# Semiotic Mediation: from History to the Mathematics Classroom

MARIA G. BARTOLINI BUSSI, MARIA ALESSANDRA MARIOTTI

This article starts with a cognitive analysis of an imaginary debate - reconstructed with excerpts from historical sources - concerning the shape of particular sections of a right cone and a right cylinder. Our analysis, based on the theory of figural concepts (Fischbein, 1993), suggests the following hypothesis:

> Where conic sections are concerned, a rupture between the figural and the conceptual aspects is to be expected and is not easily repaired (even by formally-educated university students)

We carried out an exploratory study with expert university students in order to validate this hypothesis, exploring how students could reseal the rupture and restore a sense of unity between the figural and conceptual components. In particular, what kinds of geometrical conceptions, if any, were these students able to mobilise? After reporting the findings of the study, we shift to the didactic plane and suggest certain tools of semiotic mediation (Vygotsky, 1978) which could be introduced in order to enable the students to achieve the conceptual oversight that is possibly lacking.

## 1. The task

We have constructed an imaginary debate, drawing on excerpts taken from historical sources from different ages: it concerns the shape of a particular conic section. Please read it carefully.

Serenus: Since I know that many expert geometers think that the transverse section of the cylinder is different from that of the cone which is called 'ellipse', I have thought that they must not be allowed to make such a mistake. Actually, it is absurd that geometers speak about a geometrical problem without giving proofs and are attracted by appearances of truth rather than the spirit of geometry. However, since they are convinced of that and I am convinced of the contrary, I shall prove *more geometrico* that both solids have a section of the same kind, indeed identical, provided that the cone and the cylinder are cut in a suitable way. (*Sections of the Cylinder and the Cone*, 4th century AD)

Witelo: All the ellipses that are sections of the acute-angled cone are larger in the side closer to the base of the cone: this is not true for those obtained as sections of the cylinder. It happens because of the sharpness of the cone and the regularity of the cylinder. In fact, on the one side, let us consider the intersection of the axis of the cone with a line perpendicular to a side of the axial triangle. If we draw a circle on the cone with that centre and we imagine a cylinder with this circle as the base, it is evident that the bottom piece of the cone is external to the cylinder whilst the top piece is internal. Hence, the bottom part of the conic section contains the bottom part of the cylindrical section, whilst the top part of the cylindrical section contains the top part of the conic section. On the other side, the two parts of the cylindrical section are equal because of the regularity of the solid and the equality of the angles with the axis. Hence, the thesis follows (*About Perspective*, ca 1200 AD)

Dürer: I do not know the German names of the (conic) sections, but I propose calling the ellipse egg-shaped curve, as it is identical to an egg (*Treatise on Measuring by Rule and Compasses on the Line, on the Plane and on Every Body, 1525*)

Guldin: It is necessary to avoid the mistake of those who think that the (conic) ellipse is narrower in the part close to the vertex of the cone and larger in the part close to the basis of the cone: on the contrary, they are very similar. (*Centrobaryca,* 1640)

What do you think? How could you convince an interlocutor (e g. a peer of yours) whose opinion is different from yours? And how could you convince a high school student?

## 2. A cognitive analysis of the historical debate

The above imaginary dialogue reconstructs some elements of a historical debate between voices from two antagonistic and complementary worlds: those of theoretical and practical geometry (Balacheff, 1997).

## Theoretical geometry

Since the age of Apollonius, a deep understanding of the properties of conic sections has been achieved. However, most of the properties were expressed through relationships, which are neither immediately related to the shape of the cone to be cut nor to the shape of the section. For instance, the theory of Apollonius was based on proportions and applications of areas, by means of which the very 'symptoms' (i.e. characteristic properties) of conics were expressed in the secant plane, despite the initial 3-D approach. In the seventeenth century, this process was based on calculations and culminated in the algebraic representation of conics by equations. A simple and meaningful link between the 3-D and the 2-D approach to conics was looked for by mathematicians for centuries until Dandelin (in 1822) succeeded in relating the focal properties of a conic to a configuration with a conic section and two spheres tangent to the cone and to the secant plane, where the points of tangency are the very foci (see Courant and Robbins, 1941, p. 199).

## Practical geometry

Conic sections were also studied with the purpose of applications, such as setting sundials, constructing burning mirrors and drawing in perspective. Because of the modelling process, the 3-D generation of conics was focused. Witelo and Dürer belonged to this tradition. In particular, Dürer applied the graphic method of double projection to conic sections, used in artists' workshops and in architectural practice. However, in spite of this astute (and correct) method, one that was reconsidered in the late eighteenth century by Monge (1799), Dürer drew an egg-shaped curve (see Fig. 8 on p. 34 which is taken from Dürer, 1525), instead of an ellipse, probably deceived by arguments similar to the ones expressed by Witelo. Guldin bore witness to the longevity of the misconception more than a century later.

The above outline shows the existence of two relatively independent worlds, which came in contact with and nourished each other from time to time. However, this relative independence from each other created the conditions for the birth of autonomous manners of viewing and styles of reasoning. One example is actually provided by the case of (conic) ellipses, confused in practical geometry with egg-shaped curves.

In addition to the historical point of view, the relationship between the arguments used in theoretical and in practical geometry seems interesting to investigate from a cognitive perspective. As mentioned earlier, we try here to analyse it here in terms of Fischbein's theory of *figural concepts*.

The geometrical concept of *cross section* (i.e. the figure obtained as the intersection between a plane and a surface or possibly the solid bordered by the latter) proves to be difficult in general (Mariotti, 1996), in spite of its apparent simplicity. Intersection is easily defined in set-theoretic terms, but to investigate the shape of the figure obtained as a cross section requires taking into account the complex of geometric properties of the two sets that are being intersected and the relationships between them. The contribution of the figural component, based on perception, does not always help: in the most common case, it is necessary to forget the global appearance of the solid and infer from 'outside' what happens 'inside'.

For instance, if we think of a cube cut by a plane oblique with respect to all of its edges (see Fig. 1), it is natural to 'see' squares and even rectangles as resulting from such a cross sectioning, but other polygons are hard to imagine. What proves difficult to overcome are the possible conflicts between the figural aspect, deeply affected by the global shape of the object and its components (such as all the faces of the cube are squares which lie on planes that are either parallel or perpendicular to each other), and the conceptual aspect, which concerns the properties of the intersection between the set of points of the cube and the set of points of the given plane.

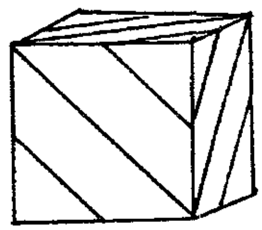

*Figure 1*

On the one hand, there are many implicit properties of the solid which become determinate in order to characterise the shape of the cross section, but cannot be immediately grasped, or which conflict with perceptual attributes. On the other hand, the shape of the solid itself can conceal other properties: for instance, the symmetry of a section is scarcely recognisable since symmetry (unlike other geometric properties) is neither an invariant of intersection nor easily related to the properties of the two sets to be intersected.

The above analysis shows that the solution of a cross-section problem implies the contribution of high-level geometric conceptions, which are supposed to be more problematic in practical geometry, where adequate geometric concepts may not always be available.

For instance, the very argument offered by Witelo is based on the concepts of contained-containing: the contrast between them suggests an asymmetrical relationship between two parts of the same section. Although the properties used are correct, the reference to contained-containing is not sufficient to infer all the properties needed to state the regularities of the conic section such as the double symmetry - two orthogonal axes of symmetry - exactly like the cylindrical section.

Drawing on the above analysis originating from a historical case, we hypothesised that even experts might not be able to fuse the rupture which exists between figural aspects and the available conceptual tools; the presence of a conflict may block reasoning and prevent one from reaching the correct response.

## 3. An exploratory study

Our investigation aimed to explore what kind of geometric conceptions, if any, students were able to mobilise in order to overcome the problem. We wished to observe how they succeeded in defending a position under the pressure of an imaginary debate, in order to bypass the rupture and harmonise the figural and conceptual aspects. If the hypothesis given at the outset were to be validated, a new didactic problem could have arisen:

> How might the teacher help the students to repair the possible rupture and to achieve (reconstruct) the harmony between figural and geometric aspects?

Fourteen students from two courses on 'Elementary mathematics from a higher standpoint' (i.e. on epistemology of mathematics for prospective teachers), taught by the two authors in the 3rd - 4th years of mathematics, agreed to take part in an afternoon problem-solving session in their respective universities (denoted here as A and B). Both groups had already taken two one-year courses in geometry - i. e. nearly 300 hours - including linear algebra, vector, Euclidean, affine and projective spaces and the algebraic study of conics and quadrics. In these courses, they had information about the intersections of a cone and a cylinder with a plane. Hence, they could be considered experts: there was certainly no doubt about the fact that in both cases it is possible to obtain an ellipse

They were divided into small groups (two trios - A2, B5 - and four pairs - A1, B1+B2, B3, B4), provided with individual copies of the text given here in section 1 and asked to produce only one answer for each group. (In one case, a pair did not succeed in reaching an agreement, and the group produced two texts - B1, B2.)

The problem-solving session lasted two hours. We collected seven written protocols (A1 A2; B1 B2 B3 B4 B5) together with drawings and paper models. The groups realised various and different explorations. We shall give only a few details

## Witelo's and Dürer's arguments

Some groups guessed the reasons for the mistake and tried to find the bug. We shall focus here on three different issues:

**Reasoning by symmetry**

Some drawings (e.g. Figure 3) and the transfer of the property of symmetry from the cone to the section might suggest the idea that the centre of the conic section (if any) is the intersection of the axis of the cone with the secant plane. One group (A1) tried to find a symmetry between the two parts of the section which are on the different sides of the triangle of the paper model (Figure 2). When the model failed, they drew a right cone and a right cylinder with the same base and a secant plane and tried to show that the two sections are concentric ellipses, by estimating their 'distance' from the axis of the cone. But by imagining increasing the inclination of the secant plane, they saw the 'distances' changing in a different way: on the one side the distance increases, on the other the distance decreases.

This was even more puzzling because it stressed the difference between the cylindrical and the conic sections contrary to their desire to eliminate differences. However, it convinced them that the ellipses might be not concentric, i.e. that the centre of symmetry (if any) might be out of the axis of the cone.

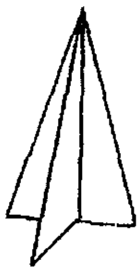

*Figure 2*

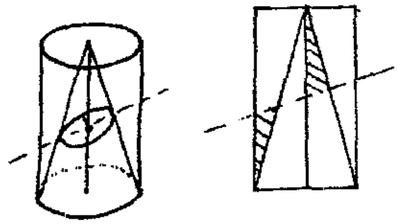

*Figure 3*

**Contained–containing reasoning**

One group of students (B3) drew a right cone and a right cylinder with the same axis as described in Witelo's argument. Then they drew the sections with the same plane. They obtained two closed curves intersecting in two points (Figure 4). The particular configuration might explain, in their opinion, why there is no contradiction between having a 'true' ellipse for the conic section that is contained in the cylindrical section on the top side and that contains the cylindrical section on the bottom side.

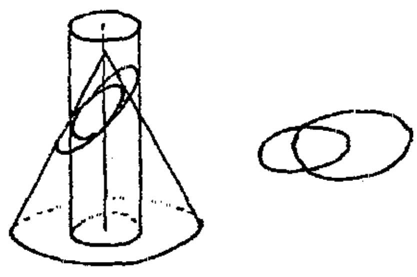

*Figure 4*

**Limit case reasoning**

One student (B2) explained Dürer's mistake by means of a limit argument. He drew the section of a regular pyramid and a regular prism with a hexagonal base by means of orthogonal projections and commented:

> It is evident that the two sections are not of the same kind. In fact, whilst the prism section is symmetrical, this is not true for the pyramid section. We can imagine that this disproportion is maintained when, by increasing the number of sides, the pyramid and the cone approach the cone and the cylinder [... In the case of the cone] perception might suggest a non-symmetrical curve [i.e. Dürer's egg]. The mistake is reinforced by the common pointwise construction of sections that is usually made by choosing the vertices of a regular polygon on the circle.

However the student did not try to explain the apparent conflict between the symmetry of the section of the cone and the asymmetry of the section of the pyramid.

## Models

The students referred both to 3-D models and 2-D models (drawings).

### 3-D models

In one case (A1), a pair of students started to cut, fold and glue paper to make some models. They were very hopeful because:

> a good paper model could be useful to convince Witelo practically, as Witelo is speaking in practice and not in geometry. He is not providing any proof.

The most promising model, one which was handled for a lot of the time, consisted of two congruent isosceles triangles stuck together orthogonally along their heights (see Figure 2 above). The two students tried to describe with hands or pencil the section in this model, but without success. They hoped to find some symmetry between the two parts of the section that are on the different sides of one triangle but they did not succeed. This practical experience helped them later to exit from a blind alley (see the previous sub-section). Two groups of students (B3 and B5) suggested using concrete wooden models to explain the result to younger students too.

### Drawings

From the very beginning, while reading the given text, all the students started drawing. Since the problem concerns a 3-D configuration, different kind of drawings were produced, according to their previous experience in high school: however, in most cases the scant - if any - mastery of effective drawing abilities did not allow them to find conclusive evidence. For instance, a group (A2) focused attention on only one position of the secant plane from the very beginning (Figure 3): the slight inclination of the plane suggested a false symmetry; hence, for the whole session, they tried to prove a false conjecture without success.

## Proof

Only one student (B2) introduced deductive arguments in his answer.

### Synthetic

He remembered a way to relate the focal property of the ellipse as a locus of points to a conic section, by drawing Dandelin's configuration (Figure 5). He said he had seen a model once and was struck by the ingenuity of the method.

### Analytic

No student tried to approach the problem by means of analytic geometry, even if some groups turned to equations later. Yet only student B2 succeeded in proving that *cutting a cone with a suitable plane, the conic section is an ellipse*,

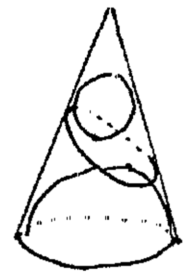

*Figure 5*

by a fusion of synthetic and analytic tools: the synthetic part allowed him to avoid the problematic recourse to the system of the equations (in three variables) of the cone and of the plane. By shifting his reasoning to the secant plane only, he introduced a local system of Cartesian co-ordinates. In this system, using arguments of synthetic geometry, the equation of the section was calculated correctly and reduced to the known canonical form. (See Appendix 1 for details.) But he was not satisfied by this proof *more analitico* (as he called it):

> because it cannot say anything to a person - like Witelo - who does not know ellipses as equations

## 4. Discussion

Despite its small size, this exploratory study confirmed our hypothesis stated at the outset. Our empirical data highlighted the rupture between the figural and the conceptual aspects in the field of conic sections and the great difficulties met in overcoming it. The students had information about conics and nobody doubted the actual shape of the cross section. Yet only one student (B2) succeeded in proving that the conic section under scrutiny was an ellipse. All the students looked for a direct argument against Witelo, that is an argument that would not break the link with the figural aspect. The group B3 argued against Witelo's argument based on set inclusion without offering a complete proof. A lot of students expressed their faith in concrete models.

Unfortunately, it is not possible to argue against and correct Witelo's argument solely by refining perception: the ellipses are not 'seen' as ellipses (as is shown by Knorr (1992) in his beautiful analysis of the ancient drawings of circular wheels and elliptical shields, depicted according to Euclid's *Optics*) and direct measurement with comparison of segments might be impossible on the wooden models proposed by (but not available to) the students.

Actually, a direct argument adjusting Witelo's figural argument to a correct geometric one does not exist. It is impossible to argue against Witelo's argument by means of a direct geometrical argument which mantains the unity between the figural and conceptual aspects. The shape of the ellipse can be characterised by elements which are intrinsic to the curve, but ones which are not directly related to its origin as a cross section.

One could defend the elliptic form of a particular conic section in different ways, by producing rigorous proofs:

- in the style of Apollonius, by proving that the section has two orthogonal conjugate diameters (i.e. axes of symmetry);
- in the style of Serenus, by constructing (and proving the construction) a cylinder with the same section as that of a given cone;
- in the style of post-Cartesian geometers by using analytic geometry (Herz-Fischler, 1990).

All the students had been trained in analytic geometry, yet they (with only one exception) did not use equations effectively.

The cause could be the difficulty in managing 3-D analytic geometry, but also the voluntary choice of excluding an inappropriate way of opposing Witelo's argument Even the sole student (B2) who constructed a synthetic-analytic proof was dissatisfied, thinking that it would not have convinced Witelo. Actually, in the sequence of algebraic manipulations of a formula, it might be impossible to keep track of a sequence of geometric steps to which the formal manipulations should correspond, because in order to make the best of the potentialities of algebraic language, it is often necessary to break from time to time the reference to the meaning of the algebraic transformations, and so employ steps with no geometric meaning at all

Although the task did not propose it explicitly, the students seemed to enter of their own accord into a voices-and-echoes game (Boero *et al.*, 1997). According to their model, students tried to:

> make connections between the voice of his/her own conceptions, experiences and personal senses, and produce an *echo*, i.e. a link with the voice made explicit through a discourse. (p 84)

In particular, they tried to express *dissonance* with Witelo's voice by appropriating it and refining it without success

## 5. The didactic problem: towards a solution

The study we have carried out might be easily generalised. We have anecdotal evidence that the task of entering the imaginary historical debate remains puzzling and confusing, even when more expert subjects are involved. In particular, we have presented the task to audiences consisting of expert teachers of mathematics, in teacher training courses, and even of expert mathematics educators, at international conferences

Although puzzling, Witelo's arguments are not sufficient to make students reject the well-known property (this could happen with younger students, however), but they are sufficient to fuel unending discussions and systematic (and unsuccessful) attempts to attack the problem of the double symmetry of the section in a direct way. Yet we have not found a single discussion of this problem in the textbooks on conics, not even in those texts that introduce conics through their spatial genesis.

This absence might be explained by reference to the standard model of teaching, one that leaves little space, if any, for students' experiments, conjectures, discussions and references to commonsense knowledge. In other words, the problem is not supposed to exist. In fact, it is hidden under a cover, represented by the statement 'cutting a cone we obtain three different types of sections, called ellipse, parabola and hyperbola', that is expected quickly to be accepted before starting the study of these sections by means of completely independent tools (such as metric plane definitions and equations).

But when the cover is taken away by a puzzling task, such as the one we have designed, a didactic problem arises:

> how might the teacher assist students in healing the rupture and resetting the harmony between the figural and conceptual aspects of the task?

We have already hinted at two synthetic arguments which might be expressed in elementary terms and provide specific geometric frames for conceptualising the complex configuration of a cone cut by an 'oblique' plane Within these frames, it is possible to give a correct response without breaking the link between the figural and conceptual aspects:

(1) Dandelin's theorem, whose proof is based on the insertion of two auxiliary spheres, tangent (internally) both to the cone and to the secant plane.

(2) Dürer's method of double projection, with a revision that precludes Dürer's mistake.

Both of these arguments are presented in Appendix 2.

These arguments, corresponding to particular geoemtrical conceptions, can be offered as means of intellectual control and they represent good examples of tools of semiotic mediation, as Vygotsky (1978) called them According to Vygotsky, a *tool of semiotic mediation* is a sign directed towards the mastery or control of behavioural processes, someone else's or one's own. In his approach to the problem of teaching and learning, Vygotsky distinguished between the function of mediation of technical tools and that of psychological tools (or signs or tools of semiotic mediation). Both are produced and used by human beings and are part of the cultural heritage of humankind The former aim to 'master and triumph over nature'; the latter are directed towards the mastery or control of behavioral processes, someone else's or one's own.

Both technical and psychological tools form an integral part of social activity. In addition to language examples, Vygotsky himself gave examples of psychological tools drawn from mathematics, for instance, various systems for counting. What makes them tools for semiotic mediation is the fact that they were produced and are still used to evaluate quantity, but, at the same time, they function, in the solution of problems, to organise and control behaviour

A typical, general feature of tools of semiotic mediation is that they are to be introduced intentionally into activity from the outside: they are not produced by the individuals who use them, but they are offered by somebody else, such as the teacher, by means of a system of external signs, such as books, models, guided practices. They have the effect of regulating immediate reactions based on commonsense or perceptual appearances. Later, when the students have appropriated these tools, the external guide may disappear and the process may be activated by the students themselves.

Coming back to our example, we observe that in both cases a deep transformation of the original figure is required. Dandelin's theorem is really ingenious and creates a visible link between the conic section and its foci. But two auxiliary spheres are to be introduced into an already complex configuration. Moreover, Dandelin's method does not address the issue of symmetry directly, relating it to the well-known double symmetry of the plane curve defined by means of the metric focal property

The revised Dürer method requires splitting the object into three different views (the vertical, the horizontal and the section planes) and considering the complex relationships among them all. The appearance of the figure can be overcome by the method of double projection, which offers a conceptual tool for the production of a quick proof of the double symmetry. It is not certain, however, that a rigorous proof, like the one we have suggested, is enough to clear up all the doubts raised by Witelo's arguments.

It is well-known (see Hoyles and Jones, 1998) that students seem to be more convinced by empirical arguments than by rigorous proofs. However, by means of the method of double projection described in Appendix 2, it is possible to understand how, by varying R′, the decreasing radius of the circular horizontal section is compensated for by the decreasing distance from the axis of the cone. This intuition can be fostered by a pointwise-correct construction of the width of the section for two points R and S, symmetrical with respect to the mid-point of CD. However, a decisive step seems to be the transformation of this pointwise construction into a dynamic ('continuous') one, for instance by means of dynamic geo-metry software like Cabri. By means of this simulation, it is also possible to understand how the puzzling case of pyramids can be harmonised with the case of cones

We have illustrated (in Appendix 2) two synthetic arguments that might offer the means to mend the rupture between figural and conceptual aspects in the case of conic sections. Yet, surely their status in mathematics is different: whilst Dandelin's theorem is an elegant yet isolated way to solve a problem, with no possible value in further problems, Dürer's method comprises the origin of descriptive geometry, developed by Monge more than two centuries later.

Students are not expected to be able to discover either of the arguments without any help. In our small-scale experiment, the only student who referred to Dandelin's theorem wrote of having seen a model of Dandelin's configuration in an exposition of mathematical models. In general, who has the responsibility to introduce such arguments into the classroom? Surely it is the teacher. The teacher's role as a guide in the teaching-learning process has often been placed into shadow by the radical constructivists who have focused only the students' active role in 'discovery'. The example we have presented in this article is only a paradigmatic case, one among several, of a situation where the students by themselves are expected to face stalemate. If they do not yet know the correct property, they may agree with Witelo and Dürer; if they do, they may even then fall into a conflict not so easily overcome

The key role of the teacher is then twofold: on the one hand, to create a problem situation where students can live the conflict; on the other, to introduce the students to suitable means in order to master the conflict and achieve a new conceptual control. As this example shows, the history of mathematics can play different yet related roles. The history suggested a paradigmatic case of a conceptual mistake, based on a rupture between the figural and the conceptual aspects. At the same time, meaningful historical sources inspired an involving educational task, which placed the students in the situation of trying to uncover a hidden conflict and enter into a dialogical game that might be useful to overcome the conflict. And finally, the detection in history of tools of semiotic mediation (the analytic and synthetic arguments) suggested effective ways to do so

## Acknowledgements

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## Appendix 1 The scheme of the proof of the student B2

Let us see now that if a cone is cut by a suitable plane an ellipse is found and not an 'egg-shaped' curve. Choose a Cartesian system of co-ordinates (see Figure 6 below). We are trying to write the canonical equation that shows us that it is actually an ellipse.

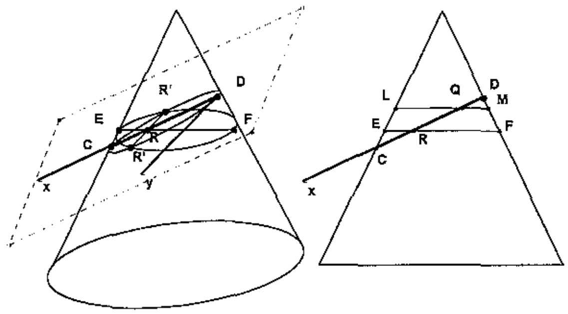


*Figure 6*

$$DR = x_R \quad RR' = R'' = y_R \quad OA = J$$

As R′ is on the cone, it lies on a circle that is in a plane perpendicular to the axis of the cone. The triangle ER′F is right-angled and so by Euclid's theorem:

$$FR \cdot RE = RR'^2 = y_R^2$$

If Q′ is another point of the section, we have in the same way:

$$DQ = x_Q$$

$$MQ \cdot QL = QQ'^2 = y_Q^2$$

But the triangles DFR and DMQ are similar, hence:

$$FR = k\,RD = k\,x_R$$

$$MQ = k\,QD = k\,x_Q$$

and CRE and CQL are similar, hence:

$$ER = h\,CR = h\,(J - x_R)$$

$$LQ = h\,CQ = h\,(J - x_Q)$$

For both R and Q (hence for any point of CD) we have:

$$y^2 = k\,x\,h\,(J - x) = kh\,x\,(J - x)$$

$$y^2 - khJx + khx^2 = 0$$

$$\frac{y^2}{\frac{khJ^2}{4}} + \frac{\left(x - \frac{J}{2}\right)^2}{\frac{J}{4}} = 1$$

that is, an ellipse with centre (J/2, 0).

## Appendix 2 Dandelin's theorem and Dürer's revised method

### Dandelin's theorem

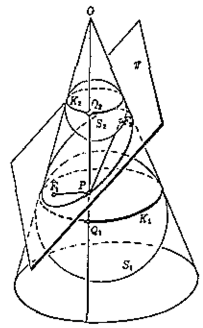

*Figure 7*

We refer to a beautiful argument given in 1822 by the Belgian mathematician G. P. Dandelin. The proof is based on the introduction of the two spheres $S_1$ and $S_2$ which are tangent to the plane $\pi$ at the points $F_1$ and $F_2$ respectively and which touch the cone along the parallel circles $K_1$ and $K_2$ respectively.

We join an arbitrary point P of E to $F_1$ and $F_2$ and draw the line joining P to the vertex Q of the cone. This line lies entirely on the surface of the cone and intersects the circles $K_1$ and $K_2$ in the points $Q_1$ and $Q_2$ respectively. Now $PF_1$ and $PQ_1$ are two tangents from P to $S_1$, so that:

$$PF_1 = PQ_1$$

Similarly:

$$PF_2 = PQ_2$$

Adding these two equations, we obtain:

$$PF_1 + PF_2 = PQ_1 + PQ_2$$

But $PQ_1 + PQ_2$ is just the distance $Q_1Q_2$ along the surface of the cone between the parallel circles and is therefore independent of the particular choice of the point P on E. The resulting equation:

$$PF_1 + PF_2 = \text{constant}$$

for all points P of E is precisely the focal definition of an ellipse. E is therefore an ellipse and $F_1$ and $F_2$ are its foci (Courant and Robbins, 1941, p. 200)

### Dürer's revised method

Dürer's method for drawing conic sections is rooted in the tradition of classical studies. Actually, in Apollonius too, the study of the symptoms (i.e. the characteristic properties) of conics was based on the study of the relationships between line segments lying in distinct planes, i.e. the base plane, the plane of the axial triangle, and the secant plane. What is new in Dürer is the transposition into geometry of a technique taken from the everyday practice of stone cutters (Pfeiffer, 1995). The 3-D object is represented in the plane by means of two orthogonal projections, the first onto a horizontal plane and the second onto a vertical plane, that is rotated around the common line L until it lies down in the horizontal plane. In this way, a point of the 3-D space is represented by (split into) two points of the plane where the representation is made: the line joining the two points is perpendicular to L.

Dürer uses this technique, without further justification, to give a method for drawing conic sections. The justification might be added as follows (we shall describe only the case of an ellipse).

Take a cone obtained by rotating a right-angled triangle around one of its non-hypotenuse sides (like in Euclid's definition). Consider a secant plane s, not parallel to the base plane b and cutting the cone along two opposite generators. If R is the intersection line of s and b, consider the line $A^\circ B^\circ$ through the centre of the base circle perpendicular to the line R. Then take the plane t containing $A^\circ B^\circ$ and the vertex V of the cone. The triangle $A^\circ VB^\circ$ is called the 'axial triangle'. Consider the line DC that is the intersection of the axial triangle with the secant plane s.

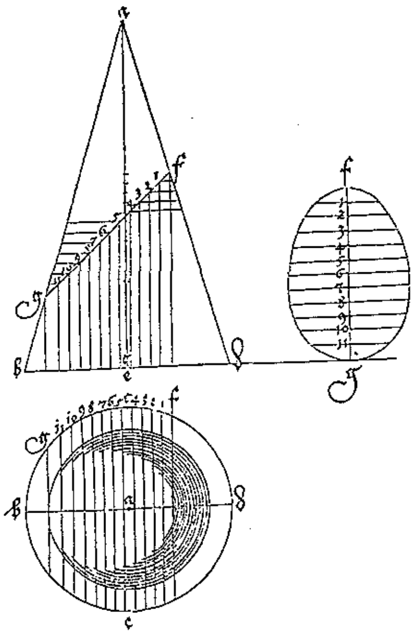

*Figure 8*

Dürer starts by dividing this line DC into twelve equal segments by means of eleven points (numbered from 1 to 11). For each point, he repeats a construction, based on the method of double projection, to find the distance of the points of the sections that are on the cone at the same quote as the chosen point.

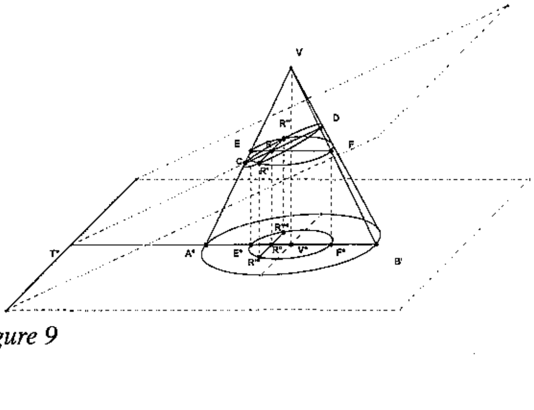

*Figure 9*

The method is the following. Consider the point R. To find the width of the cone at the quote of R one projects orthogonally on the base plane the circle obtained by cutting the cone with an horizontal plane through R. The segment $R^\circ R'$ on the base plane represents the width of the section at the point R, along a line perpendicular to DC in the secant plane.

Dürer aligns the major axis of the section vertically alongside the section of the cone, and marks off the various widths as he finds them. As Pedoe (1983) observes, the method is correct, but even a small slip of the ruler or compasses might produce an error, like the error of drawing an egg-shaped curved, if no conceptual control is mobilised.

Using the method of double projection, it is not so difficult to be convinced that the line CD is an axis of symmetry for the section: actually, the plane $\tau$ is a plane of symmetry of the cone and the plane $\sigma$ (through DC) is perpendicular to it. Hence, if a point R′ lies on the section, its symmetrical point R″ (with respect to DC) lies on $\sigma$, and on the cone too. Yet the existence of a second axis of symmetry is troubling. Actually, this axis crosses the line DC in the mid-point O, which does not lie on the axis of the cone, hence does not fit in with the symmetry of the solid. It would be more natural to guess that the intersection of DC with the axis of the cone might be the centre of symmetry of the section; as it is not, it would be easy to conclude that there is no centre and hence only one axis of symmetry.

It is possible to prove rigorously that a second axis of symmetry exists, by using the method of double projection. If the construction is done well, for two points R and S symmetrical respect to the mid-point of DC, the widths $R^\circ R'^\circ$ and $S^\circ S'^\circ$ are perceptually equal, even if they are obtained from two circles of different radii.

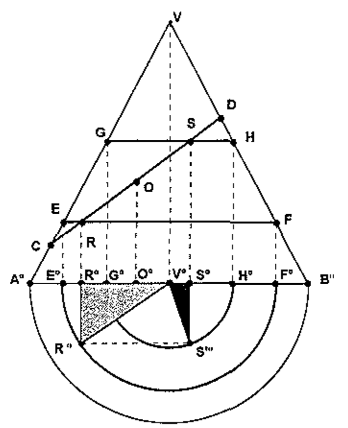


*Figure 10*

*The proof that $R^\circ R'^\circ = S^\circ S'^\circ$*

Let R and S be two points of CD symmetrical with respect to O (the mid-point). EF and GH are the diameters of the circles, obtained as horizontal sections of the cone at the quotes of R and S. For each point X, we shall denote by $X^\circ$ the orthogonal projection of X onto the base plane. In the figure above, only half of the base circle is drawn.

The triangles $V^\circ R^\circ R'^\circ$ and $V^\circ S^\circ S'^\circ$ of the base plane are right-angled Hence, by Pythagoras' Theorem:

$R^\circ R'^{\circ 2} = V^\circ R'^{\circ 2} - V^\circ R^{\circ 2} = V^\circ E^{\circ 2} - V^\circ R^{\circ 2}$

$= (V^\circ E^\circ - V^\circ R^\circ) . (V^\circ E^\circ + V^\circ R^\circ)$

$= E^\circ R^\circ \quad F^\circ R^\circ$

$S^\circ S'^{\circ 2} = V^\circ S'^{\circ 2} - V^\circ S^{\circ 2} = V^\circ H^{\circ 2} - V^\circ S^{\circ 2}$

$= (V^\circ H^\circ - V^\circ S^\circ) . (V^\circ H^\circ + V^\circ S^\circ)$

$= H^\circ S^\circ \quad G^\circ S^\circ$

To show that $R^\circ R'^\circ = S^\circ S'^\circ$, it is sufficient to show that:

$E^\circ R^\circ . F^\circ R^\circ = H^\circ S^\circ . G^\circ S^\circ$

i.e. ER FR = HS GS

i e. the proportion ER : GS = HS : FR holds.

But this proportion follows from the relation:

ER : GS = CR : CS = DS : DR = HS : FR

(based on the hypotheses - CR = SD and RO = OS - and on the similarity of the pairs of triangles ECR / GCS and HDS / FDR)

The following figures show how the construction can be made dynamic by means of Cabri (Figure 11) and how the case of pyramid is related to the case of cone (Figure 12).

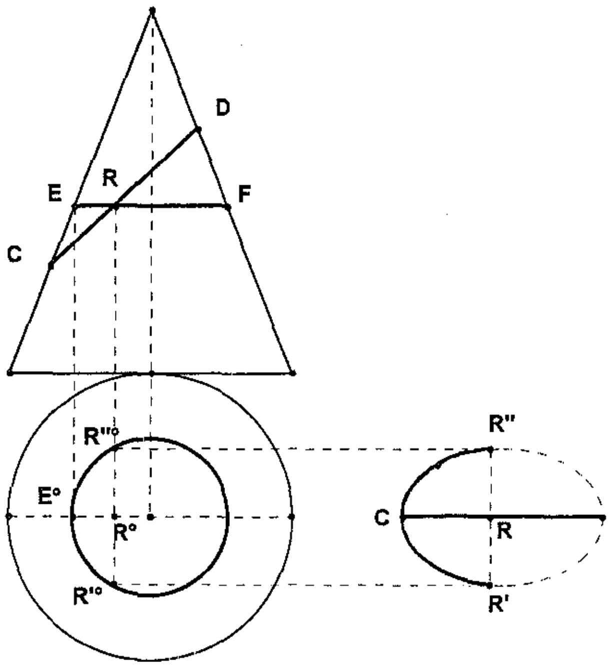


*Figure 11*

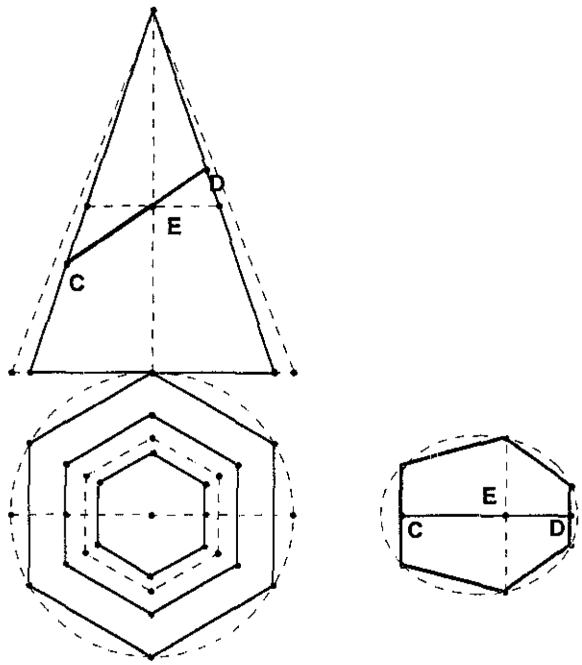


*Figure 12*