Problem Solving or Mathematics

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PART 1

There is a lot of discussion going on about how to teach problem-solving, some of it concerned with the metamathematical views of the person teaching — or following the course — see, for instance, S. Lerman [6] and E. Blaire [3], some focussing on what differentiates an expert from a novice — see the thought-provoking papers of A. Schoenfeld [7], [8], [9], among others.

My own concerns are limited to how to give a course on "Problems in Elementary Mathematics" to future secondary-level teachers in mathematics The target audience is third year undergraduates in mathematics, and what I would like them to get out of this course is more of a feel for how mathematics is done, not taught.

The course is based on solving problems thrown at me, on the blackboard — sometimes together with the students, sometimes by myself. This does not mean that general ideas or methods are not discussed, they are. But let me behave here as I would in the classroom: in three different parts, of which this is the first, you will find condensed versions of classroom experiences. Numbers in □ at the left refer to the analysis of the solution — the last section in each paper Taken together these three last sections are supposed to give a general idea of the teaching method followed

The problem

The subject being discussed was the use of generalisation in mathematics. So one of the brighter students proposed:

S: Let's start from the bisector It is the locus of points equidistant from two intersecting lines (the *two* bisectors of course) and the usual immediate generalization is to look at points equidistant from two planes Couldn't we instead look at points equidistant from two skew lines?

T: Call the desired locus S. An obvious point of S is the midpoint M of the common perpendicular. (Figure 1)

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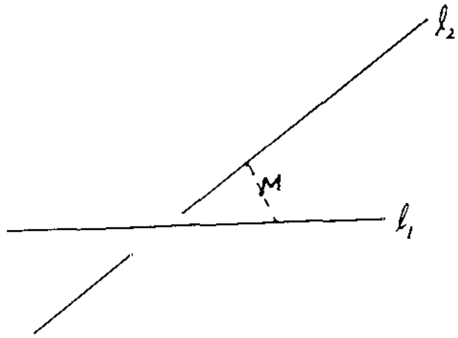

Figure 1

Let's take a particular case to see if we find any other obvious points. Look first at the case of perpendicular lines — obviously we get a parabola in the plane through l_1 which is perpendicular to l_2 (Figure 2). This gives us two parabolas already. (Figure 3).

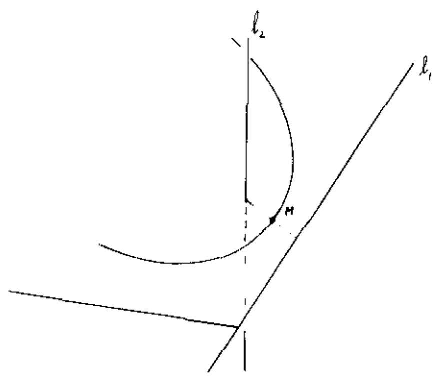

Figure 2

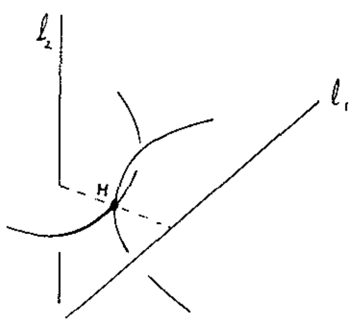

Figure 3

T: We could use "brute force" — switch to analytic
2 geometry and just compute. The two lines can be taken as $x = y = 0$, and $z = 0$, $x = a$.

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Points at a distance d from $x = y = 0$ satisfy

$$x^2 + y^2 = d^2$$

Points at this distance from $z = 0$, $x = a$ satisfy

$$z^2 + (x - a)^2 = d^2$$

Therefore our locus has the equation

$$x^2 + y^2 = z^2 + (x - a)^2$$

- a quadric. Not surprising, since for two intersecting lines we also have a second degree locus.

S: ....

T: *Two* lines so a second degree locus.

T: Looking at $x^2 + y^2 = z^2 + (x - a)^2$, I feel like rewriting it

5

$$z^2 - y^2 = x^2 - (x - a)^2$$

or $(z + y)(z - y) = a(2x - a)$

which shows that points on the line

1

$$x - y = \lambda a$$
$$x + y = (1/\lambda)(2x - a)$$

lie on our quadric — which therefore admits two families of lines

(The other family is

$$x - y = \mu(2x - a)$$
$$x + y = (1/\mu)a\,)$$

The general case could be treated much the same way, the computations will be slightly more complicated. No point in doing that.

T: Could we find these lines in a more "geometric" way? We didn't look at the symmetry properties. If we take

6 the parallels l_1' and l_2' to the given skew lines through the midpoint of their common perpendicular, then the bisector of the angle between those two lines will be a line p which is symmetric with respect to our two lines. In fact we have two lines like that, and both are axes of symmetry for the whole configuration. Points on those lines are obviously part of our locus. (Figure 4).

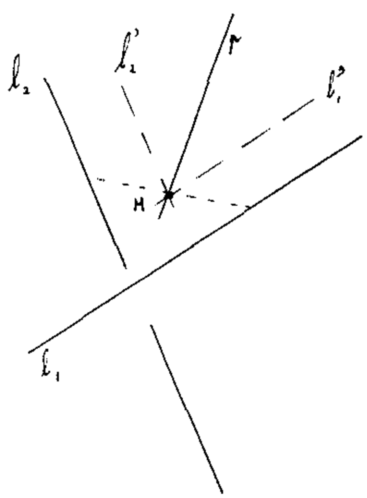

Figure 4

It is easy to construct more points of our locus. Through any point P on l_2, (Figure 5) draw a parallel to l_1, namely m. Let d be the distance between l_1 and m. In the plane defined by m and l_2 draw a parallel to l_2 at distance d, then R, the point on that parallel and on m, is equidistant from l_1 and l_2.

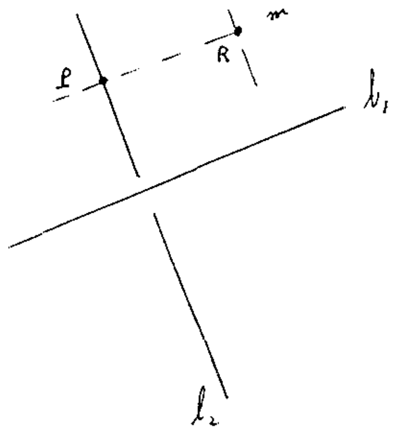

Figure 5

Using one of our axial symmetries we get one more point on the locus: R'. But Q, on the axis, is also a point of our locus. Therefore the line RQR' contains 3 points of our locus. If we could prove that our locus is a quadric, we would know that the whole line RQR' is part of our locus. (Figure 6)

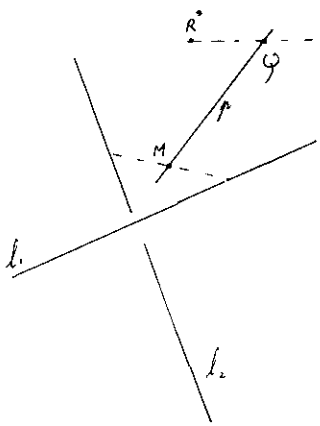

Figure 6

S: Aren't you turning things around?

T: No. And it is certainly worthwhile, to mix "pure" geometric constructions with analytic geometry. Let

3 us prove the following.

Lemma I The locus S is of the second degree (or less) We will do this formally: The square of the distance of a point (x, y) from a given line is an algebraic expression of the second degree in x and y (Indeed, from the line $x = 0$, $y = 0$ it is given by $d^2 = x^2 + y^2$, for any other line use a cartesian change of coordinates in euclidean 3-space This is a linear transformation on the coordinates). Setting two such expressions equal leads us to an equation of at most degree two. We can now go on, in an equally formal way.

Lemma II On S there are two families of lines, each parallel
1 to a given plane. Indeed the construction given above gives us two families of lines, each perpendicular to one of the axes of symmetry

Theorem: S is a parabolic hyperboloid.
It is a surface of degree 2, with two families of lines, each parallel to a given plane.
Obviously M is its saddle point.

Analysing the solution

Let me remind you first that this is a condensed protocol. Even a casual glance will show that all the teaching moves discussed by N Hadar and R. Hadass in [4] have been used — and this is important, since teachers have a tendency to teach the way they have been taught. But the important points—those that have been signalled—are the following ones:

1 Develop new mathematical content
2 Switch from one branch of mathematics to another
3. Insist on the formal proof process
4. Generalize and particularize
5 Use free association
6. Mention — and use — general mathematical ideas and structures

Now in [5] Hatfield distinguishes between teaching for problem-solving, teaching about problem-solving, and teaching via problem-solving. In our course on "Problems in Elementary Mathematics" we try to integrate all of these approaches—and this is why all the preceding six elements are an integral part of our teaching conception.

There is no need to explain points 1 and 2. Whenever the formation of future teachers is discussed, the lack of breadth of their mathematical knowledge is deplored. So every occasion should be taken advantage of to round-up their knowledge—sometimes, like here, of parts of mathematics not taught at universities any more, like classifying quadrics, or finding lines on quadrics

Balacheff in [1] concludes that when teaching by the discovery method students tend to neglect the need for formal proof and are at a loss when asked to translate their investigations into proofs. This is, I think, our common experience. Therefore point 3 is quite important. More about this can also be found in J. Baylis article "Proof—the essence of mathematics" (see [2])

It is standard procedure nowadays to particularize situations and look at simple cases. Sometimes at the expense of the opposite operation, namely generalising. This being one of the main tools in research it should be used as much as possible within the given context. Other examples underlining the uses of generalization will be found in Parts 2 and 3 of this paper.

Free-association. Who ever heard of using free-association in mathematics? It is called, more modestly, remembering similar situations. But research mathematicians do much more than remember similar situations. This is much too restrictive and hampers our problem-solving ability. Certainly looking at any mathematical situation, property, or object, which in some features is associated to the problems now considered is mathematically fruitful.

The last point—using general mathematical ideas and structures needs no further elaboration

References

1 Balacheff, N [1982] *Recherches en Didactique des Mathématiques* 3, 261-304
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3. Blaire, E [1981] *Int J Math Educ Sci Technol* 12, 147-
4 Hadar, N. and Hadass, R. [1981] *Int J Mah Educ Sci Technol* 12, 101-108
5. Hatfield, L L [1978] Heuristical emphasis in the instruction of mathematical problem-solving: rationale and research MPS edited by L L Hatfield (ERIC)
6 Lerman, S [1983] *Int J Math Educ Sci Tehnol* 14, 59-66
7 Schoenfeld, A [1980] *Am Math Monthly* 87 794-804
8 Schoenfeld, A [1982] *Jl Res Math Educ* 13, 31-49
9 Schoenfeld, A. [1983] in Lesh and Landau, *Acquisition of Mathematics concepts and processes*. Academic Press

PART 2

This second protocol of a lecture (?) given in a course on "Problems in Elementary Mathematics" to future secondary-level teachers in mathematics has as starting point the "principle of the student"—meaning that whenever a student takes an examination in mathematics he cannot fail, since during the past trimester—or semester, or whatever period is examined—he has only been taught a limited number of new notions and techniques; so that, when confronted with an examination question he only has to go over the list, and select the right technique(s) and notion(s). This is—in a way—akin to the mechanistic "method of description" of S.P. Kalomitsines in [1]

In fact it is also a question of being aware of other than strictly mathematical clues, like after which chapter you find a specific problem. How different from research or real life—where you don't always know if the right technique for solving a particular problem is at all available. You often have to devise a new method by yourself. No doubt, this is the way many developments in mathematics originate. But using givens which are not explicit in the question is a legitimate and—for the student—an important procedure. As you will see the teaching-concepts in this lesson use this principle, but are more wholistic than that

The problem

S: I found the following problem in a geometry text:

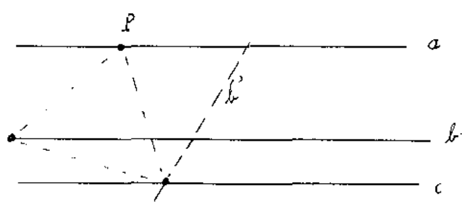
> "Given three parallel lines, construct an equilateral triangle having one vertex on each line."

T: After which chapter?

[7] S: After a chapter on isometrics.
T: Well *(explains the "principle of the student")*, we have to find an isometry.
S: By the way, obviously you can select one of the vertices.
T: Good. We could look for a convenient isometry in a systematic way:

Isometry	Characteristic elements
Translation	One direction Given distance
Rotation	One point (centre) Given angle
Reflection	One line (axis)

S: (*After some prodding*) It looks like a case for rotation; after all we have one vertex (= special point) and an angle (= 60°).
T: What can we rotate?
S *and* T *arrive at the following solution:*
Choose a point P on line a, rotate line b by 60 degrees around P, obtaining line b'. The point of intersection of the lines b and c is another vertex of the triangle.



(Rotate back to find third vertex)

Figure 1

T: This is a very elegant solution. But we arrived at the
[6] solution because we were looking for an isometry. However we could have another look at the problem and see if we cannot come up with another solution.
S: Let us look at the figure we obtain if we suppose that the problem is solved:

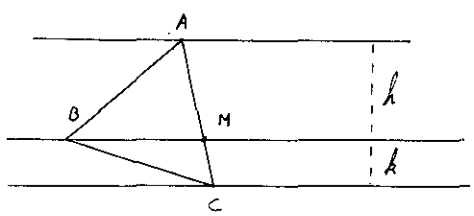


Figure 2

Obviously $AM/MC = h/k$ (h and k distances between parallels).

T: We can easily construct a figure similar to the one we want: Take *any* equilateral triangle ABC, find M between A and C so that $AM/MC = h/k$.
Join M to B.
Draw parallels through A and C to MB.

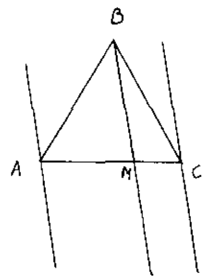


Figure 3

T: Now that we have a figure similar to the one we want
[6] the problem is in fact solved. *Goes on to discuss the general method of constructing a figure similar to the*
[1] *wanted one as a first step.*
T: Let me change the conditions of our problem slightly:

[4] "Given three concentric circles, construct an equilateral triangle having one vertex on each circle."

Do the same methods work?
S: Yes. We can certainly use the rotation. Only this time we get more solutions.
T: In every case?

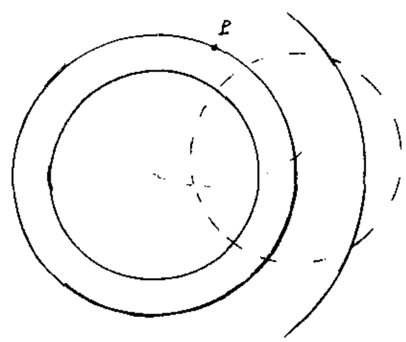


Figure 4

T: Can you use the "general similarity method"? *(Well, can you?)*

The discussion is more involved in this case. There are cases without solutions, and other cases with "more" solutions than in the case of parallel lines. The teacher can go from here, if he wishes, into comparing infinities of solutions, parameters, and so on—or into comparison of pencils of lines and pencils of circles.

T: Let us modify the problem again:

[4] Given any 3 lines construct an equilateral triangle with one vertex on each line.

S: *(At first)* This is completely different We cannot choose a vertex at random.

T: Can't you, then? *Shows that "in general" you can solve in exactly the same way*

The discussion, although straightforward, involves a large number of cases; if the lines form a triangle we have to look very carefully at the case where one of the angles is 120°, for instance.

By now everybody is tired of the problem, and going on and changing the problem again—like considering three parallels in 3-space—would probably be a bad idea.

Analysing the solutions

The numbers in □ refer to the following points—most of which we discussed already in Part I

1 Develop new mathematical content
4 Generalize and particularize
6 Mention—and use general mathematical structures
7 General problem-solving procedures

The last point is new—and delicate. Indeed much has been written—books, articles and computer programs—emphasizing and using general methods. Often the examples that occur are trick examples, like the famous problem of covering an incomplete chess-board by dominoes. My own feeling about these artificial problems is that of course they are very good for impressing your students but only teach them that they are inadequate and that mathematics is full of theatrical tricks. Which it is not. Therefore general methods should be shown to work for legitimate mathematical problems only.

Also being systematic is not being mechanical. Computer programs solve problems, but not yet in as interesting and aesthetic a way as we do. Our main advantage (for the moment) is to eliminate blind alleys quickly and to use our associative powers to generate new ways of looking at a problem. This is what we should try to impart to our students

Reference

[1] Kalomitsines, S P [1983] *Educational Studies in Mathematics* 14, 251-274

PART 3

This third and last condensed protocol of a lecture in a course on "Problems in Elementary Mathematics" for future secondary-level teachers in mathematics is less interesting than the preceding ones, since its mathematical content is run-of-the-mill stuff. But it is not less important. Indeed I have been irritated for years by what I take is a misreading of Polya [4], [5] and a complete disregard of Lakatos [2], [3], namely the insistence on finding patterns, from kindergarten on. Now patterns are important, but only if you prove that indeed the pattern sensed is the only one that agrees with the data. We all know that, but I'm afraid it is being disregarded in teaching at the primary level, and often at the secondary level too. We ask students for an "explanation why the rule works" and then, knowing how difficult it is to do that, forget about it, or give it a low priority. So when teaching future teachers, I make it a point to go over it again. See [1].

The problem

T: The problem we are discussing today is from an algebra text:

"Sum to n terms the series
$2\cdot5+5\cdot8+8\cdot11+\ldots$"

T: How do we start?
S: Find the general (or nth) term
T: Fine. Why not.

After some discussion S comes up with

$$A_n = (3n-1)(3n+2)$$

Comment: Technical skills can be developed here, asking, for instance, what A_n could be for the series

$3\cdot7+7\cdot11+\ldots$

T: Are you sure?
S: Of course.
8 T: I claim that the formula is

$$A_n = 10 + (n-1)30 + (n-1)(n-2)9 + (n-1)(n-2)(n-3)6105\cos(n\pi/127)$$

1 S: ....
T: Check for $n = 1, 2, 3$. You do get 10, 40, 88 ... don't you?
S: All right. But you must admit that our formula is "what is meant in the exercise." Anyway it is simpler
T: Let me show you another example:
How would you continue the series:
1, 2, 4, 7, ...
1, 2, 4, 8, ...
S: 1, 2, 4, 7, 11, 16 ....
1, 2, 4, 8, 16, 32 ....
T: Fine. How do you continue the series
1, 2, 4 ......
with a 7 or an 8? Is either one of them simpler?
1 S: ....
T: How then is the series well defined?
3 S: Obvious—by specifying the general term.
T: Is there no other way?
S: *After examples, prodding and general discussion, discovers (or remembers) the "constructive" way of defining a series (by recurrence, defining a term by manipulations on preceding terms).*
6
T: We still have to find the sum to n terms
S: Since $A_n = (3n-1)(3n+2)$

$$\sum_1^n a_i = 9\sum_1^n i^2 + 3\sum_1^n i - 2n$$

Comment: This is certainly not immediately put into this form. The correct way of using Σ's should be insisted upon. No sloppy Σa_i should be allowed. Why $-2n$?

T: Very good. That is what I call solving by "brute force". Does anybody know of another way of summing this series?

S: ....

T: Did you ever come across a relationship between two series

6 $u_1 = v_1 - v_2$

1 $u_2 = v_2 - v_3$

$u_n = v_n - v_{n+1}$

S: So what?

T: Look at the sums.

S: Find the following way of summing the series:

$$2 \cdot 5 = 1/9\,[2 \cdot 5 \cdot 8 - (-1) \cdot 2 \cdot 5]$$

$$5 \cdot 8 = 1/9\,[5 \cdot 8 \cdot 11 - 2 \cdot 5 \cdot 8]$$

etc.

$$(3n-1)(3n+2) = 1/9[(3n-1)(3n+2)(3n+5) - (3n-4)(3n-1)(3n+2)]$$

$$\text{sum} = 1/9[(3n-1)(3n+2)(3n+5) + 10]$$

T: Could we use similar methods for summing to n terms the series

$$\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \cdots + \frac{1}{(3n-1)(3n+2)}\ ?$$

5 S: Rediscover the difference method, after all

$$\frac{1}{(3n-1)(3n+2)} = \frac{1}{3}\left[\frac{1}{3n-1} - \frac{2}{3n+2}\right]$$

$$\text{Sum} = \frac{1}{3}\left[\frac{1}{2} - \frac{1}{3n+2}\right]$$

T: What happens when n becomes very large?

Analysing the solution

The numbers in □ refer to the following points

1 Develop new mathematical content

3 Insist on the formal proof process

5 Use free association

6 Mention and use general mathematical ideas and structures

8 Use counterexamples

These have all been examined in Parts I and II — except the last one, counterexamples. The main use of counterexamples in teaching now is either when the discovery method is used—i.e. in the lower grades—or as far as a proof by reductio ad absurdum may be considered to be basically using a counter-example. But more positive uses should be made of this notion, as done in this lesson.

Two more remarks: first I would like to insist on the necessity of increasing the quality of our secondary-level teachers, by developing more of an understanding of mathematical content which is relevant to what they teach, and also of the unifying basic ideas of mathematics. And secondly, an overview of the diverse aspects of problem solving. A very good bibliography can be found in Sahu [6]

References

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In proof of the eternity of mind Descartes and others had adduced a pure concept of being whose fundamental quality was existence itself; yet, says Leibniz, it is not all certain whether what is thinkable has also real existence: of this type are such concepts as the number of all numbers, infinity, smallest, largest, most perfect, totality and other notions of that sort, which are not by their nature self-evident, and become fit to use only when clear and unambiguous criteria for their existence have been established. It all amounts to our making a truth mechanically, as it were, reliable, precise and so irrefutable: that this should at all be possible is an all but incomprehensible sign of grace.

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