some mathematicians believe that convincing and explaining are the *only* purposes of proof. These proofs provide knowledge about the body of mathematical truths - one uses accepted definitions and axioms to verify that or understand why a particular statement is true. In this section, I describe two types of proofs whose purpose is not to gain knowledge about mathematical truths, but rather to show why we use the axioms and definitions that we do or to illustrate techniques that we can use in conjunction with these axioms and definitions to construct proofs Although proofs of this type are rarely published in mathematical journals, they play a prominent role in the university mathematics classroom.

Proofs that justify the use of a definition or axiomatic structure

The first type of proof that I will discuss is a proof that justifies the use of a definition (or an axiomatic structure) An overview of this type of proof purpose is given in Figure 3 In proofs that convince or explain, the assumptions that the prover makes are generally accepted, while the theorem he or she proves is in some way in doubt. In proofs that justify the use of a mathematical structure, it is the assumptions that one is making that are questionable, but the conclusion is regarded as obvious.

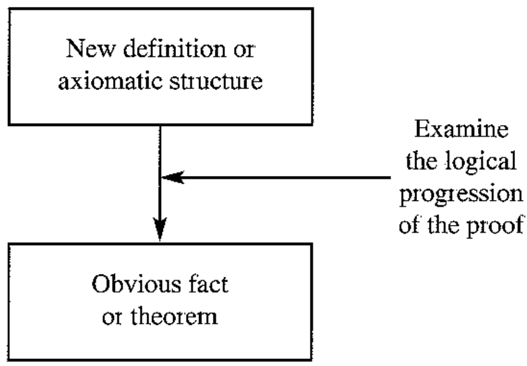

Figure 3: A proof that justifies structure

Most mathematical theories (at least prior to the foundations movement in the late nineteenth century) do not begin as formal mathematical systems, but rather as intuitive notions However, there is usually some point where intuition is no longer sufficient and the mathematical community sees a need to formalize or 'rigorize' the theory. When this occurs, mathematicians propose logical systems or definitions to model the formerly more intuitive theory Whether or not a mathematical structure is accepted as a representation of an intuitive theory depends, in part, on how well it captures the intuitive essence of the theory. In particular, if an axiomatic system is a reasonable model for an intuitive theory, one should be able to use this system to derive all the obvious facts of this theory A proof that justifies the use of a mathematical structure shows that the mathematical structure can do just that.

Perhaps this type of proof can best be illustrated by an example. In the late nineteenth century, Peano proposed a set of axioms to define arithmetic. Using these axioms, Peano and his colleagues proceeded to 'prove' theorems such as two plus two equals four and addition over the integers is associative. The purpose of these proofs was not to convince mathematicians that these facts were true; such facts were never in doubt in the mathematics community Nor was the purpose of these proofs to explain why these statements were true. Peano's proofs were lengthy, technical, difficult and decidedly non-intuitive Furthermore, childhood experience is sufficient for most to understand why two plus two equals four.

The purpose of these proofs was to convince the mathematical community that Peano's system of arithmetic was a reasonable one. Peano could have created any logical system that he wanted and called it 'arithmetic'. However, if this system failed to account for the essential properties of arithmetic (e.g. addition and multiplication are well-defined, associative, commutative binary operations and multiplication distributes over addition), it would not have gained any currency within the mathematical community By proving that this system possessed all the properties a proper arithmetic *ought* to have, mathematicians were more likely to accept Peano's system as a reasonable model of arithmetic and thus a valid topic of mathematical investigation.

Proofs that illustrate technique

The fourth category of proofs that I propose is *proofs that illustrate technique* An overview of this type of proof purpose is given in Figure 4. In these proofs, the prover begins with an established set of definitions and axioms and concludes with a theorem. Whether or not the theorem had been previously proven is irrelevant, but generally the theorem is at least an intuitively obvious result. The purpose of this proof is to show one's audience (usually students) *how* one can prove these *types* of statements The individual who reads or studies this proof is asked to examine its general form and is expected to be able to use this form to produce similar proofs.

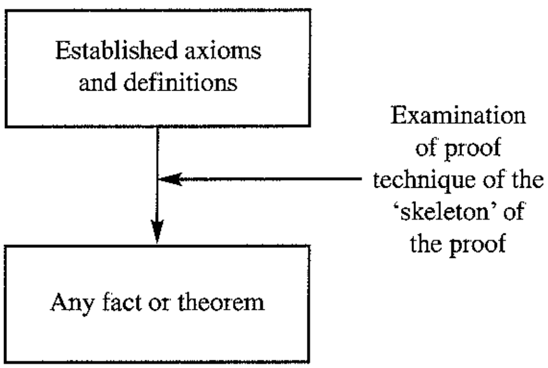

Figure 4: A proof that illustrates technique

Consider the following proposition: $f(x) = x^2$ is a continuous function This proposition, or some similar proposition, is usually proved to students in their first real analysis course. It would be misleading to tell the student that the purpose of this proof was to convince them that or show them why this theorem was true. Since calculus, students have been told that polynomial functions are always continuous. Upon inspection of its graph, students can see that the function $f(x) = x^2$ is well-behaved and does not suffer any discontinuities due to jumps, gaps, vertical asymptotes or wild oscillations. What the instructor hopes the students gain from this type of proof is how to construct analogous 'delta-epsilon' proofs in similar circumstances

At this point, I should stress that the categories that I present are not absolute. Some proofs can fall into multiple categories; others might fall into none For instance, a presentation of Euclid's proof that there are infinitely many primes can simultaneously serve three distinct purposes: the proof can convince students that this proposition is true, show them why it is true, and illustrate a proof by contradiction Also, which category a proof belongs to is contextual and depends on the presenter's intentions and which audience is studying the proof and for what purpose A proof can convince an undergraduate that a theorem is true, illustrate a proof technique to a graduate student and explain why a theorem is true to a mathematician.

3. Suggestions for pedagogy

A summary of the four types of proofs that I described is presented in Figure 5.

	Type of proof			
	Proofs that convince	**Proofs that explain**	**Proofs that justify structure**	**Proofs that illustrate technique**
Hypotheses	Accepted	Accepted	In doubt	Accepted
Result	Validity in doubt	Not intuitively obvious	Accepted	Irrelevant
Level of rigor	Very rigorous	Not necessarily rigorous	Very rigorous	Rigorous
Reader should focus on	Line-by-line justification	General	Line-by-line justification	General
Salient features	Logical progression	General structure of the argument	Logical progression	Reasoning behind the argument
The reader understands the proof if he or she can	State how each line logically follows from the previous work	Describe the proof in his or her own words	State how each line logically follows from the previous work	Produce a similar proof in an analogous situation
Example	Heine-Borel theorem	Intermediate theorem (in elementary calculus)	2 + 2 = 4 using Peano s arithmetic	$f(x) = x^2$ is a continuous function

Figure 5: A summary of proof attempts described in this article

Note that what the salient features of a proof are and how the proof should be read depend on the purpose of the proof being presented. Hence, proofs offered with different purposes should be presented and studied in different ways. However, instructors often treat all proofs alike, especially in advanced mathematics courses. I believe that this is a mistake. When presenting a proof, instructors should be explicitly aware of the purpose(s) of the proof and, furthermore, should make the students explicitly aware of this. I offer two arguments in defense of this claim.

First, according to Harel (1998), a primary reason that students dislike advanced mathematics is that they feel no intellectual need to justify the statements that are proved in their courses Since much of advanced mathematics classes consists of proving seemingly obvious statements, students are lead to believe that the process of proof is purely pedantic. This point can be best illustrated by the following example. Imagine that you are an undergraduate in an analysis class and you are introduced to Dedekind's model for the real numbers in the form of 'Dedekind cuts' Using this model, the instructor proves that addition over the real numbers is commutative

If you believe that proofs are solely to convince and explain, it would be difficult to see much if any value in this proof. Prior to this lesson, that addition over the real numbers is commutative has always been accepted as an established fact, and indeed you and the instructor have both felt free to use this fact without justification. Further, the proof that the real numbers commute under addition using Dedekind's model provides little in the way of intuition. What the proof essentially boils down to is that since the rational numbers commute under addition, so do the real numbers. Why should the fact that the rational numbers commute under addition be any more intuitive than the fact that the real numbers do? Of course, the point of this proof is *not* to establish that addition is commutative over the real numbers, but rather to show that Dedekind's model can account for this fact.

The purpose of the proof is not to provide knowledge about addition; it is to provide knowledge about how Dedekind's definition of the real numbers relates to the real numbers as we know them (and you can bet students will feel an intellectual need to understand this). In this sense, the proof is quite important. If the students were told that this was the point of the proof, they might be more receptive to it.

Second, by identifying the purpose of a proof, the instructor and the students can pay more attention to its most relevant details. When presenting a proof, the instructor should know the answer to these three questions. Should the students examine the logical validity of the proof, its general structure or both? Will the student be asked to reproduce an analogous argument in a similar circumstance? Which is of primary importance: the proof itself, the result of the proof or both? [2]

The answers to these questions have obvious pedagogical consequences For example, when presenting a proof that $f(x) = x^2$ is a continuous function, the instructor should be aware and explicit that the purpose of the proof is to show a technique, but the result itself is of little value. Thus, the students should be made aware that they will be asked to produce a similar argument and the instructor should explicitly describe the reasoning behind the proof.

In contrast, proofs of Cauchy's theorem in abstract algebra (i.e. that if a prime p divides the order of a finite group G, then G necessarily has a subgroup of order p) are typically non-intuitive. They usually involve a subtle technique that the students would not be expected to come up with and this technique is generally not useful in constructing other proofs. (For instance, consider McKay's clever proof of Cauchy's theorem using equivalence classes of p-tuples of group elements. This proof can be found in Hungerford, 1974, p. 92.) Thus, students should focus on the logical validity of the proof of this important result, but need not be concerned with how the ingenious author of the proof came up with his or her argument. I examine one more example in detail below.

A proof of the intermediate value theorem is often presented to first-year calculus students. In this case, I use the word 'proof' loosely, as the arguments presented to the students usually display a (necessary) profound lack of rigor [3] The 'proof' is, in fact, an intuitive argument based on a picture of a hypothetical function's graph. The purpose of this argument is to explain why the intermediate value theorem is true, but is not to convince students (in a mathematical sense) that the theorem is true. However, this message is rarely conveyed to students and too often students believe that the argument that they observe is a valid proof. Dreyfus (1999) argues that this may be one reason that students will produce proofs with an inappropriate lack of rigor in their future courses. Hence, instructors should be clear on whether the proofs they present are intended to be convincing in the mathematical sense

In summary, proofs presented in the mathematics classroom often serve purposes other than convincing and explaining. An awareness of this can allow students to focus on a given proof's most salient details, so they can get the most out of the proof's presentation and this awareness can improve students' appreciation for the role of proof in mathematics

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Notes

[1] This example nicely illustrates how the purpose of proofs can be contextual. The authors of the proofs of Sylow's theorems in the 1950s used them to explain why Sylow's theorem (to the mathematical community) was true However, an instructor can use this proof for the dual purposes of convincing and explaining when teaching an undergraduate abstract algebra course.

[2] Students typically do not have a good grasp of which classroom theorems are important and which are not For instance, in a recent study, an abstract algebra student was asked to prove that a particular group was abelian. The student attempted to do so by recalling the theorem, "If every element in a group G has order 1 or 2, then G is abelian" (Weber, 2001). I suspect that this student observed a proof of this theorem, but misunderstood its purpose, appearing to believe that this theorem was a problem-solving tool. Of course, while this theorem is true, it will rarely be useful in constructing future proofs.

[3] This is not an indictment of textbook writers or calculus instructors The proof of the intermediate value theorem depends on the completeness of the real numbers and this concept is too advanced for most students in an introductory calculus course. For instance, consider $f(x) = x^2 - 2$ as a function from the rational numbers to the rational numbers. In this case, the intermediate value theorem fails: $f(0) = -2$, $f(2) = 2$, but $f(x)$ has no roots between 0 and 2 ($\sqrt{2}$ is irrational and hence not in the domain of f) This is only possible because the rational numbers are not complete.

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