Beyond a Psychological Approach: the Psychology of Mathematics Education*

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I. INTRODUCTION

The major goals of the international research group "Psychology of Mathematics Education" (PME) are to promote international contacts and exchanges of scientific information in the psychology of mathematics education, and to promote and stimulate interdisciplinary research in this field [Nesher and Kilpatrick, 1990]. In this context the relevance of pieces of research dealing with didactical phenomena is not clear. From my experience as a member of PME since its birth in 1976, I can witness to the fact that that we are very often embarrassed by the status to be given to research projects dealing with classrooms and didactical processes. Beyond academic considerations of the quality of such research projects, their relevance with respect to psychology as a discipline is always problematic. It is this problem that I would like to consider here in so far as I consider it essential for our research community, and, more generally, for the development of relationships between research in mathematics education and research in psychology.

As a starting point I will consider one of the basic hypotheses of research in mathematics education: the constructivist hypothesis. "Hypothèse d'un sujet qui explore activement son environnement, qui participe activement à la création de l'espace, du temps, de la causalité" [Inhelder and Caprona, 1985, p. 8]. This hypothesis has been, and is still, largely discussed in the group. I would like to consider it again to show how it calls for a step beyond a psychological *problématique* in order to understand the nature of the complexity of mathematics learning in a didactical context.

The starting point for the developmental process according to the constructivist hypothesis is the experience of a contradiction which is likely to provoke a cognitive disequilibrium: it is the overcoming of such a contradiction which results in new constructions [Piaget, 1975]. This process concerns the learner as an individual. As a result she will have her own understanding of some piece of knowledge. But this could turn into a problem since this understanding must be open to exchange and collaboration with others. This difficulty can be overcome only once the student's understanding "has been discussed and checked by others," as Sinclair [1988] quoting Piaget [1965] reminds us. Actually, this social dimension is always present in the mathematics classroom in so far as all the way along the teaching process the learner interacts with other students and with the teacher. My position is that the relevance of our psychological approach depends on our capacity to integrate this social dimension into our *problématique*.

More precisely, the psychological relevance of the social dimension lies in two characteristics of mathematics learning in a didactical context:

(i) Students have to learn mathematics as social knowledge; they are not free to choose the meanings they construct. These meanings must not only be efficient in solving problems but they must also be coherent with those socially recognized. This condition is necessary for the future participation of students as adults in social activities.

(ii) After the first few steps, mathematics can no longer be learned by means of interaction with a physical environment but requires the confrontation of the student's cognitive model with that of other students, or of the teacher, in the context of a given mathematical activity. Especially when dealing with refutations, the relevance of its resolution is what is at stake in the confrontation between two students' understandings of a problem and its mathematical content.

The consequence of the recognition of the crucial role played by these two characteristics is that we have to take the study of didactical phenomena as a paradigm of psychological research in mathematics education.

To support this position I will take the results of the research I have made on teaching and learning problems of mathematical proof [Balacheff, 1988a]. In particular I have studied the complexity of students' treatment of refutation within the context of solving a given mathematical problem. I have then considered the same question in a didactical context. In both cases my analysis focussed on the respective contributions of the cognitive and social dimensions in the determination of the phenomena observed.

II. THE PROBLEM OF CONTRADICTION

II.1. Awareness and treatment of a contradiction

Following Grize [1983] I consider that a contradiction exists only if it has a witness. That means that a contradiction does not exist by itself but only with reference to a cognitive system A contradiction may be recognised by the teacher in a student's solution to a problem but ignored by the students On the other hand, a contradiction may exist for the learner where none exists for the teacher. In both cases, the fact that a contradiction is elicited or not depends basically on the knowledge of the one who claims that a contradiction exists or does not.

According to Piaget [1974, p. 161] the awareness of a contradiction is only possible at the level at which the subject becomes able to overcome it Actually, to become aware of a contradiction means to become able to raise the question of the choice between two propositions: an assertion and its negation. Whatever the choice, it implies that the formulation of the assertion is available and that the subject can construct and express its negation Thus consciousness of a contradiction depends on two constructions, and the subject can only be aware of the contradiction when he becomes able to carry out these two constructions. Indeed, even when a contradiction has been identified, overcoming it may only be possible after a long process

On the other hand, a contradiction exists only with reference to a disappointed expectation, or with reference to a refuted conjecture. The potential existence of an assertion is not sufficient, to use a metaphor of Taine[1]: it is necessary for it to come to the forefront of the scene. Piaget himself remarks that to become aware of a contradiction is far easier when it appears between an expectation and an external event which contradicts it [ibid]. The existence of such an expectation means that the subject is actually committed to an assertion and is able to sidestep its action In other words, she is able to consider it as a possible object of thinking; even more, as an object of discourse At this point action is no longer just carried out Being the product of a choice, it is considered in terms of its validity and the adequacy of its effect. That is to say, action is then related to an aim, and a contradiction becomes apparent when this aim is not fulfilled. The choice which has been made and the conditions under which the action has been performed are then put in doubt.

I would like, then, to state the following conditions as being necessary conditions for the emergence of the awareness of a contradiction:

(i) *The existence of an expectation or an anticipation*

(ii) *The possibility of constructing and formulating an assertion related to this expectation, and the possibility of stating its negation*

In mathematics counterexamples are the critical indications of the possible existence of a contradiction Taking as a basis the model proposed by Lakatos [1976], we can differentiate the implications of a counterexample depending on whether what is considered is the conjecture, its proof[2], or the related knowledge and the rationality of the problem-solver himself The schema[3] (Figure 1), showing a conjecture and its proof as the product of both the knowledge and the rationality of a subject, outlines the main possible consequences of a counterexample. Of course the conjecture may be rejected, but other issues are possible, for example: to inspect the proof carefully in order to elicit a possible lemma and then to incorporate it as a condition in the statement of the conjecture; or to question

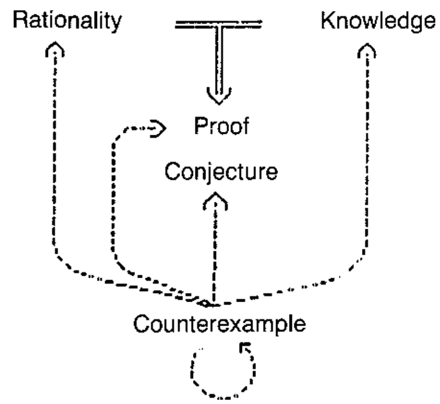

Figure 1

the rationale of the proof or the underlying knowledge; or even to dismiss the counterexample as a "monster". Also — and it is what we expect in so far as we are concerned about learning — such a process is likely to provoke an evolution of the knowledge related to the proof[4].

The analysis Lakatos [1976] provides of the development of mathematical knowledge leaves out a question that is essential for the teacher and the educational researcher: *what determines the relevance of a particular choice for overcoming the contradiction brought forth by a counterexample?* It is this question that I have investigated by means of an experimental study that I will now briefly outline.

II.2. Students' treatment of a counterexample

II.2.1. The experimental setting

In order to explore students' behavior when faced with a counterexample, I have used a situation involving social interactions which encouraged the confrontation of different viewpoints about the solution of the problem and, hence, a verbal exchange making explicit the possible refutations and how to deal with them, thus showing the proving process underlying the solutions proposed by the students. Pairs of 13-14 year-old students were required to solve the following problem[5]:

> *Give a way of calculating the number of diagonals of a polygon once the number of its vertices is known*

The answer to this question was to be expressed in a message addressed to and to be used by other 13-14 year-old students.

The invoked communication structures the student pairs' activity, and more particularly it solicits a verbal formulation of the counting procedure. This is something that students do not normally do straightaway even if they are technically capable of it At the same time the desire to supply the "other 13-14 year-old students" with a reliable technique is likely to make the student pairs pay more attention to its formulation Lastly, the two students have access to as much paper as they want but, on the other hand, to only one pencil. This constraint reinforces the cooperative nature of the situation while at the same time giving us more direct access to the dynamic of the two confronted knowledge systems, especially in the case of decision making.

The observer intervenes only after the students have claimed that they have produced a final solution. At this stage he abandons his stance of neutrality and asks the students to deal with the counterexamples that he offers. Thus there are *two different phases* during the observation, one quasi-independent of the observer and the other with a strong observer-student interaction.

II.2.2. Students' solutions and their foundation

The observed problem-solving procedures are closely linked to the meaning students give to the "objects" *polygon* and *diagonal*. Particular, the interpretation *regular polygon* for "polygon", and *diameter* for "diagonal" leads students to the conjecture $f(n) = n/2$[6] Three types of solution have been proposed which "effectively" provide the required number of diagonals. They are:

$f(n) = f(n-1) + (n-2)$

$f(n) = (n-3) + (n-3) + (n-4) + \ldots + 2 + 1$

$f(n) = (n.(n-3))/2$

Indeed, these are not the ways students expressed them but display their form

The rational bases for these conjectures are more often than not empirical But even so, the nature of the underlying proving process can vary considerably from one solution to another. Naïve empiricism and generic examples are dominant[7]

The analysis of the students' dialogue shows that the lack of an operative linguistic means is one of the major reasons for the absence of proofs at a higher level. For example, the use of a generic example indicates the willingness of the students to establish their solution in all its generality, but this willingness is hampered by the absence of an efficient linguistic tool to express the objects involved in the problem-solving process and their relationships

I must emphasize that this complexity is not only linguistic but also has cognitive origins: that is, the complexity of the recognition and the elicitation of the concepts needed for the proof. Linguistic constructions and cognitive constructions are dialectically related during the problem-solving process. Let us take the example of $f(n) = (n-3) + (n-3) + (n-4) + \ldots + 2 + 1$. In this case students have to express an iterative process and to control it (number of steps, completion of the computation), but at their level of schooling they do not have the necessary conceptual tools To them, the use of a generic example seems the best means to "show" the computation procedure and to justify it

Some of the students mention the need for a mathematical proof but they do not try to produce one. In fact they stay at a level of proving which is consistent, on the one hand, with their level of uncertainty and what they think is required by the situational context, and on the other hand, with the cognitive and linguistic constructions they are able to perform [Fischbein, 1982; Balacheff, 1987b].

II.2.3. The treatment of counterexamples

Most types of treatment of a refutation, as mentioned above (Figure 1), have been observed But we have to make a distinction between treatments which follow an *analysis* on the part of the problem-solver and those which are merely *ad hoc* treatments of the conjecture in order to save it at all costs.

Important differences exist between "phase I" and "phase II", but these differences are not the ones one could possibly foresee *a priori*. It is in phase I that rejection of the conjecture is dominant, not in phase II where one might have expected that a counterexample proposed by the observer would provoke a rejection of the conjecture. On the other hand, rejection of the counterexample is dominant in phase II as well as its being treated as an exception. Two explanations can be proposed for this phenomenon:

— The rejection of the conjecture appears in cases of extreme fragility which are verified in only one case: $f(n)=n$ (verified by P_5), $f(n)=2.n$ (verified by P_7), $f(n)=n^2$ (verified by no polygon, but conjectured by Hamdi and Fabrice....).

— The rejection of the counterexample happens when it is opposed by a strong conjecture whether or not the latter is strongly established (with respect to the conceptions of the students) or correct

In the following I will present in some detail the categories of students' treatment of a refutation.

Rejection of the conjecture

The analysis of this type of response to a counterexample involves two different behaviors:

— The immediate rejection of the conjecture, as soon as the counterexample has been produced. This decision is more often than not taken on the basis of naïve empiricism. This behavior is coherent in so far as the observation of a few polygons was sufficient to construct the conjecture, one counterexample is sufficient to dismiss it.

— The rejection of the conjecture after an analysis of the possible origin of its refutation This analysis provides students with new elements which they can use to restart the search for a solution to the problem.

Modification of the conjecture

The modification of the conjecture when faced with a refutation appears to be equally possible in both the phases of the experiment. Three main types of such a treatment can be distinguished:

— An *ad hoc* modification which consists in a direct adaptation of the conjecture to superficial features, observed by the students, of the relations between the expected result and the one indicated by the counterexample. The origin of the new conjecture is related to the observation of only one case, thus it has a naïve empiricist foundation.

— A modification which follows an analysis of the sources of the refutation with reference to the foundation of the conjecture.

— A modification which consists in reducing the domain of validity of the initial conjecture to a set of objects which excludes the counterexample, and in constructing a solution specific to the objects the class of which the counterexample is considered a good representative

The counterexample considered as an exception

This type of treatment of a counterexample is quite rare. It appears just once in the first phase of the experiment A possible explanation may be the belief that a mathematical assertion must not suffer any exception.

Introduction of a condition
The decision to introduce a condition in the statement of the conjecture follows two types of strategies:

— Some students try to find a condition by means of an analysis of the counterexample in order to bear down on the class of objects of which it is a representative

— Others try to find a condition by means of an analysis of the refuted conjecture and its foundations But the decision to introduce it seems impermissible to some students because of the experimental contract[8].

The definition revisited
Most of the students posing the problem of what a polygon is, or trying to elicit a definition of a polygon or of a diagonal, are those whose conjecture is $f(n) = n/2$. Their uncertainty about what a polygon is is noticeable from the very beginning of the experiment, but it is only after a refutation that some of them clearly state this problem of definition.

Rejection of the counterexample
This way of treating a counterexample is the most widely distributed among the observations I have made. On the other hand its presence is more frequent in phase II of my experiment during which the students were perhaps more eager to defend their conjecture against refutation by the observer. But not all these reactions should be considered the same. Three main categories can be differentiated:

— The counterexample is rejected after an analysis which reveals that it actually does not refute the conjecture: it is shown that the refutation is made on the ground of a misinterpretation The possibility of such an analysis was opened up by the example of a polygon P_5 with three aligned (but not consecutive) vertices, proposed by the observer

— The counterexample is rejected with reference to a precise conception of what a polygon is: a conception which may have been previously formulated as a definition.

— The last category is the one Lakatos named "monster-barring". The counterexample is in this case rejected by students without any further consideration of the definition or any debate about their conceptions. It is, in particular, the category for the triangle because of its lack of diagonals

II.3. Some conclusions on students' treatment of refutations

What the Lakatos model [1976] describes is the complexity of overcoming contradictions in mathematics due to the diversity of possible ways of dealing with a refutation. For students also the *voies royales du progrès* do not exist. They have at hand more than one way to cope with a contradiction, but some of the possible ways they can choose may not be acceptable with respect to the teaching target

From our research, three factors appear to determine students' choice in their treatment of a refutation:

Analysis with reference to the problem itself
It gives a central place to discussion of the nature of the objects concerned, and thus on their definition. This analysis potentially leads to any type of treatment which could be considered suitable, none of them privileged *a priori*. The choice the students make can be understood only in the light of a local analysis or of the specific character and knowledge of each of the individuals. The type of treatment can change in the course of the problem-solving process: modification of the definition followed by the introduction of a condition or a modification of the initial conjecture when the conceptions have been stabilized. The origins of the choice between the introduction of a condition and the search for a specific solution, and the modification of the conjecture, cannot be traced in the data gathered. What can be conjectured is that when the refutation might brush aside a wide range of objects (with respect to the students' conceptions), then a modification of the conjecture is decided which consists in a specific solution.

Analysis with reference to a global conception of mathematics
This could be a serious obstacle to some of the treatments of a refutation: refusal to treat the counterexample as an exception, refusal to accept a solution which cannot be expressed by a unique formula, etc. This global conception of mathematics has its origins mainly in the interactions between the teacher and her students. We can speak of the emergence of a didactical epistemology of mathematics as the result of the everyday practice of the classroom micro-society. Recognizing this, Lampert [1988, p. 470] reports that "changing students' ideas about what it means to know and do mathematics was a matter of immersing them in a social situation that worked according to different rules than those that ordinarily pertain in classroom, and then respectfully challenging their assumptions about what knowing mathematics entails". Such a project is at the core of many attempts to modify classroom practices in order to obtain a modification of students' view about what mathematics consists of [e.g. Arsac et Mante, 1983; Capponi, 1986]. Shoenfeld [1987], exploring the issue of mathematics as "sense-making", shows that these kinds of social considerations are not recent, citing for example classroom discussions promoted by Fawcett [1983]. What is new now seems to be the consensus of the mathematics educators' community about the fact that issues related to the social dimension must be considered more carefully Evidence of the fruitfulness of using social settings to facilitate mathematics learning consists mainly of stories about the success of such attempts But, as we know, such settings do not work with every teacher or in every classroom Research is needed to strengthen the foundation of such approaches.

Analysis with reference to the situation
What is in question is mainly the way students view (or "read") the situation which leads them to favour some specific treatments of the counterexample (a definition game, a riddle game in which they abandon their solution quite easily) or which constitutes an obstacle to others (the refusal to introduce a condition because it has not been stated in the problem statement) The students' view of the situation is mainly a consequence of what has been stated about the characteristics of the situation, but also of what they think about it and which has never been stated explicitly. The idea of an "experimental contract" tries to catch this phenomenon by referring to a negotiation, more often than not implicit, which defines the situation for the learner and which contributes to giving meaning to its

behavior. The theoretical concept of "didactical contract", coined by Brousseau [1986], is a tool to use in coping with this phenomenon, which plays an essential role in the mathematics classroom. Its importance lies in the fact that it can affect students' construction of meaning in so far as "les sujets en viennent toujours à élaborer des connaissances propres à la situation" [Inhelder and Caprona, 1985, p. 15]. This remark has some implications for our research methodologies since our interpretations of observed students' behaviors might be relevant, or valid, only if we can say what was the game they were playing.

III. DEALING WITH CONTRADICTIONS IN THE MATHEMATICS CLASSROOM

To base the learning of mathematics on the students' becoming aware of a contradiction requires that we take into account the uncertainty about the ways they might find to overcome the contradiction. If, as I believe, we cannot assume that there is strict determinism in cognitive development, what can be the extent of the role played by a particular situation? Rather, the teacher's interventions will be fundamental. The way she manages the teaching situation may bring the students to see that their knowledge and the rationality of their conjectures must be questioned and perhaps modified because no *ad hoc* adaptation of a particular solution, or its radical rejection, can by itself lead to a conceptual advance.

Mathematicians, Lakatos considers, share almost the same rational background. In the case of students the background is not the same: naïve empiricism, or pragmatic validations, can be the basis for their proofs and can constitute the roots of their beliefs in the truth of a statement. How can we escape the fact that when faced with a counterexample produced by the teacher, students claim that it is only a particular case when, in fact, what they should question is the naïve empiricism on which their conjecture is based? At a higher level of schooling this problem can still appear: a student may be discussing the legitimacy of a counterexample when it is his or her understanding of the related mathematical knowledge that should be being questioned.

On the other hand, in the teaching situation, the existence of a referent knowledge (the scientific knowledge or the knowledge to be taught) constrains the teacher to decide whether a fact is contradictory or not with respect to this knowledge. The problem for her, then, is to make the student recognise this contradiction that she alone initially sees.

The didactical question thus becomes two-fold: first, *what are the conditions necessary to engender, on the part of the student, awareness of a contradiction?* And, second, *what are the conditions under which the student can resolve it properly?*

Actually, these are genuine psychological questions addressed to our community. To tackle them we need to go beyond a classical cognitive problematic and to take into account both the complexity of the didactical situation and the specificity of mathematics as a content to be learned. Following this path, the work I have done confirms the benefit of social interaction as a didactical tool, but it also clarifies its limits in helping students to make sense of mathematical knowledge [Balacheff, 1989b].

III.1. Social versus mathematical behaviors

Mathematical behaviors are not the only ones likely to appear in social situations, and in some circumstances they can even be almost completely replaced by other types of interactional behavior. My point is that social interaction might become an obstacle when students are eager to succeed, or when they are not able to coordinate their different points of view, or when they are not able to overcome their conflict on a scientific basis[9]. In particular these situations can favour naïve empiricism, or they can justify the use of a crucial experiment in order to obtain an agreement, instead of proofs at a higher level [Balacheff, 1988a].

Some people may suggest that a better didactical engineering would allow us to overcome these difficulties; indeed, much progress can be made in this direction and more research is needed. But I would like to suggest that "argumentative behaviors" (i) are always potentially present in human interaction, and (ii) that they are genuine epistemological obstacles[10] to the learning of mathematics. By "argumentative behaviors" I mean behaviors by which somebody tries to obtain from somebody else an agreement to the validity of a given assertion, by means of various arguments or representations [Oléron, 1984]. In this sense, argumentation is likely to appear in any social interaction aimed at establishing the truth or falsehood of something. But we do consider that argumentation and mathematical explanation are not of the same nature: the aim of argumentation is to obtain the *agreement* of the partner in the interaction but not in the first place to establish the truth of some statement. As a social behavior it is an open process, in other words it allows the use of any kind of means; whereas, in mathematics, we have to accept the requirement that the use of knowledge must fit into a common body of knowledge on which people (mathematicians) agree. As an outcome of argumentation solutions to problems are proposed but nothing is ever definitive [Perelman, 1970, p. 41].

In so far as students are concerned we have observed that argumentative behaviors play a major role, pushing back other behaviors like the ones we were aiming at. Clearly enough this can be explained by the fact that such behaviors pertain to the genesis of a child's development in logic: children very early experience the efficiency of argumentation in social interactions with other children, or with adults (in particular with parents). So it is quite natural for these behaviors to appear first when what is in debate is the validity of some production, even a mathematical one.

What might be questioned is perhaps not so much the students' rationality as a whole, but the relationships between the rationale of their behaviors and the characteristics of the situations in which they are involved. Not surprisingly students first refer to the kind of interaction they are already familiar with; argumentation has its own domain of validity and of operationality as all of us know.

So, in order to teach mathematics successfully the major problem appears to be that of *negotiating* the acceptance by students of new rules, but not necessarily to obtain their rejection of argumentation in so far as it is perhaps well adapted to other contexts. In particular, mathematical proof should be displayed "against" argumentation, bringing students to the awareness of its specificity and its efficiency in solving the kind of problems we have to solve in mathematics.

Negotiation is the key process here, for the following reasons:

— First, because the teaching situation cannot be delivered entirely "open" to the students otherwise many of them will not understand the point and will get lost. The following quotation from Cooney [1985, p. 332] makes it clear: "Maybe not all of them but at least some of them felt "I am not going to participate in this class because you [referring to the teacher] are just wasting my time". It is so ironic because if I was doing the type of thing they wanted me to do, they would be turning around in their seats and talking". Where Cooney sees a "no-win situation", Brousseau [1986, p. 119-120] suggests theorizing a phenomenon specific to didactical situations: the "paradox of the devolution of situations". That is, all that the teacher undertakes in order to obtain the expected pupil behavior tends to deprive this behavior of its mathematical meaning; but on the other hand, if the pupil refuses information coming from the teacher then the didactic relationship is broken.

— Second, because of the rules to be followed the true aim of the teacher cannot be stated explicitly. If the rules for the interaction are explicitly stated then some students will try to escape them, or to discuss them just as many people do with a legal point. And because interacting mathematically might then become "mastering a few clever techniques" which would turn into objects to be taught, just as teaching "problem solving" has often become teaching quasi-algorithmic procedures [Schoenfeld, 1985].

III.2. Efficiency versus rigor

Even if we are able to set up a situation whose characteristics promote content-specific interaction with the students, we cannot take for granted that they will engage in "mathematical debate" and that finally they will act mathematically.

A particularity of mathematics is the kind of knowledge it aims at producing. Its main concern is with concepts specific to its internal development. There is evidence that Egyptians used intellectual tools in practical situations for which we now have mathematical descriptions, but the birth of mathematical proof is essentially the result of the willingness of some philosophers to reject mere observation and pragmatism, to break off perception (*le monde sensible*), to base knowledge and truth on Reason. This was an evolution, or a revolution, of mathematics as a tool — encapsulated in practice — into mathematics as an object made available by itself — through a constructive process — for the purpose of explicit considerations, that as a consequence brought a change of focus from "efficiency" towards "rigor".

It is a rupture of the same kind that takes place between "practical geometry" (where students draw and observe) and "deductive geometry" (where students have to establish theorems deductively). In numerical activities, like the one reported by Lampert [1988], the same rupture happens when students no longer have to find some pattern out of the observation of numbers but have to establish numerical properties in their "full" generality (using letters and elementary algebra).

We have here to realize that most of the time students do not act as *theoreticians* but as *practical men*. Their task is to give a solution to the problem the teacher has given to them, a solution that will be acceptable with respect to the classroom situation. In such a context the most important thing is to be effective. The problem of the practical man is *to be efficient* not *to be rigorous*. It is to produce a solution not to produce knowledge. Thus the problem solver does not feel the need to call for more logic than is necessary in practice.

That means that beyond the social characteristics of the teaching situation we must analyse the nature of the target it aims at. If students see the target as "doing" more than "knowing", then their debate will focus more on efficiency and reliability than on rigor and certainty. Thus again, argumentative behaviors could be viewed as being more "economic" than behaving mathematically, while allowing students to feel good enough about the fact that they have completed the task.

To all these problems I see possible ways of solution in the study and the better understanding of the phenomena related to the didactical contract, the condition of its negotiation (which is almost essentially implicit) and the nature of its outcomes: the devolution of learning responsibility to the students. We cannot expect "ready-to-wear" teaching situations but it is reasonable to think that the development of research will make available some knowledge which will enable teachers to face the difficult didactical problem of managing the life of this original cognitive society: the mathematics classroom.

Notes

[1] Quoted by Hadamard [1959, p. 34].

[2] By "proof" we mean a discourse whose aim is to establish the truth of a conjecture (in French: *preuve*) not necessarily a mathematical proof (in French: *démonstration*)

[3] Indeed such a schema just sketches the range of possible consequences, but it is sufficient to give an idea of what we call the openness of the possible ways to treat a refutation.

[4] What Lakatos called a *proof-generated concept* [Lakatos, 1976, p. 88 seq.].

[5] Observations, which began in 1981, were carried out mainly during the first semester of 1982. Fourteen pairs were observed for 80-120 minutes. For a complete report on this research see [Balacheff 1988a] or [Balacheff, 1989a] for a detailed report in English

[6] "f" is the name we will use for the function which relates the number n of vertices of a polygon to the number of its diagonals

[7] By *naïve empiricism* we mean the situation in which the problem-solver draws the conjecture from the observation of a small number of cases (for example that $f(n)=n$ because P_5 has 5 diagonals). We use the expression *crucial experiment* when the problem-solver verifies the conjecture on an instance which "doesn't come for free"; here the problem of generality is explicitly posed. The *generic example* involves making explicit the reasons for the truth of the conjecture by means of actions on an object which is not there in its own right but as a characteristic representative of its class. The *thought experiment* invokes action by internalising it and detaching it from a particular representation. (For a precise definition of the different types of proofs, see: Balacheff [1987a], [1988a] and [1989a].)

[8] The relationships between the actors involved in the situation (the two pupils and the observer) are determined by rules which are both explicit (like what is said about the task) and implicit (like those suggested by the aim of the experiment). All these rules constitute what we call the "experimental contract" [Balacheff and Laborde, 1985]

[9] I mean, a content-specific basis

[10] The notion of "epistemological obstacle" was coined by Bachelard [1938], and then pushed to the forefront of the didactical scene by Brousseau [1983]. It refers to a genuine piece of knowledge which resists the construction of a new one, but such that the overcoming of this resistance is part of a full understanding of the new knowledge

References

Arsac, G ; & Mante, M [1983] De "problèmes ouverts" dans nos classes du premier cycle. *Petit x*, **2:** 5-33

Bachelard, G [1938] *La formation de l esprit scientifique*. Paris: Vrin, 1975

Balacheff, N. [1987a] Processus de preuve et situations de validation *Educational studies in mathematics*, **8:** 147-176

Balacheff, N. [1987b] Cognitive versus situational analysis of problem-solving behaviors *For the learning of mathematics*, **6** (3): 10-12

Balacheff, N [1988a] *Une étude des processus de preuve en mathématiques chez des élèves de collège*. Thèse d'état. Grenoble: Université Joseph Fourier

Balacheff, N [1988b] Aspects of proof in pupils' practice of school mathematics. *In* Pimm, D (Ed.) *Mathematics, teachers and children*. London: Hodder and Stoughton. pp 216-235

Balacheff, N [1989a] Treatment of refutations: aspects of the complexity of a constructivist approach to mathematics education *In* Von Glasersfeld, E (Ed.) *Constructivism in mathematics education* (in press)

Balacheff, N [1989b] Benefits and limits of social interaction: the case of teaching mathematical proof *In* Bishop A S Mellin-Olsen, & J Van Dormolen (Eds.) *The genesis of didactical knowledge in mathematics education* (in press)

Balacheff, N ; & Laborde, C [1985] Langage symbolique et preuves dans l enseignement mathématique: une approche socio-cognitive *In* Mugny, G (Ed.) *Psychologie sociale du développement cognitif.* Berne: Ed Peter Lang

Brousseau, G [1983] Les obstacles épistémologiques et les problèmes en mathématiques. *Recherches en didactique des mathématiques* **4** (2): 164-198

Brousseau, G [1986] Basic theory and methods in the didactics of mathematics. *In* Verstappen, P F L (Ed.) *Second Conference on Systematic Co-operation Between Theory and Practice in Mathematics Education*. Enschede: Institut voor Leerplanonwikkeling pp 109-161

Capponi, M T ; et al [1986] *Apprentissage du raisonnement* Grenoble: IREM de Grenoble

Cooney, T J [1985] A beginning teacher's view of problem solving *Journal for Research in Mathematics Education.* **16**: 324-336

Fawcett, H P [1938] *The nature of proof.* 1938 NCTM Year Book New York: Columbia University Teachers College

Fischbein, E [1982] Intuition and proof. *For the Learning of Mathematics*. **3** (2): 9-18, 24

Grize, J B ; Piéraut-le Bonniec [1983] *La contradiction* Paris: PUF

Hadamard, J [1959] *Essai sur la psychologie de l'invention dans le domaine mathématique*. Paris: Gauthier-Villars

Hanna, G [1983] *Rigorous proof in mathematics education* Curriculum Series, **48**. Toronto: The Ontario Institute for Studies in Education

Inhelder, B ; & de Caprona, D [1985] Constructivisme et création de nouveautés. *Archives de psychologie*. **53** (204): 7-18

Lakatos, I [1976] *Proofs and refutations* Cambridge: Cambridge University Press

Lampert, M [1988] The teacher's role in reinventing the meaning of mathematical knowing in the classroom *In Proceedings of the 1988 PME-NA Conference*, pp 433-480

Nesher, P.; & Kilpatrick, J [1990] *Mathematics and cognition* Cambridge University Press

Oléron, P [1984] *L augmentation*. Paris: PUF

Perelman, C [1970] *Le champ de l argumentation* Bruxelles: Presses Universitaires

Piaget J. [1965] *Sagesse et illusion de la philosophie* Paris: PUF

Piaget, J [1974] Les relations entre affirmations et négations *Recherches sur la contradiction*. Vol. 2 Paris: PUF

Piaget, J. [1975] L équilibration des structures cognitives. *Etudes d'épistémologie cognitive* Vol XXXIII Paris: PUF

Schoenfeld, A H [1985] When good teaching leads to bad results: the disaster of well taught mathematics courses. *In* Peterson, P L ; & Carpenter, T L (Eds): *The study of students' thinking during instruction in mathematics* (to appear)

Schoenfeld, A H [1987] On mathematics as sense-making: an informal attack on the unfortunate divorce of formal and informal mathematics *In* Perkins, D N ; Segal, J ; & Voss, J (Eds) *Informal reasoning in education* Hillsdale: Erlbaum

Sinclair, H [1987] Constructivism and the psychology of mathematics *In* Bergeron, J C ; Herscovics, N.; & Kieran, C. (Eds) *Proceedings of the eleventh international PME conference* Montréal: Université de Montréal. pp 28-41

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