AN ERLANGEN PROGRAM THAT EMPOWERS STUDENTS' MATHEMATICS

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At the University of Erlangen, in 1872, Felix Klein laid out his program for classifying geometries based on properties of objects that remain invariant under transformation. For example, in Euclidean geometry, the relationships between angles and sides within a triangle remain the same when transformed by reflections, rotations and translations. The totality of those transformations forms a mathematical group in the sense that each transformation has an inverse and composing any two transformations yields another transformation within the group. When dilations are included (treating similar triangles as identical), these transformations form the 'principal group' (Klein, 1893). For projective geometry, the group would be bigger; it would contain all of the transformations in the principal group (as a subgroup), plus projections, which distort angles but preserve other properties.

In 2007 Cinzia Bonotto laid out a vision for how the Erlangen Program should be applied to curriculum and instruction in geometry. She argued that schools in Italy had focused too narrowly on isometries and similarities without attending to their group structures. For example, students might use reflections, rotations and translations to superimpose one triangle onto a congruent triangle, but they might not understand that composing two reflections generates a rotation (if the lines intersect) or a translation (if they do not). And what happens when two rotations, about two different points, are composed? Such questions pertain to the group structures that organize various transformations of the Euclidean plane.

The purpose of this article is to expand Bonotto's vision beyond the domain of geometry. In line with Jean Piaget (1970), I argue that the Erlangen program indicates the psychological basis for all mathematical objects, and I provide examples from the domains of geometry, fractions and algebra. As such, an expanded Erlangen program can do more than focus students' attention on the ways transformations are structured, as Bonotto suggested; it can empower students by revealing mathematical objects as products of students' own construction, under the power of their own mental actions.

In line with my purpose, I place a special emphasis on mathematical groups, which Klein used to classify geometries and Piaget used to describe the ways students' might structure their mental actions as operations, which are reversible and composable. However, as Piaget repeatedly demonstrated, these structures take years to develop, usually outside of students' awareness of them. Groups, then, are researcher constructs used to model students' potential constructions of mathematical objects as coordinations of their own mental actions (operations). How might groups of operations describe the psychological basis for mathematical objects like triangles, fractions and functions?

The principal group

Groups are sets with a mathematical structure that satisfies the following four properties:

> Any two elements of the set can be composed and the result of composition is an element of the set;
>
> The set contains an identity element that, when composed (in either order) with another element of the set, results in that other element of the set;
>
> Every element of the set has an inverse element in the set so that composition (in either order) results in the identity element;
>
> Composition is associative.

The Erlangen program directly pertains to groups whose elements are spatial transformations. For example, consider reflections in the plane. Each reflection is its own inverse (satisfying Property 3) because composing a reflection with itself (performing the reflection twice) has no effect on the plane; in other words, it results in the identity transformation. For reflections to form a group, this identity transformation must be included (satisfying Property 2). Now consider what happens when two different reflections are composed (see Figure 1).

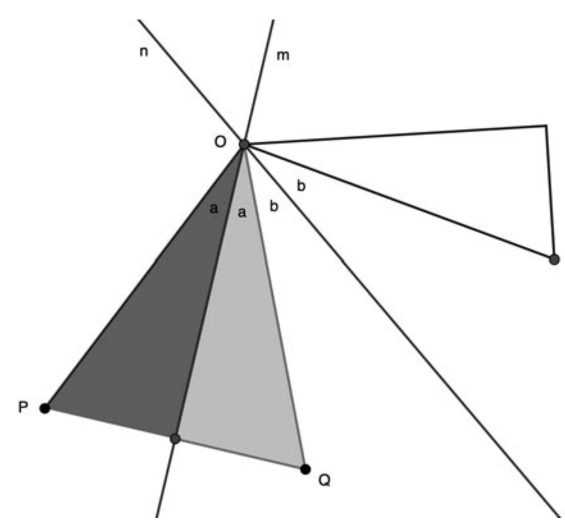

Figure 1. Composing two reflections.

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The triangles in Figure 1 illustrate the effects of two reflections and their composition. First, reflecting the dark gray triangle over line *m* results in the light gray triangle. Then, reflecting the light grey triangle over line *n* results in the white triangle. More than that, we can see that the point of intersection of the two lines, O, does not move during either of the reflections, so it will stay fixed under their composition. We can also see that any other point, P, will form an angle, *a*, with point O and line *m*. When P is reflected over line *m*, to point Q, the same angle will be formed on the other side; and Q will have the same distance to O that P had. Likewise, when Q is reflected over line *n*, angle *b* will be replicated and the distance to O preserved. Thus, we can see that the result of composing reflections about two intersecting lines is a rotation about their point of intersection by an angle that is twice the angle between them. If the lines of reflection do not intersect (*i.e.,* are parallel), their composition will yield a translation.

Because the composition of two reflections might form a rotation or a translation, those transformations must be included in the group as well (by Property 1). As we can imagine, translations have inverses and composing two translations yields another translation. We can also see that rotations have inverses (by rotating in the opposite direction), but we need to consider what happens when rotations about different points are composed (see Figure 2). Here, the pedagogical merits of Bonotto's (2007) argument become apparent.

If we focus on the structure of transformations, as Bonotto suggested, we can decompose every rotation into two reflections, where the lines of reflection intersect at the point of rotation. Thus, composing two rotations amounts to composing four reflections. Moreover, for each rotation we can pick any pair of lines that intersect at the point of rotation and have an angle between them that is half the angle of rotation. For the rotations about A and B, as shown in Figure 2, let us choose pairs of lines that have *l,* the line through AB, in common. The rotation about A is the composition of reflecting over *m* and then reflecting over *l*. The rotation about B is the composition of reflecting over *l* and then reflecting over *n*.

The structure for composing transformations informs us that the composition of the two rotations is the composition of four reflections, about *m, l, l* and *n*. Here, we need to know that the composition of reflections is associative (Property 4): although changing the order of the reflections might change the final result (for example, *ml* and *lm* would be rotations in opposite directions), collapsing any neighboring pair of reflections into a single transformation will not. In particular, composing *l* with *l* yields the identity transformation so that *mlln* is reduced to *mn*—a rotation about O. Thus, we can see that composing two rotations yields a new rotation (when *m* and *n* intersect), or a translation (when they do not).

We can demonstrate associativity by observing that every reflection takes points in the plane to other points in the plane, in one-to-one correspondence. If the first reflection maps one copy of the plane, P, onto another copy of the plane, P′; and, likewise, if a second reflection maps P′ onto P′′ and a third reflection maps P′′ onto P′′′; then pairing any two neighboring reflections simply cuts out the intermediate copy (*e.g.,* composing the latter two reflections takes P′ to P′′′) so that the end result is always a transformation from P to P′′′.

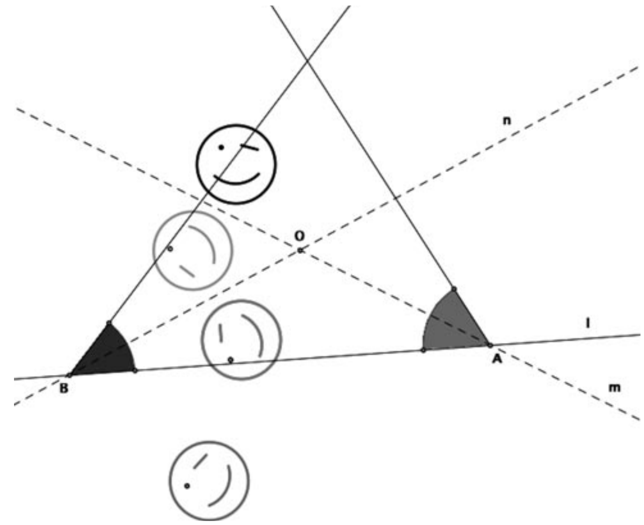

Figure 2. Composing two rotations.

We now know that the collection of transformations generated by composing reflections satisfies all properties of a group. This group includes reflections, rotations, translations, and their combinations (*e.g.,* glide reflections, which involve reflecting and then translating, or *vice versa*). In accord with Bonotto's argument, the group structure helps us understand how rotations, translations and their combinations are reducible to the reflections that generate them. In accord with Klein's argument, the group of transformations defines a geometry whose objects (*e.g.,* angles and triangles) are invariant under transformations in the group. If we consider similar objects (*e.g.,* similar triangles) to be the same, then the group of transformations would also include dilations.

> For geometric properties are, from their very idea, independent of the position occupied in space by the configuration in question, of its absolute magnitude, and finally of the sense [orientation] in which the parts are arranged. The properties of configuration remain therefore unchanged by any motions of space, by transformation into similar configurations, by transformation into symmetrical configurations with regard to a plane (reflection), as well as by any combination of these transformations. (Klein, 1893, p. 218)

A Piagetian perspective on the principal group

With its focus on transformations and the dynamic origins of mathematical objects, the Erlangen program fits nicely within Piaget's (1970) structuralist epistemology of mathematics, as recognized by Piaget himself:

> The various kinds of geometry—once taken to be static, purely representational, and disconnected from one another—are thus reduced to one vast construction whose transformations under a graded series of invariance yield a 'nest' of subgroups within subgroups. It is this radical change of the traditional representational geometry into one integrated system of transformations which constitutes Felix Klein's famous Erlangen Program. The Erlangen Program is a prime example of the scientific fruitfulness of structuralism (p. 22).

In Piaget's structuralism, mathematical objects consist of coordinations of operations: composable and reversible mental actions, including geometric transformations such as reflections. Research in cognitive psychology identifies the mental action of reflection as a psychological primitive (*e.g.,* Barlow & Reeves, 1979) and we have seen that compositions of reflections generate the group of isometries in the plane (reflections, rotations, translations and glide reflections). Thus, a Piagetian perspective on the Erlangen program demonstrates the psychological basis for Euclidean geometry. The following example illustrates its psychological power.

Consider a triangle as a particular coordination of rotations, A, B and C (see Figure 3). Rotations are transformations that constitute elements within the principal group, which describes all Euclidean objects, but the particular object under consideration—the triangle represented in Figure 3—arises from a particular configuration of the three particular rotations. To study their composition, consider their effect on the bottom side of the triangle. Ignoring changes in length, rotation A transforms this side to the left side of the triangle; rotation B transforms that side to the right side; and rotation C transforms it back to the bottom side—now facing the opposite direction. Thus, we see that the combined effect of composing the three rotations is a rotation of 180 degrees. This intuitive argument about the sum of angles in a triangle appeals to nothing but our own operations, using the figurative material only to keep track of their particular configuration (like a group action on a set).

We can see the triangle sum argument as an application of the Erlangen program when we consider subgroups of the principal group. Specifically, we can consider the subgroup of rotations and reflections through a given point. Note that this group is isomorphic to the quotient group formed by treating transformations as identical when they differ by a translation or dilation. Thus, the three rotations under consideration (A, B and C) would be treated as rotations about the given point—rotations that, together, would form a straight angle of 180 degrees. In this way, the Erlangen program points not only to a classification of geometries but also to the psychological power of our own mental actions. I argue that this psychological power extends much further than geometry.

Piaget used mathematical groups to describe the structure of students' logico-mathematical operations across several domains, including children's constructions of space and number. He described the organization of operations that children rely upon to construct space as a group of displacements: "It is because the group concept combines transformation and conservation that it has become the basic constructivist tool" (Piaget, 1970, p. 21). As members of a group, these operations had to satisfy two fundamental criteria: the operations needed to be composable with other operations (Property 1); and they needed to be reversible (Property 3).

Composability

The possibility of combining two operations to form a third operation organizes all three within a structure for operating. Likewise, mathematical objects, such as shape and number,

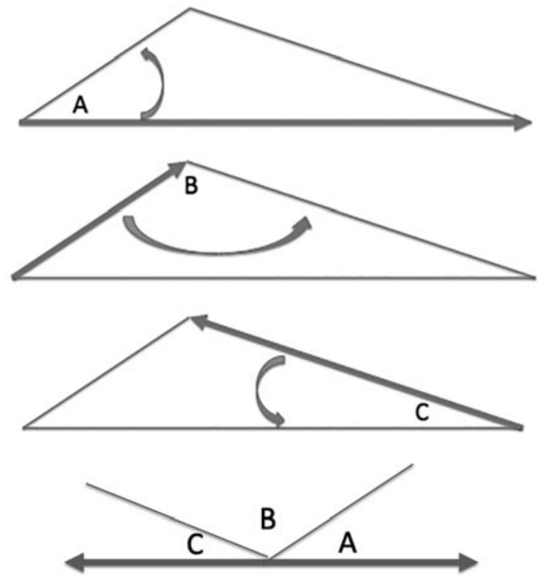

Figure 3. Sum of angles in a triangle.

are related within a space of like objects. "Integers do not exist in isolation from one another, nor were they discovered one by one in some accidental sequence and then, finally, united into a whole" (p. 7). For instance, the operations that define 7 as a number are inseparable from those that define other natural numbers: 7 is two more units of 1 beyond the collection of five 1s that define 5; *i.e.,* 7 is a composition of five 1s and two 1s. Ulrich (2015) has exquisitely elaborated on the operations children use to construct and transform numbers—operations like unitizing, partitioning and iterating. In doing so, she further demonstrates the psychological basis for numbers, as well as their organization within a system; *i.e.* children's number sequences.

The possibility of composing operations within a system allows us to transform mathematical objects (as coordinations of operations themselves) while preserving the structure of the system as a whole. In Klein's terms, it empowers us to transform a manifoldness while holding certain properties of its objects invariant; object permanence and the conservation of volume are Piagetian examples of this Erlangen principle. Furthermore, composing operations enables us to extend our ways of operating through long chains of operations that extend shape and number beyond the finite bounds of experience.

Reversibility

The second fundamental criterion of logico-mathematical operations is that they can be undone by other operations within the system. Reversibility provides the ever-present possibility of returning to a starting point and carrying out the same operation (or chain of operations) again, with assurance of the same result.

> An operation is a perfect regulation. What this means is that an operational system is one which excludes errors before they are made, because every operation has its inverse in the system (*e.g.,* $+n - n = 0$) or, to put it

differently, because every operation is reversible, an erroneous result is simply not an element of the system (if $+n - n \neq 0$, then $n \neq n$). (Piaget, 1970, p. 15)

Together, properties of composability and reversibility distinguish particular mental actions—like partitioning and rotating—as mathematical operations. Because mathematical objects consist of coordinations of such operations, these two properties also distinguish mathematics as a field of study while emphasizing its psychological basis. As such, students need not settle for learning mathematics; by constructing and coordinating their own composable and reversible mental actions, they are constructing mathematics.

Expanding the Erlangen program to fractions and algebra

Whereas Klein (1893) used groups of transformations to classify geometries, Piaget (1970) used the transformations themselves to describe the psychological basis of mathematics. Both perspectives rely on organizations of transformations to emphasize the dynamic nature of mathematical objects. Here, I extend that dynamic vision to fractions and algebra. In fact, Klein himself envisioned such extensions: "I did not conceive of the word geometry one-sidedly as the subject of objects in space, but rather as a way of thinking that can be applied with profit in all domains of mathematics" (Klein, 1923, as cited in Gray, 2015, p. 63).

Fractions: the splitting group

In geometry, shapes—with properties like area, angle measure and curvature—are the objects of study. Triangles, for example, consist of particular configurations defined by coordinated operations that include rotating through angles, as we saw with angle sums (see Figure 3). Numbers, and fractions in particular, can be defined similarly. Every fraction arises through a particular pairing of a partitioning and an iteration—elements from a group action on a unit (Norton & Wilkins, 2012). Consider, for example, the fraction 5/3 acting on the unit segment (see Figure 4).

$$I_5 \circ P_3\ [0,1] = \left[0, \frac{5}{3}\right]$$

Figure 4. 5/3 as a coordination of partitioning and iterating, acting on the unit segment.

The unit segment, [0, 1], serves as a whole of length 1 that we can partition into three equal parts: P_3[0, 1]. This partitioning results in a segment of length 1/3, which we can iterate 5 times: I_5[0, 1/3]. That iteration results in a segment of length 5/3: [0, 5/3]. All fractions (positive rational numbers) can be constructed in this way, as a coordination of partitioning and iterating the unit segment. The set of all possible pairings of partitions and iterations form a group, satisfying Properties 1-4: the composition of any two pairs forms another pair (like fraction multiplication); there is an identity element (I_1P_1); every element of the group has an inverse (reciprocal fractions); and composition is associative. Norton and Wilkins (2012) referred to this group as 'the splitting group'.

Within the expanded Erlangen program, the splitting group defines a space of mathematical objects with certain properties. The splitting group is isomorphic with the group of positive rational numbers under multiplication, but in line with the Erlangen program, the former focuses on transformations, whereas the latter focuses on the figures on which they operate. The entire space of segments might be transformed by a particular partitioning or iteration, but the relations between the segments—the unit segment (whole), unit fractions of the whole (partitions of the whole) and non-unit fractions (iterations of unit fractions)—would remain invariant, just as relationships within and between triangles remain invariant under transformations from the principal group. This explains why, for example, we have equivalent fractions (or commensurate fractions; Steffe 2004); 2/3 includes the same relationships as 4/6. Also, when the whole is transformed, the sizes of all of its fractions change in proportion to it. The whole (unit) is simply a reference frame for operating, just as it is in geometry.

Consider now, the case of fraction multiplication, more appropriately framed as fraction composition (Hackenberg & Tillema, 2009). Because each fraction is itself a composition of a partitioning and an iteration from the splitting group, the composition of two fractions is of the form IlP-kImPn. Note, then, that the success of the usual fraction multiplication algorithm owes to the associativity and commutativity (not a property of all groups; *e.g.*, the principal group is not commutative) within the splitting group: $I_lP_kI_mP_n=(I_lI_n)(P_kP_m)$. Note also that this composition involves five levels of units, including the whole, which explains why fraction multiplication is so difficult for students to conceptualize. (Hackenberg & Tillema, 2009).

Finally, consider the challenge of fraction division. As coordinations of particular operations within the splitting group, fractions include only one operation—composition. The meaning of division is what children understand it to mean—reverse multiplication, or composition with the inverse element within the group. If I_mP_n is an element from the group, then its inverse would be I_nP_m. Once again, the challenge of computing fraction division lies in the transformations between all of the units, including those involved in finding the inverse.

Fractions are not associative under division; *e.g.,* 1/2/3 is an ambiguous expression (is it 1/6 or 3/2?). However, when we understand fraction division as the composition of a fraction with the inverse of another fraction, we rely on associativity of the operations within the splitting group. Associativity in the splitting group derives from associativity of its generators, partitions and iterations, just as associativity of the principal group derives from associativity of reflections: partitions and iterations transform the space of line segments through a one-to-one correspondence.

Algebra: invariant relationships among covarying quantities

We have seen that an Erlangen perspective on fractions knowledge highlights the importance of units coordination—the transformation of units via operations like partitioning and iterating. In algebra, the situation becomes more complicated because the units being transformed are unknown, or even variable (Hackenberg & Lee, 2015). Only relationships between variables remain invariant, as described by equations and their graphs. Importantly, graphs necessitate a reference frame (*e.g.,* choice of axes), which introduces the possibility of geometric transformations of the frame (*e.g.,* the x and y-axes might be switched). However, the relationships between objects (variables) in this domain should be independent of (invariant under) choice of reference frame, just as they are with fractions and triangles.

Consider the case of the function $f(x) = -2x^2 + 3$, over the real numbers. The equation for this function describes both a transformation and an invariant relationship between the variables represented by x and y. In other words, as x varies in the given domain (real numbers), y co-varies in a manner prescribed by the equation. Research on co-variational reasoning testifies to the complexity of the situation (Carlson, Jacobs, Coe, Larsen & Hsu, 2002).

We can represent the function as a transformation of points from one line to another (see Figure 5). The independent variable x varies within the first copy of the line (left side of Figure 5) through all possible movements on the line. The function itself is a particular coordination of operations from a group of transformations, just as fractions are particular coordinations of partitions and iterations from the splitting group. In the case of linear functions, the group contains all translations of the line, reflections over any point on the line, and dilations of the line; in other words, the group contains a one-dimensional version of the principal group. The function represented in Figure 5, $f(x) = -2x + 3$, coordinates particular operations/transformations of dilating (by 2), reflecting (over 0) and translating (to the right by 3 units).

Linear transformations also include projections of the line to any point on the line. For example, $f(x)=1$ represents the projection of the line to the point $x=1$. In Norton (2016) I argued that although such functions do not have inverses, the mental action of projecting a line into a point does have a reverse mental action—sweeping out a line from a point to create length. Note here that this extension to projections parallels Klein's extension of the principal group to the group of transformations that define projective geometry.

In line with the Erlangen program, I have described linear functions as invariant relationships between variables that co-vary via transformations from a group. Here, the utility of the expanded Erlangen program lies in its focus on those transformations rather than pairs of points, (x, y), or static graphs. This distinction, between functions as dynamic objects versus static figures, has profound consequences for mathematics education, as seen in research on co-variational reasoning and emergent (versus static) shape thinking (Moore & Thompson, 2015). Many students consider graphs as static shapes to associate with equations, or as point plots. In contrast, mathematics educators emphasize the need for graphs to emerge from co-varying quantities, where one varying quantity is transformed into another, as prescribed by an equation.

Consider the example of a parabola. Historically, Greeks defined a parabola as a locus of points in the plane whose distance from a given line was equal to their distance from a given point not on the line. Descartes turned geometric problems like this one into algebraic problems by taking given

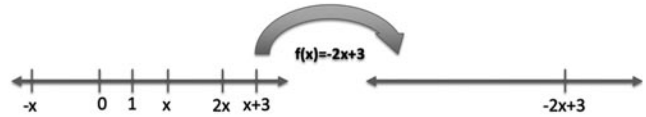

Figure 5. A function transforming points from one line to points on another line.

lines to define a reference frame. He would drop a perpendicular line from the given point to the given line and define the (perpendicular) distance between the given point and the given line as a unit of 1. He would then allow for a new point, x, to vary on the given line, taking on values defined by its distance from the perpendicular line, measured in units of 1. Hence, the geometric problem is reduced to finding an equation that transforms those values of x into values of y (distances along the perpendicular line) in a way that satisfies the invariant condition. In Figure 6, we see that this condition is satisfied by the distance formula (or Pythagorean theorem): $(y - 1)^2 + x^2 = y^2$. Algebraic manipulation leaves the relationship invariant when the equation is transformed into $y = x^2 + 1/2$. Thus, the familiar equation for a parabola is not merely the description for a *u*-shape; it describes how two distances co-vary while maintaining an invariant relationship.

Moore and colleagues (*e.g.,* Moore & Thompson, 2015) have documented pre-service teachers' confusion when the reference frame is altered by, for example, switching the x and y axes (reflecting over the line $y = x$). Such transformations do not alter the invariant relationship between x and y from which the graph emerges. Although we have conventions for choosing reference frames, functions and their graphs are independent of those choices. They are even independent of choice of unit. For example, consider what would happen if the function operated on the value of $-2x + 3$ rather than the value of x. The function f would transform those values in the same way it would transform values of x.

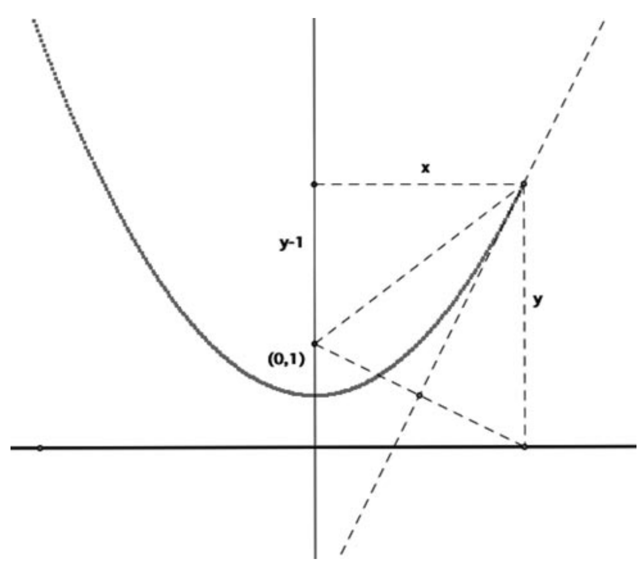

Figure 6. Geometric construction of covariation in a parabola.

The function would not change, only the reference frame. If we reframe that transformation in terms of x, we get a new composition of operations: the dilation, reflection and translation described previously, and those that define the function, f. The graph of that composed function, $f(x)=(-2x + 3)^2 + 1/2$ would look different because it would explicitly reference x rather than $-2x + 3$, but would, nevertheless, include the transformation of $-2x + 3$ as well. Indeed, this is the basis for transformational geometry.

Conclusions and implications

The Erlangen program focuses our attention on the dynamic operations that define mathematical objects rather than the static figures that we use to represent them. For example, triangles can be understood as a coordination of transformations that include three particular rotations. While triangular figures help us keep track of their coordination, composing the rotations themselves allows us to see that they sum to π without relying on an axiomatic system or even rules for Euclidean construction. To be sure, formal mathematics has its own power, especially in codifying coordinations of operations and offloading their demands on working memory, but students should appreciate that mathematics exists first in their own minds, as a product of their own mental activity.

A Piagetian perspective on the Erlangen program expands it beyond geometry while framing mathematics as a product of reversible and composable mental actions (*qua* operations). In each domain highlighted here—geometry, fractions and algebra—there is a principal group that organizes operations, their inverses and their potential compositions. The expanded Erlangen program uses each of those groups like a group action on a set. In the case of fractions, the splitting group might act on the set of segments. Although transformations of segments help us keep track of their effects and invariances, the focus is always on the operations themselves. Whether we consider triangles, fractions, or functions, each mathematical object arises from a particular coordination of particular transformations from the relevant group. Thus, while the Erlangen program empowers students' mathematics, it also demonstrates the unity of mathematics.

The program comes at a cost, not only in terms of the cognitive demand of coordinating the operations that define mathematical objects, but also in constructing those operations as reversible and composable mental actions. Although even young children have access to some mental actions, like reflecting, students may require instructional support for reversing and composing them. In the expanded Erlangen program, the importance of such development is not restricted to geometry; it extends to all of mathematics and includes mental actions like partitioning and iterating, which children might need to develop through sensorimotor activity.

Along with his paper describing the Erlangen program, Klein included a separate paper articulating his vision for mathematics education. Concerns he expressed there and then, in the nineteenth century, echo in present-day frustrations expressed by many mathematics educators: "Instead of developing a proper feeling of mathematical operations, or promoting a lively intuitive grasp of geometry, the class-time is spent learning mindless formalities or practicing pretty trivialities that exhibit no underlying principle" (Klein, as cited in Rowe, 1985, p. 127).

An expansion of Klein's program implies not only the need for different approaches to curriculum and instruction but also the need for new lines of research in mathematical cognition. This research would include studies on the mental actions that undergird mathematical objects and constitute their groups of operations, across every domain of mathematics; and studies on instructional conditions that support constructions of those operations and their coordinations. Research papers cited here exemplify some budding efforts in those directions, especially in the domains of fractions and algebra. If seen to fruition, those efforts might yield a paradigm shift in the nature of mathematics itself.

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