'Magical' Moments in Mathematics: Insights into the Process of Coming to Know

MARY BARNES

At the end of a mathematics lesson, Naidra, an eleventh grade student, was asked to reflect on what he had learned and how this learning had come about. He talked about the advantages of working in a group and "getting your ideas out into the open" The following exchange then took place:

N: . it can be thought-provoking for everybody concerned

I: So, was there anything particularly thought-provoking today? .. I wonder if there was any particularly significant moment during the lesson, anything particular that someone said, because we could play back a little bit of this to you on the video

N: No, I really don't think anything magical happened today.

I: No? Do magical things sometimes happen in lessons?

N: Just, um, flashes of understanding can happen, and that's, it's great when it happens

I: . Can you think of a time when that happened, a sudden flash of understanding?

N: Oh, it's happened a lot It doesn't have to happen in a lesson. It's just, you suddenly understand something that you didn't follow, and lots of different things can spark that off And when it happens, it's good

This exchange aroused my curiosity and interest, and prompted me to investigate 'magical' moments - to seek to understand what they are and how they happen, to look for examples and to reflect on their impact on students' mathematics learning.

Describing 'magical' moments

References in the literature

That sudden flash of understanding which Naidra described as magical is often referred to in the literature as the 'Aha!' or 'Eureka!' experience. It has frequently been described by mathematicians writing about the creative process. Gauss spoke of "a sudden flash of lightning" (cited in Hadamard, 1945, p. 15). Hadamard, drawing on the ideas of Poincaré, described stages in the creative process as preparation (a period of intensive conscious work on a problem), incubation (a period of rest or relaxation away from the problem) and illumination (when a solution suddenly becomes clear). Poincaré also linked mathematical insight to an appreciation of the aesthetics of mathematics Polya (1965/1981) spoke of:

> a sudden clarification that brings light, order, connection and purpose to details which before appeared obscure, confused, scattered and elusive (p 54)

Davis and Hersh (1980) refer to:

> the flash of insight, the breakthrough, the 'aha,' symbolizes that something has been brought forth which is genuinely new, a new understanding for the individual, a new concept placed before the larger community (pp. 283-284)

More recently, Rota (1997), like Poincaré, linked insight with the appreciation of mathematical beauty:

> we think back to instances of appreciation of mathematical beauty as if they had been perceived in a moment of bliss, in a sudden flash like a light bulb being lit The effort put into understanding the proof, the background material, the difficulties encountered in unraveling an intricate sequence of inferences fade and magically disappear the moment we become aware of the beauty of a theorem The painful process of learning fades from memory and only the flash of insight remains. (p. 130)

Finally, many of the mathematicians interviewed by Burton (1999a, 1999b) described, sometimes in almost lyrical terms, feelings of joy, excitement or euphoria which accompanied the sense of 'Aha!' Making a new discovery or finding a new connection were compared with a light switching on, climbing a mountain and seeing the view from the top, finding a new path through unfamiliar terrain or seeing how to fit pieces into a jigsaw. Similar responses were reported by the scientists studied by Shaw (1994) who identified these feelings as belonging to a key stage in the creative process. Like Poincaré and Hadamard, Shaw called the stage when everything falls into place 'illumination' and gave a list of associated positive emotions including: "orgasmic, free, walking-on air, [...] euphoric, high, happy" (p. 6)

Only a few mathematics educators have written about the 'Aha!' experience, but those who have done so have stressed its importance Krutetskii (1976) reported the joy of creation experienced by mathematically gifted children:

> this joyous sense of a small discovery does not just come down to simply experiencing a successful result but includes a feeling of satisfaction from the awareness of the difficulties that have been overcome, that one's own efforts have led to the goal (p 347)

McLeod (1989), writing about affective responses to mathematical problem solving, claimed that although frustration is the most common emotion encountered, the 'Aha!' experience is also perceived very intensely He subsequently stressed the positive emotions which accompany the moment of insight, suggesting that:

> emotional responses can play a significant role in students' learning of mathematics (1992, p 583)

Papert (1980) followed Poincaré in linking mathematical insight and intuition with pleasure and an appreciation of beauty He described the excitement and pleasure people feel when they reach a key stage in solving a problem, even if they do not know where the process is leading (p 202). Dreyfus and Eisenberg (1986) also linked the excitement of insight to an appreciation of the aesthetics of mathematical thought They listed 'surprise' among the important factors contributing to the aesthetic appeal of a solution or proof, and suggested that: "the 'aha' of problem solving activities should be exploited" (p 9) And finally Burton (1984), writing about problem solving, associated curiosity with wonder and pleasure:

> The process is initiated by encountering an element with enough surprise or curiosity to impel exploration of it [. . Getting a sense of pattern] releases the tension into achievment, wonder, pleasure, or further surprise or curiosity that drives the process on (p 40)

A characterization of 'magical' moments

Naidra, in the interview quoted earlier, described these moments as sudden flashes of understanding, adding "It's great when it happens", suggesting they were highly motivating Simon, another student from the same class, was asked about such moments and commented:

> It happens, it happens in most projects, when [ ] you're just looking and looking over again, and then it just happens and you're on your way again I mean, you can just keep throwing forward ideas and stuff and then one - like a word or something - it might just click, I don't know, it's just really that

I noticed that the students' descriptions had a great deal in common with those I had found in the literature. This led me to formulate the following characterization of a 'magical' moment:

1. There is a claim to a sudden realisation of new knowledge or understanding Usually this new knowledge is 'seen' with great clarity, or experienced with a high degree of confidence or certainty
2. The realisation of new knowledge is accompanied by a positive emotional response, which may be described variously as joy, delight, pleasure, excitement, triumph, satisfaction, surprise or relief

For the students, the sudden realisation of new knowledge was expressed when they spoke of "a flash of understanding", "it might just click" and "different things can spark that off" The positive emotional response was evident from the repetition in Naidra's statements: "It's great when it happens" and "When it happens, it's good" Simon's emotional response gives a strong impression of anxiety followed by relief when "you're on your way again"

I recognize that while an 'Aha!' in problem solving is sometimes accompanied by feelings of certainty, on other occasions it may be very tentative - just a glimpse of a pattern, or an intuitive idea of a possible path worth exploring, which may or may not provide a useful approach to the problem I have chosen to focus on those occasions when there is a conviction of certainty - a way has been found to bridge a gap, a new link has been forged, everything seems to fit into place or a solution is clear Such occasions may best be described as illumination or insight rather than intuition Even though the conviction of certainty may not always turn out to be justified, it is this type of experience that Naidra appeared to have in mind when he spoke of magical things happening and which I wish to explore

In what follows, I first give some examples of magical moments observed in one particular classroom, before reflecting on these examples from the point of view of the learner, focusing on why these incidents might be regarded as important and how they came about Following this, I give a description of the classroom context in which these incidents occurred, and use this to formulate some hypotheses about what mathematics teachers might do to encourage magical moments Finally, I draw some tentative conclusions and suggest some issues which I believe warrant further research

Classroom examples of magical moments

The observations reported here took place in an eleventh grade class (students aged 16-17) in a co-educational high school in Melbourne The students in this class had chosen to study an integrated mathematics course which included some introductory calculus I videotaped a few lessons in this classroom as part of a larger study of the negotiation of meaning in mathematics, details of which, including methodology and data collection procedures, can be found in Clarke and Kessel (1995) and Clarke (1998) After my curiosity about magical moments was aroused by Naidra's remark, I modified the questions asked in subsequent interviews and reviewed the taped lessons to investigate the occurrence of this phenomenon

To illustrate such magical moments observed in the classroom I have, for reasons of brevity, chosen examples drawn from a single lesson The group whose work is reported consisted of two boys, Simon and Naidra, and two girls, Maria and Lida [1] Lida was a recent migrant to Australia, having arrived from Europe about two months earlier Naidra had transferred from another class at about the same time. Simon and Naidra had worked together before but never with either of the girls Lida's English was still very limited and she appeared to lack confidence She said little, but assured me that she was able to follow the discussions in the group She claimed that she said nothing because she felt no need to – she agreed with what the others were saying.

The problem

The class was working on *The Theatre Confectionery Pack* problem (Williams, 1990, pp 27-28). Briefly, each group was given a sheet of paper 20 cm by 30 cm, with instructions on how to fold it into an open box suitable for holding candy Then they were asked to work out how to vary the positions of the folds to obtain a larger box This problem is a more complex variant of the standard open box problem familiar to calculus students Figure 1 shows the diagram given in the instructions, with variables introduced by the group added Figure 2 shows a completed box as seen from below

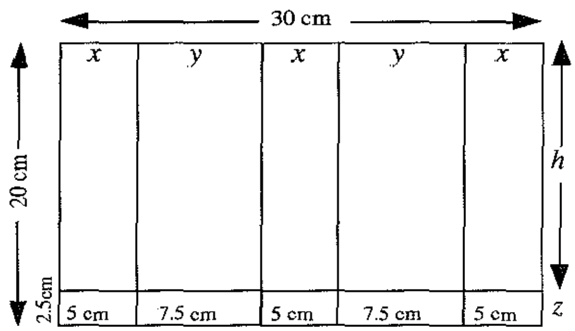

Figure 1 Net for the confectionery box

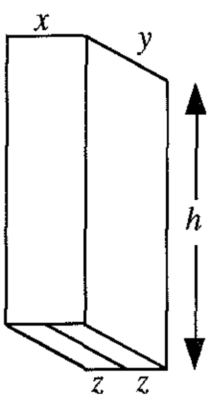

Figure 2 The finished box

It is important to note that these students had never had a formal lesson on maximum-minimum problems. This task was the third in a series designed to introduce them to such pro-blems and to have them develop techniques for solving them and reflect on the concepts involved. Earlier in the year, the class had worked on a project to find gradients of graphs. As a result of this, they knew the definition of a derivative and how to find derivatives of polynomials, but their only knowledge about finding maxima and minima was what they had worked out in the course of the previous two tasks They had been given no rules or procedures, nor had the teacher attempted to summarise for the class what they had learned.

Outline of the group's solution process

A summary of the steps the group took to solve the problem is included to assist the reader to follow the excerpts from the transcript in the examples which follow

- During the previous lesson, the group had made a box, following the instructions and using the measurements given
- At the beginning of the lesson reported here, they introduced variables x, y, h, and z, as shown in Figure 1, and used these to write down the equations $3x + 2y = 30$ and $h = 20 - z$. They wondered aloud whether x and y were related to z and h in some way
- Next they rewrote the first equation in the form $y = -\frac{3}{2}x + 15$ and graphed it, but seemed unsure what the graph told them
- Simon noticed that they had two linear equations, and began to solve them simultaneously – ignoring the fact that the variables in them were different. He simply changed h to y and z to x. After a while, he realised that this gave him meaningless results
- Meanwhile, Maria noticed that z always had to be half of x if the two flaps were to fit together properly to form the base She made a couple of attempts to point this out, and eventually the others listened to her. This was a key step which set them on a pathway towards a solution of the problem
- They realised they could draw up a table of values of x, y, z, h and V (the volume of the box) and used this in a systematic way to find the maximum volume
- As they calculated entries for the table, they became aware that only some values of the variables made sense in the context of the problem This then led them to interpret the equation $y = -\frac{3}{2}x + 15$ and its graph as giving information about the domain of the independent variable.
- Finally, they decided to use calculus to get a more precise answer, and after some discussion they derived an equation for the volume as a function of x. They were expanding this in preparation for differentiating it when the lesson ended

Magical moment 1: It's half of x

When the students were looking for relationships among the variables x, y, h, and z, Maria had an idea. After a couple of false starts, she managed to make herself heard [2]

M: It's half [*pointing to the model*] because like, it seems like, [*indistinct*] you see how, here, this is half of the base -

S: Is that just a coincidence?

No further attention was paid to Maria's remark, and she did not pursue the matter Six minutes later, the group was still stuck, and discussed making another box with different dimensions. They went back to the instructions, and Simon read them aloud. He held the box and turned it over and over, while the rest watched. Maria tried again.

M: I just thought, it's got to be half of *x*, so it will fold up.

S: It goes, hang on, is this *x*? [*4 seconds pause, while he studies the model*]

M: [*Indistinct*]

S: You're right, it is [*Looks towards Maria and nods emphatically*]

N: [*To Maria*] So *z* has to be half of *x*.

S: [*To Naidra, excitedly*] Because look, this is *z*, and this is *x*, so it's always going to be half [*Turns to Maria*] It's not a coincidence. Well done!

From Simon's reaction, it is clear that this was a magical moment for him. He finally understood what Maria had been saying and acknowledged this. His positive emotional response was indicated by his excited manner of speech and the way he went on to explain the whole thing to Naidra, even though Naidra had already acknowledged that he understood it.

Why did the boys not grasp this idea when Maria first pointed it out? My initial thought was that they had not paid much attention to what Maria said - perhaps they were not expecting her to make a helpful observation, because they did not know her well, and were unaware of her mathematical and problem-solving abilities, or perhaps they held preconceptions about gender and mathematical competence But this interpretation does not stand up to scrutiny. Simon's remark "It's not a coincidence" indicates that he had indeed paid attention to what Maria said and had kept it in mind during the ensuing six minutes of discussion.

An alternative interpretation is that when Maria first made her observation, the boys were not ready to take it in. The fact that Maria did not respond to Simon's question "Is that just a coincidence?" may indicate that she herself was unsure of the validity or significance of her claim. It is worth noting that they did not have a diagram like Figure 2 to study, but were working from Figure 1 and their model, which had not been labelled with the variables *x*, *y*, *z*, and *h*. When they began to think about how they could alter one dimension of the box, they re-read the instructions and studied the model more carefully They then began to wonder about how any change would affect the other dimensions, and at this point Maria's suggestion may suddenly have made more sense

Magical moment 2: Restricting the domain

Very early in the session, the group derived an equation connecting *x* and *y*, expressed it in the form $y = -\frac{3}{2}x + 15$ and drew a graph of this line in the *x*-*y* plane.

S: Um, but I don't know what the hell that gives us

N: Well, it's the possible widths, that it can be

It could be inferred from this that Naidra had realised that the graph could be used to work out the domain of *x*. When the teacher joined the group, Simon explained it to her, saying:

S: ... and we've got a graph, to show us where, if you pick an *x* value or a *y* value, you can find out what the other is.

This suggests that Simon saw the graph as useful only for finding *x* from *y*, or *vice versa* - he had just used it in this way to check their working. Since he was normally an articulate student, I assume that if he were aware of the more complex understanding hinted at by Naidra, he did not at that moment see it as relevant to the problem

Some twenty minutes later, the group had worked out how to express all the variables in the problem in terms of *x* and were drawing up a table of values Simon discovered that $x = 16$ gave a negative value for *y*.

S: And now I find out that *y* equals negative nine, which is impossible. Which means there's a domain and a range, *x* can't go over ten

N: Of course it can't, it's unrealistic.

This was the first mention of the technical terms 'domain' and 'range'. For Simon, it appears to represent a shift from regarding the relationship simply in terms of ordered pairs of numbers to invoking some of the language and concepts associated with the idea of a function. The tone of Naidra's reply seemed to indicate that this was obvious to him and that Simon had said nothing new.

Nearly two minutes later, the following exchange took place:

S: Why can't it go above ten? [*5 seconds pause*]

N: Cos you get a negative *y*

Later, the teacher joined the group again and asked them to explain what they were doing. Once again, Simon was the spokesperson and explained what they had found out about the domain

T: Why does it have to be between zero and ten?

S: Because there's three of them, and if it was ten, there'd be thirty, and there's no room for *y*s, and anything above. And on the graph it shows our domain, ... and our range, *y* can't be bigger than fifteen

Naidra later described this series of exchanges as giving him insight It seems here as though Simon and Naidra were thinking aloud, as they jointly constructed a deeper understanding of the connection between the dimensions of the box, the possible y values and the graph. This new understanding made such an impression on Naidra that, when it was the group's turn to report one interesting thing they had discovered to the rest of the class, this is what he suggested they should talk about. He mentioned this when interviewed

> I: I'm interested in moments when something became clear, or when you suddenly made sense of something. ... Can you tell me if that happened?
>
> N: Oh, it happened on a few occasions.
>
> I: .. Can you tell me about one of them?
>
> N: OK, how about what I said when I was at the board, um, restricting the domain.
>
> I: Restricting the domain, yes. Now how did that become clear to you?
>
> N: Well, Simon was the one who came up with it, and he just, I guess he just realised that if you solve for y, that'll give you a rule, make it possible to graph it, and therefore be able to see where there's a restricted domain. .. Yes, well, once Simon showed it to us, and we could all see what he was doing with it, it just became clear.
>
> I: So was there something particular made it clear?
>
> N: Um, sometimes, it just presents itself, I think. And, this was one of those times, it just became clear, when it wasn't a second before. That's what happened.

The description Naidra has given satisfies my criteria for a 'magical moment': he claims to have achieved a new understanding, experienced it as "clear, when it wasn't a second before" and his emotional response is evidenced by his desire to explain the newly grasped idea to the rest of the class.

When the group first discussed the meaning of the equation, Naidra appeared to understand better than Simon what it told them, and said so on three occasions. But from his own account it was not until later, when he heard Simon explain it that he realised the full significance of the idea. In the initial exchanges, this incident may not seem like a 'magical moment'. It appears that coming to know can sometimes be so gradual that students fail to recognize *that* they know or *what* they know. Yet, when it is finally seen clearly, it is experienced as a moment of illumination, and the former uncertainty is completely forgotten. In this case, although Naidra himself first articulated the idea that the equation told them about "the possible widths", it was not until Simon linked this with the formal language of domain and range and the physical shape of the box that he experienced a moment when "it just became clear".

Magical moment 3: Too many variables

Early in the discussion, Simon was worried that they had too many variables

> S: we can't have three variables in an equation.
>
> N: Why can't you?
>
> S: No, I mean, in a graph like, you can't graph something with three variables [*Naidra nods*] you have to get it down to two, so you have a y and x [*gestures, begins to draw*]

Much later, after the group had realised that z is half of x, they began to discuss making a table so as to calculate the volume systematically. Simon began to draw up the table using values they had already calculated. Naidra interrupted him.

> N: Could I suggest that we .. choose one variable to work it around, and then work from the lowest to the highest possible one, using integers in the table. In that way, we get a really good table, you know, that we can see
>
> S: We could, yeah, okay, I was just - so like, to start with our height as being 1, and then do two, and then three, and then get up to nineteen. Okay, we'll do that. I'll just work out these two volumes. How much was this?
>
> N: No, hang on, I think we should work around x, because um z is x over two [*1 second pause*] alright? ... We can say that x is the variable and the rest are dependent variables.
>
> S: Oh but what about y? I think -
>
> N: Yeah, y is a dependent variable, cos if you take, if you work //around x
>
> S: //Oh yeah, but I mean, you can say, there's another way of saying it, that if you work around y, you can work out the rest of them then. If I say y equals five -
>
> N: But z is not y over two
>
> S: You can work out what x is, through y, and then work out z, but yeah, we'll work around x, which is probably more neater, but I'm saying the only thing that we can't work around is height, although, you could probably get to all answers through height //as well
>
> N: //Yeah.
>
> S: But I mean, x is probably the easiest, yeah

When they started on the table, Simon had not fully thought through how they would do it. His initial proposal was to start with height. This was contradicted in his later statement "but I'm saying the only thing we can't work around is height", which he then immediately contradicted again. Simon, it seems, was only thinking of filling in numbers in a table.

Naidra raised the discussion to a more formal, theoretical level by bringing in the idea of (independent) variables and dependent variables This may have helped Simon to conceptualise the relationships between the variables in a different and more powerful way. By the end of the extract, Simon was quite clear that any of x, y, z or even h could be treated as the independent variable, and the others expressed in terms of it But he was still apparently thinking of a chain of relationships which enabled each variable to be calculated from the next. As the next extract shows, he had not realised that this implied that V, the volume of the box, could be expressed as a function of x alone (or of any of the other variables by themselves) Naidra, too, was probably thinking in terms of a chain of relationships. His concern to "work around" x seems to have been largely because he thought it would be more efficient

The group proceeded with their table of values and used it to find an approximate maximum volume Finally, they began to think of calculus and decided "We definitely have to do a derivative".

> S: Yes, but, we need an, an equation for the graph, //which we don't have.
>
> N: //Yeah, that's it, exactly
>
> S: Do you want to try and graph the whole thing, try and, because we've got three variables, that's what I don't -
>
> N: We don't have three variables
>
> S: Oh, because we can do x over two [*Very excitedly*] Look what we can do
>
> N: //What can we
>
> M: //That's for z
>
> N: All right, I'm looking
>
> S: Well basically volume, equals, x, times y, times h. So volume, if we want just one, we can relate it all to one variable, that's x and then we'll have two variables to graph
>
> [*Simon then explained how to write an equation for the volume in terms of x alone*]
>
> S: So that's what y equals -
>
> N: Ohhh, that's brilliant -
>
> S: and this is what height equals and our x value -
>
> N: Oh, that's, that's brilliant -
>
> S: Now we've got two variables, we can graph it! Just expand it, and we can graph it! Just expand it -
>
> N: Oh, that is beautiful.

In the interview that followed, Naidra was asked about this incident.

> I: OK, now there was another time, when you suddenly said, "Oh, that's wonderful, that's brilliant, beautiful!" Now, what was that?
>
> N: Um, I think that was Simon again, um, we, I could see where the volume was going to be x times y times twenty minus z [*He gave a detailed explanation*] . . Simon saw that, came up with it, and then I saw it . he explained it to me.

This was a magical moment for both Simon and Naidra. For Simon, this is clear from the statement "Look what we can do", his excitement as he said it, the many repetitions, and the way he kept on talking over Naidra's interjections, until in the end he had explained the whole thing twice. And Naidra's response leaves us in no doubt as to how he felt.

Notice that it was Naidra who first pointed out that they could treat x as the independent variable and the rest as dependent variables Simon then took this up and developed it, saying that any of the three variables could play the role of independent variable. Later, when they began to derive an expression for the volume, Simon's feeling that they had too many variables returned, and Naidra had to remind him that this was not so. But when Simon adopted his suggestion, the final result came as a revelation to Naidra. In the interview he was certain that the whole thing was Simon's idea. It is as though he "knew" at some preliminary level, but did not know that he knew until he heard it explained by Simon.

Was this a magical moment?

As the students made a table, starting with $x = 1$ and increasing by one at each step, they noticed that y, z and h changed by constant amounts from row to row but V did not. Naidra suggested that this was because V was a cubic function When the table was complete, Naidra looked at it and said:

> Um, that's beautiful, that table, it really is, it really gives you insight

and about 30 seconds later:

> But I've got a very happy feeling about this problem, I do, I've got to say.

These reflective statements express aesthetic appreciation and very strong positive affect, but the claim to new knowledge is less clear. Naidra's use of the word 'insight' seems to be more a reflection on what he had observed and commented on earlier than a claim to a new understanding at that instant. Thus, it may not be a magical moment according to my definition, although it certainly demonstrates the close relationship for this student between learning, aesthetic appreciation and positive emotions

Reflections on the learning context

Should we value magical moments?

Are magical moments important enough for us to be concerned about recognising and encouraging them? The interview evidence suggests they are motivating: recall Naidra's remark in the first interview "It's great when it happens". Boaler (1997) also observed excited and enthusiastic responses from students working on open-ended projects when they made breakthroughs (see, for example, p. 53).

I believe that the feelings of pleasure and excitement which these events produce are of value both for their own sake, and because they may influence students' continued participation and performance in mathematics. Burton (1999b) noted the importance of moments of insight for the mathematicians in her study:

> Mathematical activity for many of these mathematicians was driven by curiosity and the resultant pleasure when something was resolved [...] Satisfaction was associated with finding a pattern, making a connection, eradicating a difficulty - coming to know [...] Far from understanding being something which is only driven by knowledge, there is both a *need* to know and an associated *pleasure* in knowing which is its own reward. (p. 29; *italics in original*)

She also reported that the excitement and euphoria experienced are "what holds them in mathematics" (Burton, 1999a, p. 135). If these feelings are so important for professional mathematicians, we may surmise that similar feelings will have a powerful influence on those learners who experience them.

I earlier quoted McLeod (1992):

> emotional responses can play a significant role in students' learning of mathematics. (p. 583)

He explained that although an emotional response may be fairly transient, if students repeatedly encounter the same or similar emotions in mathematics, it can lead to the development of positive attitudes. Attitudes are more stable than an emotional experience, however intensely felt. Naidra's "happy feeling" quoted above seems to illustrate the effect of repeated positive emotions beginning to produce positive attitudes. The importance of developing such attitudes is stressed in recent curriculum documents (e.g. AEC, 1991; NCTM, 1989), and the model of academic choice developed by Meece and her colleagues (1982) relates positive attitudes to choices about future participation in mathematics.

The confidence and clarity with which new knowledge is perceived in a magical moment are important. Rather than interpreting explanations given by other people, the students have constructed the knowledge for themselves - they have 'authorship' of the ideas. This provides an important element of self-affirmation and encourages them to have confidence in their capacity to master new mathematics in the future. It may also mean that the new knowledge is better understood and recalled than knowledge achieved without the same struggle and flash of insight. Boaler's (1997) study supports this suggestion. Her data on long-term retention showed that students taught through open-ended projects had superior long-term retention to those taught using a traditional textbook-based approach (pp. 73-74). As one of the students she interviewed claimed:

> it's more interesting and I think you tend to remember it more when you've found things out for yourself. (p. 60)

Since magical moments are intensely-experienced events, it seems likely that individuals will store them as episodic memories, and the nature of these episodes means that they are intrinsically linked to the semantic knowledge involved. Research by Mackenzie (cited in White, 1981) suggests that this may aid retention of the material that has been learned.

> Mackenzie found that a geography excursion which emphasised the formation of episodes and their linking with facts and skills had a remarkable effect on the retention of knowledge. (White, 1981, p. 232)

However, there is need for caution here. White (1982, 1989) also found that recall of episodes was most strongly influenced by their "vividness *and rarity*" (my italics). This suggests that the effectiveness of magical moments in promoting recall of the mathematical ideas involved would depend on whether each moment was sufficiently novel to be distinguishable from other such moments, and so count as 'rare'.

Can we recognise them?

It is not always possible to identify magical moments from transcripts of a discussion, or even from viewing the videotape record. First, it can be difficult to be sure that a student has grasped a new idea. The student may appear to be stating new knowledge, but, as in the second and third incidents discussed, may not realise its significance until later - possibly when another person restates it and points out its relevance or links it to other knowledge. An expression of excitement or triumph may be the best indicator that a dimly-understood idea has suddenly become clear, but these emotional responses are not always evident to an observer, because some students do not display their feelings overtly. For example, when interviewed after the lesson, Lida (the student who was not confident about her English) mentioned the moment when it suddenly became clear to her why z was half of x as giving her insight. Yet at the time she said nothing, and her facial expression and body language did not reveal her thoughts.

Thus, identifying magical moments can be problematic even when, as here, students are articulating their ideas freely in small-group discussions. It may be even more difficult in classes taught using more traditional approaches. Anyone interested in investigating these experiences would have to rely largely on self-report data. Teachers may find it hard to recognise them unless students give cues.

Finally, magical moments can sometimes be misleading to problem solvers - they may think they have seen the light, but discover later that they were mistaken. Although this did not happen in my examples, it is certainly a possibility.

How do magical moments happen?

I noticed that in each of the examples cited, the moment of insight was associated either with *changing the way of looking at the problem*, or with *making a connection between the immediate problem and more general mathematical concepts or structures*, or both.

For example, the first magical moment involved seeing the box in a new way - instead of concentrating on the net diagram given in the instructions, the students began to study in detail the box they had made, and this helped them to see that when the two little flaps, each of width z, were folded up, they met in the middle, and together equalled the width of the box. The second magical moment was associated with a shift from thinking of the equation they had derived as giving information about pairs of numbers to noting how it related to the way the paper had to be folded, and to seeing the equation as a function and linking it with general properties of functions such as domain and range And in the third example the students made a shift from seeing the set of relationships they had derived as rules for calculating entries in a table to thinking of them as functions, linking this to the idea of independent and dependent variables, and realising that they could use these relationships to express the volume as a function of a single variable Thus, magical moments appear to involve seeing things from a new perspective, and may also be associated with increased formalisation.

Received wisdom from many mathematicians (e g. Hadamard, 1945; Davis and Hersh, 1980) suggests that, after an initial period of intensive work on a problem, one can put it aside, relax and think of other things, and a moment of insight may come "out of the blue" Rota (1997), on the other hand, stresses the process of struggle, eventually culminating in a flash of insight, and suggests that the effort and the difficulties may often be forgotten in the euphoria of the 'Aha!' experience.

The incidents I have described fit better with the description given by Rota, although it is possible that occasions when insight came "out of the blue" may have occurred out of school or at other times when the students were not being observed. As Naidra remarked, "It doesn't have to happen in a lesson". According to their own reports, the students I observed certainly experienced the magical moment as a sudden insight, but the process of groping towards it was quite long and confused. Understanding seemed to develop through uncertainty and struggle, but there came a moment at the end when everything was suddenly clear In each example described, a process of preparation seemed to be necessary before the insight could occur, but the data are insufficient to indicate what forms of preparation might be most helpful

Reflections on the pedagogical context

The teaching approach in the class observed

To simplify reporting, the examples described were drawn from the work of just one group, during a single lesson They illustrate what can happen in the circumstances of this particular classroom, but tell us nothing about how often magical moments might occur in classrooms using different teaching approaches. Nor, indeed, do they indicate how frequently magical moments happened in the class observed, although Simon's claim that "it happens in most projects" and Naidra's "it's happened a lot" certainly suggest that the environment in this classroom was conducive to their occurrence Similar events observed at other times and with other groups in the same class provide further confirmation that the examples described were not isolated incidents.

While many instructional strategies may lead to the occurrence of magical moments, the teaching approach employed in the classroom studied was sufficiently structured to warrant a description of its key features, outlined below; further details can be found in Williams (1994, 1997).

Challenging problems

The usual procedure in the class was for groups of four students to work for three or four lessons on a challenging, open-ended investigation The teacher designed these tasks with the aim of enabling students to develop for themselves the mathematics needed, by working on the task in small groups and then discussing and comparing their ideas with the rest of the class. Before such a task began, there was no formal teaching of the mathematical content. At most 10 minutes might be used to introduce the topic.

Group composition and teamwork

The teacher decided on the composition of the groups for each task, taking into account the complexity of the task and her knowledge of the personalities and mathematical capabilities of the students Groups were expected to work together as teams and to resolve their problems by collaborative efforts

The teacher's role

As groups brainstormed and developed their ideas, the teacher moved around, listening to and sometimes joining in group discussions. She asked questions to elicit progress, to help students clarify their understanding, to prompt them to elaborate their explanations, to help groups refocus and so facilitate a breakthrough, to encourage generalisation or extension of the original problem, or to elicit evaluation; but avoided answering questions, giving hints or indicating whether an explanation or an approach to a problem was 'correct' She made a habit of asking a question and then walking away, leaving the group to think about it for themselves

Reporting

After about twenty minutes' work on a problem, each group was asked to report its progress to the class as a whole. The teacher decided the order of reporting, and at times nominated which parts of their material groups were to report on, to ensure that each group would have something new to contribute. After a reporting session, students returned to their groups for further work on the problem, and the cycle continued. The reporting session allowed students to see how other groups approached problems and to realise that there is no single 'correct' method. The pace of reporting sessions was determined by the students. They were encouraged to ask questions of the student at the board, either to clarify their own understanding or because they suspected a flaw in the justification, but not to criticise until it was their own turn to present.

Related tasks

Between problem-solving sessions, the class might work on other mathematical tasks, perhaps even on a different topic. Where possible, these tasks were designed to introduce ideas which might be of help in solving the problem - but the students were not made aware of this.

Classroom norms

The teacher's approach strongly emphasised social norms which she believed would help to create a supportive learning environment in which all students would feel able to take risks. These included collaborating effectively within groups, showing consideration for other people's feelings and acknowledging other people's contributions. In addition, she stressed certain socio-mathematical norms (Yackel and Cobb, 1996), such as the need to justify assertions and an appreciation of aesthetic aspects of mathematics.

What might teachers do to promote magical moments?

Anyone may experience occasional flashes of insight while trying to solve an unfamiliar type of problem or when seeking to understand mathematics in a coherent and meaningful way. From time to time, a breakthrough will be experienced, when something is seen in a new way, and its significance becomes clear or its links with some other part of mathematics are understood. Such events will happen no matter how competent (or otherwise) the teacher or what style of teaching is used.

One might surmise, however, that magical moments would be less likely to occur in the context of expository teaching alone, where work on problems is uniformly preceded by carefully-structured preparatory explanations and guided practice. If instruction progresses by small, simple steps, and the teacher anticipates difficulties and provides immediate clarification, students will have less need to struggle and less occasion to make efforts of their own to achieve understanding and insight. Further, as Boaler's (1997) study indicates, this approach to teaching tends to discourage thinking in favour of rote memorisation and 'cue-based behaviour', in which students do what they think is expected of them instead of thinking about the mathematics involved. Under such conditions, magical moments might appear rather less often.

Aspects of the teaching approach described above may provide some clues as to ways in which insight and illumination may be nurtured. I make no claim that this is the only way to promote magical moments. I simply suggest that this approach has been observed to be somewhat effective and so may be worth further exploration.

The level of challenge in the task set seems to be critical. If the challenge is too great, many students will be unable to make headway and will likely become frustrated; with too little challenge, a solution will be too easily attained, and students will have little need to strive for insight. Since closed tasks are less likely to meet the needs of all students, open-ended problems with a variety of entry and exit points are indicated.

Interaction among students can reveal different ways of thinking about a topic and may provide for some students the key to solving a problem or understanding a new concept. This is perhaps what Simon was referring to when he said "a word or something ... might just click". If students discuss problems in small groups, the chance of any one student experiencing a flash of insight should be greater, because more talk ought to mean a greater variety of ideas being proposed.

A focus on understanding processes rather than on getting answers quickly may also be important. Each individual needs to be made responsible for ensuring that (s)he has understood and can explain the method of solution. One way of ensuring that students accept this responsibility is to require them to justify their solutions to the rest of the class, as was the normal procedure in the class I observed. If it is left to the other students rather than the teacher to validate the justifications provided, it is more likely everyone will think seriously about them. At the same time, a wider range of approaches to each problem is placed before the class. In these ways, each student's opportunity to have a moment of insight may be increased.

The class in the study had a rule "Don't spoil things for others", which I believe was directly related to the frequent occurrence of magical moments. Students were asked to avoid telling other groups how to do something until the teacher decided that it was time for sharing, and indicated how much should be shared. Co-construction was valued, but not direct telling. "We don't want you to tell us" one student said to another who had already solved a problem. In an interview, this girl recounted excitedly a moment of insight which she had had, long after most of the class had grasped the point that eluded her. She clearly felt joy in her achievement, and there was no hint that she felt ashamed of having taken longer than others to understand it.

Encouraging reflection may be another important way to facilitate magical moments. A problem which can be solved in a short time provides less opportunity for this, because one quick thinker in a group may see immediately how to solve it, and leave little for the rest to do. Extended problems which require work over a longer period provide more scope for reflection, and make it easier for less quick-thinking group members to contribute. In addition, discussions may need to focus on the processes of discovery.

Students should be encouraged to reflect, not only on the mathematical ideas they are developing, but also on how they come to know them and on aesthetic aspects of them. Making magical moments at times a focus of reflection may help students to become more aware of their own thinking processes, and so be better able to direct and control them. Naidra's observations about the table that "really gives you insight" is an example of such reflection. Teachers can encourage this by modelling it themselves, both when interacting with groups and in whole-class discussions.

Conclusion

The episodes I have described clearly have an important motivational value. Although a single event may not make a great difference to students' attitudes, the cumulative effect of many such moments has the potential to influence their dispositions towards mathematics profoundly. If we want students to learn to love mathematics and to experience joy and self-affirmation through doing mathematics, I suggest that we should endeavour to structure our classrooms so as to cultivate magical moments. In addition, there may well be some cognitive benefit. My examples indicate that magical moments may be associated with understanding an idea with great clarity, gaining a new perspective on a mathematical concept, and possibly with increased levels of formalisation and perceiving links between hitherto unrelated ideas These are, however, very tentative proposals.

On this question concerning those classroom environments more likely to promote magical moments, the literature provides some clues. In an account of events in a second-grade classroom, Cobb, Yackel and Wood (1989) reported frequent expressions of joy and excitement from children when they constructed mathematical relationships or completed solutions to problems which they had found challenging. The episodes they described strongly resemble the events which I have called magical moments, and they claim that:

> it was because the teacher and children established social norms that contrast sharply with those of typical classrooms that we observed generally desirable emotional acts (p. 119)

The social norms referred to included figuring things out for yourself, persisting with challenging tasks, not telling others the answer, refusing to accept something you don't understand, showing consideration for others' feelings, taking pride in your own achievements, valuing other people's ideas and methods, and not being ashamed of making mistakes Despite the difference in ages of the children involved and the level of sophistication of the mathematics they were doing, the classroom described by Cobb *et al.* seems to have much in common with the classroom I observed, in the way it functioned as a community of inquiry, in the social norms established, and in the frequency of magical moments This provides some support for my hypothesis that the kind of teaching I observed may help to promote magical moments.

Further support comes from Burton (1999b) who claims that the research mathematicians in her study:

> provided clear directions towards a mathematical pedagogy that could be as engaging, exciting and rewarding for learners, as it was for them in their own practices. These directions were firstly towards valuing and nurturing intuitions and also recognising the importance of making connections or links in the building of mathematical meaning. (p. 31)

She goes on to explain that learning mathematics as an activity demands a pedagogical situation:

> to which student control, enquiry, argument and justification are all necessary contributors. In turn, these learning approaches rely upon a sense of community with expectations of exploration and creativity towards coherent and connected learning in which intuition plays a central role. (p. 31)

This description fits well with the approaches I suggest to encourage magical moments.

Many metaphors have been used to describe moments of insight in mathematics: a flash of lightning, a spark, things fitting together with a click My preferred metaphor captures the idea of sudden illumination leading to deeper understanding: it is the image of someone stumbling around in a dark room, bumping into objects and failing to realise what they are and then finally finding the light switch. When the light goes on, they are not only able to see clearly where they are going, but can also see where they have been, recognise the objects they bumped into on the way and see how they are placed in relation to one another.

Acknowledgements

I should like to thank David Clarke and Leone Burton for their helpful advice and comments on earlier versions of this manuscript, and Gaye Williams for many valuable conversations about her approach to teaching and for the privilege of observing and videotaping her classroom. I am grateful to the students for allowing me to videotape their discussions and for sharing with me their reflections on their learning, and to David Clarke and Merrilyn Goos for invaluable assistance with videotaping

Notes

[1] All these names are pseudonyms chosen by the students themselves

[2] Symbols used in the transcripts of classroom discussions are as follows:

- // (in consecutive utterances) beginning of simultaneous speech;
- \- (at the end of an utterance) no gap between speakers, interruption;
- \- (in the middle of an utterance) self-interruption;
- [ ] observation from videotape or field notes

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