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On the Stability of Symbols for the Learning and Teaching of Mathematics

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This article, based on the author's Ph. D. thesis, develops an argument on the advantages of using symbols, the limitations in their use, and their stability for the learning and teaching of mathematics

The advantages of using symbols

Two examples will be used to highlight the advantages of using symbols and motivate the reader to consider my approach in discussing the stability of symbols

EXAMPLE 1

Let us consider a symbol such as:

Solve $5^x = 10$.

This symbol stands for some mathematical idea called an equation, and what we want is another mathematical idea called a solution which the above symbol has instructed us to produce. We return to the symbol system to see if manipulating the system can help us obtain a symbol which can be interpreted as a solution, a mathematical idea

The symbol can stand for something else, a relation or function called exponential: another mathematical idea which is not entirely independent of that of an equation. But, having hit on the exponential idea, we may then continue within the demain of ideas to connect it to another mathematical idea called the inverse function or logarithmic function. Next, we transfer to the system of symbols to search for a way of representing this mathematical idea so that it shows the relationship with exponential functions which we have imposed within the demain of ideas. At the purely symbolic level we then transform $5^x = 10$ to $x = \log_5 10$. We now have another symbol derived from the given symbol, and this can be related to the mathematical idea of a solution.

Now suppose the symbol

Solve $5^x = 10$

is accompanied by another symbol:

Hint: $\log_{10}(5^x) = \log_{10}10$ [Quotation from School Mathematics Project (SMP) Book 5]

This second symbol is of help only if we can handle it in such a way that an appropriate relationship exists between it and the first symbol.

We have already derived $x = \log_5 10$ without the hint as a symbol which represents the mathematical solution. The hint forces us not to limit ourselves to symbols such as definitions, but to use symbols which have to do with mathematical ideas or properties of logarithmic functions. Then we come back to the symbols and manipulate them to get

$$\log_{10}(5^x) = x\log_{10}(5) = \log_{10}10 = 1$$

$$\text{i.e. } x = 1/\log_{10}5$$

Note that the symbol $x = 1/\log_{10}5$ is not the same as the symbol $x = \log_5 10$. Do they represent the same mathematical idea — a solution for the given equation? The answer is, of course, in the affirmative because both satisfy the equation, thus

$$5^{\log_5 10} = 10 \text{ and } 1/5^{\log_5 10} = 10.$$

But this sameness we have mentioned can be used to generate more symbols, relations and ideas which may not seem to bear any relationship to the symbol and the idea we started off with. For instance, the two solutions being the same implies that

$$1/\log_{10}5 = \log_5 10 \text{ or } \log_5 10 \times \log_{10}5 = 1.$$

This is still the symbol of an equation, but it happens here to be a particular case of a more general law, namely:

$$\log_b a \times \log_a b = 1.$$

In other words, it means that when we interchange the bases and the arguments the product of the two logarithms gives 1.

There is an important relationship which runs right through this example, and it is that between the sumbols for exponential and logarithmic functions and the ideas they represent. The use of exponents to define logarithms is essential in order to connect the symbols as well as the mathematical ideas behind them.

Sometimes symbols take a form with which we cannot identify ideas known to us. When this happens, what may be intended to be a hint may give the student no room for manoeuvre, and create problems of understanding a given situation. The advantage of the hint only exists for a student when he is capable of handling the given or chosen symbols and establishing the relationships between them, and can then transfer these activities to the mathematical ideas for which the symbols stand. The process of transferral should establish a correspondence between the symbols and mathematical ideas. This is what the learning and teaching of mathematics must pay attention to. It is not an easy task. The student has to be guided in this direction, and it may require more time than we usually allow if what is needed is something close to a perfect match between symbols and the mathematical ideas they represent.

This example brings us to a serious problem in school mathematics. It is believed by some reformers that by simply connecting symbols called "topics", using some other symbols called "unifying topics", the teacher or his student will be able to remember the mathematical ideas which the connecting topics represent. In the light of the preceding example some reservations ought to be mentioned in connection with this procedure. The connected symbols presented by writers may have relationships for them which are not the same as those a student and teacher get from the symbols. The outcome in this case is a mismatch. Even if they agree on the relationship between the symbols, they may not be able to agree on the mathematical ideas represented or the relationship between the ideas. Let us take another example

EXAMPLE 2

Solve a pair of simultaneous linear equations, such as

$$4x - 5y = 5$$
$$7x - 9y = 8.$$

Sometimes students will be able to recognize the possibility that this symbol can be transformed into another symbol without changing the mathematical ideas it represents. For example, one student may transform it into a matrix equation:

$$\begin{bmatrix} 4 & -5 \\ 7 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

whereas other students will never see this possibility. another student may be able to produce three or more procedures for manipulating the given symbol because there is no restriction as to which transformation to use. He may decide to use functions and manipulate these with the aid of other symbols with the name of "inverse elements".

For the vector or matrix representation, we need to find the inverse matrix, if it exists, of the matrix of coefficients, and employ matrix multiplication to find a solution. If a hint is given to use, say, the graphical method, then a restriction is not only imposed but a problem created — that of finding an appropriate relationship between the hint and the given equations, and then to manipulate these symbols to get one which will represent the solution. When symbols are left unconnected and then manipulated, as is often done, the difficulty caused by this style of handling symbols can be grave. When there is an emphasis on relationships and the connections between symbols, we may not be any better off if the relationship we are looking for is a perfect match between symbols and mathematical ideas. I feel compelled to take exception to the popular thinking that when unity and relationship are forced on mathematical topics this implies the existence of relationships between the mathematical ideas behind the symbols, and that these ideas and topics are better retained and remembered. I prefer to think of relationships and symbols only in terms of the persons for whom the symbols stand for some mathematical ideas.

The formation of relationships, as I have said, needs some guidance. For the student this must come from the teacher and his previous experience. It is personal, and sometimes one has to search previous experience for something that will help transform the given symbol and show a relationship internal to it, or its relationship to some mathematical idea. Can there be criteria acceptable to writers, teachers and learners for forming the relationship internal to a model, or to a piece of mathematics it represents? What we may consider as acceptable for the author may not be helpful to students or teachers. When the author is the same as the teacher, the situation involves only one more element, the student, but we still have to think of criteria of acceptability for this situation, or even among a group of writers.

The integration of school mathematics

The question raised above has to do with the integration of school mathematics and the problems attached to such integration. A critical look at this situation will not be interpreted, I trust, as an opposition to the integration of school mathematics.

The introduction of notions of relations and functions creates a new conception of logarithms not known in traditional school texts, just as the introduction of coordinates creates a new conception of the cosine, sine and tangent of angles. The ratios of traditional trigonometry are transformed into graphs, functions and coordinates of points in such a way that the traditional teacher may find that trigonometry has been suppressed in the process of embedding it in other topics.

In the case of embedded schemata, the structure of a schema is given in terms of its relationships with other schemata. In some cases schemata can be embedded within themselves; subschemata are represented by the schemata in which they appear: the dominating schemata. The successive dependence of a schema on subschemata must end with atomic or primitive schemata which represent unanalysable conceptual components of human knowledge. These cannot be drawn from experience alone. Thus our entire knowledge would appear ultimately to rest on a set of atomic schemata.

It is possible for symbols to depend on subsymbols in a particular text, just as Davis and others have described the dependence of a schema on subschemata. In some other text what may appear as a superordinate topic is an isolated topic without bearing any direct relationship to any other topics. In thinking about symbols and how they are related to other symbols and to the mathematics corresponding to them, we have prepared the way for considering possible advantages derivable from the manipulation of symbols in teaching and learning. In teaching, much as we should have our mind set on isolated ideas, it should also be set on how these ideas are connected or can be connected. If we are going to use symbols for these ideas, it seems appropriate for the mind to operate with the symbols as it does with the ideas they represent. Davis *et al.* [(1978), p. 57] have made the following comment, which I think is in support of the view expressed above:

> Teaching has the highly peculiar property that one is more-or-less simultaneously trying to get the learner to develop the subschemata ... at the same time that he/she is developing the organisational schema. In fact "meaningful learning" reaches in both directions: "downward" to previous experience (which may or may not be "more concrete") and "upward" to organisational or relational schema. If the "downward" schemata are not well-developed, the student gets the feeling that he/she is dealing with meaningless "symbol pushing". If the "upward" organisational schemata are not being developed, the student cannot see "where all of this is leading", "how it all fits together"...

The point about organisational or relational schemas, as they relate to symbols, should be thought of as a problem situation in which we have selected to manipulate symbols so that they reveal features which can hardly be accessible to us if we decide to operate with the objects themselves, which the symbols represent. It is not a question of one simple symbol; it has to do with the dynamics internal to the symbols and their referents, which are mathematical ideas and objects, and the dynamics external to symbols and mathematics.

TWO ADVANTAGES

Mathematical knowledge can be assimilated in terms of its major constituents without necessary reference to the internal structure and relationship of the constituents themselves. Yet deeper understanding can be achieved if reference is made to the internal relationship between these constituents. This is one major advantage of focusing attention on the unification of mathematics. Symbols are used to represent it as one whole. But too great a stress on this without reference to the parts of both symbols and mathematics may deprive the symbols or the mathematics they represent of depth in understanding.

A second advantage is economy, which may help writers reduce the number of variables not of interest to provide space for those of interest. Mathematical structures and relationships have sometimes been given a special place for their use in the reduction of variables, symbols and their manipulation. For example, it is possible to impose the same structure on equations of the form:

$$ax = b$$
and $$AX = B$$

where the elements of the first are real numbers and those of the second are matrices. This may have, as a consequence, the suppression of the differences between the elements and the operations so as to obtain a perfect match between the two equations. This may be successful at the writing level, but can it really be achieved in a teaching situation? If it is not possible to have such a perfect match between structures and relationships of symbols on the one hand and mathematics on the other, what then are the implications of this for the teaching of mathematics from textbooks which rely heavily on structures and relationships as models for making mathematics accessible to students? We shall now turn to problems of attempting a perfect match between symbols and the objects they represent.

The texts which present this approach have the unusual feature of forcing connections and relationships between topics, and in doing this they set criteria or standards which implicity suggest that the student or the teacher has no alternative but to accept the particular relationships provided. But these symbols and their relationships are aimed at presenting a perfect match between symbols and mathematics, a match which is known only to the writer and not necessarily to the students. It can be argued that when the teacher intervenes, the intervention introduces some stability to the models which have been provided by the writer for the students. This implies a matching which may not be perfect between student and writer, but comes about because the teacher acts in his teaching as a stabilising factor. This will be the case provided the teacher is not so close to the writer or to the students as to make the stability he is supposed to provide a destabilising activity instead. His dilemma is that he has to maintain a balance between himself, the writer,and the students, in his use of a textbook. He must also maintain a balance between the set of symbols available to the each person and the ideas the symbols hold; for the students these may not all be mathematical. The student's experience may not have involved situations where only a perfect match exists between mathematical symbols and ideas, and since this requirement is an essential aspect of mathematics and mathematics teaching, a teacher who has accepted the unity given to mathematics in a textbook has to choose between following the text strictly and doing things as the writer suggests, or departing from this and working on the experience and language available to the students

The first choice amounts to encouraging misunderstanding and confusion over the representation of mathematical ideas by the symbol systems of mathematics. The teacher may insist on a perfect match to be made by the student. (From the point of view of the writer this is a reasonable thing to do.)

The second choice open to the teacher invites him to *provide* the student with experience, not necessarily mathematical, but essential for an understanding of the mathematical symbols in the textbooks and the ideas for which they stand. This kind of teaching may not lead to a perfect match, but may go a long way to develop the students' confidence to handle symbols.

If we show a strong preference for teaching relationships, it should not be for mathematical ideas only, but also for the symbols which represent these mathematical ideas. But if we take this line, how shall we handle the practical questions of writing and designing a textbook, or the classroom decisions about the use of symbols such as textbooks for teaching mathematics?

I have elsewhere [Sesay, 1980] discussed at some length the problems encountered by the SMP authors in their preparation of the SMP 0-level syllabus and textbooks.* It is not possible to include in this article an analysis of the idiosyncratic characteristics and styles of individual writers as revealed by the experimental drafts but suppressed in the published textbooks. So I will simply touch on the content of the published textbooks to support some of the points raised in the foregoing remarks on the integration of mathematics.

The content of Books 1 to 5 includes such topics as function, relations and networks, which topics are not found independently in Books T and T4. The SMP authors of Books 1 to 5 use these topics to unify the content of Books T and T4.

The topic of logarithms, for instance, was probably omitted in Books T and T4 because the authors did not want to use them as a computational tool. There was already in the textbooks a topic — the slide rule — which would do that job. Besides, the authors did not believe in emphasizing computational skills. As a consequence, logarithms were banished.

The writers of Books 1 to 5, mainly grammar school teachers, created a new conception of mathematics which did not deny logarithms a place but ensured that they were not to be introduced as computational tools. The writers suppressed this aspect of logarithms by introducing them through the example of a function whose inverse is the exponential function.

A topic such as transformation geometry in Books 1 to 5 is related in different ways to other topics like matrices, networks, and relations or mappings. Relations are the starting point for introducing networks, and networks and matrices appear as two ways of describing relations.

The relationships between these topics and their subordinate topics may not be clear for all to see how they are related to the mathematics they represent. Some topics are so embedded in others that we hardly appreciate their significance or the true mathematical ideas behind them. Another example of this embedding is found in the SMP textbooks treatment of trigonometry.

Limitations in the use of symbols

There are limitations which make the notion of a "perfect match" unattainable. The examples we are going to give support this view, and they have some implications for authors of mathematics textbooks, which force unity on mathematics and make it impossible for teachers and students to

*The School Mathematics Project is an experimental project which started in England in 1961. Books 1 to 5 cover the first five years of a secondary school course; Books T and T4 are transitional books for students beginning the Project in the third year of the secondary school (the latter were published first) appreciate the value of instrumental activities within the relational framework of mathematics provided by the texts. The suppression of instrumental activities implies the reduction of mathematics, and what the textbook represents is not sufficiently adequate. Teaching must pay attention to the advantages of using symbols as well as to the disadvantages.

FIRST LIMITATION

We begin with what Polanyi [1973, p. 88] calls "ineffable knowledge". This is a knowledge we have of something and which we can describe even less precisely than usual, perhaps only very vaguely. He uses the example of riding a bicycle to demonstrate that there are things we know but we cannot say clearly how we do them, but that such a defect will not prevent us from saying that we know what we do. A guess or conjecture which turns out to be correct, and for which language is not precise enough to demonstrate an argument in support of the conjecture, is what comes to my mind immediately, as an example of what Polanyi is saying. Often it is something we are very much engaged in, for instance teaching mathematics, but when questions are posed about mathematics teaching we are at a loss for words to describe precisely what we are engaged in.

In working with students on sequences, I usually start the topic with lots of examples, taking great care not to get down too quickly to notations, definitions and proofs of theorems. Students are encouraged to give the first few terms of a sequence. When they begin to develop confidence in this activity of listing terms of the sequence, one wants the students to be able not only to list terms of the sequence but to give, when possible, the nth term of the sequence and reduce the tedious task of listing. Most students find this very difficult to do, even though enough time has been devoted to work involving the listing of terms. Some students require considerable practice to be able to give the nth term; for others it comes immediately and almost naturally, but when asked to explain how they have arrived at the relationship, or why they think what they have arrived at is the required relationship, they have difficulties in demonstrating the relationship they have given. Modern school mathematics may give indications of showing how to deal with such difficulties, but I am not sure that the demand for reasons to explain not only what we do but why we do it will be adequate for eliminating such difficulties.

In traditional mathematics these difficulties showed themselves in the way symbols such as rules, examples, definitions and proofs of theorems were taught. The outcomes was that students were able to use the rules or prove theorems but could not possibly say anything about their knowledge, or identify the theorem or its proof when the language of the theorem was slightly altered. Although it was not within their power to say clearly how they proved a given theorem, nor how they recognized it, this would not stop them from saying that they knew how to prove the theorem and how to identify it.

The steps and arguments might be known in an instrumental manner, but they might not be known as far as the *relationship* between the steps and arguments were concerned. Teachers would say that students knew these matters even though they could not tell clearly, or hardly at all, what it was that they knew. It was difficult to test, or rather match, what was known by observed performance. It is even more difficult today to do so.

The difficulty of testing, or rather matching, what is known about these examples lies in the fact that the matching can only be done by accepting the final thing offered — a performance — as a representation of what is known without getting down to the details of the representation. In an attempt to handle this situation, modern mathematics texts demand that students give reasons for what they do. The writers of such texts are convinced that the ability to explain why the examples and principles used work will lead to understanding mathematics. But this is not always the case. While writers are interested in understanding, some teachers and students are engaged in work that will lead to an awareness of a kind of knowing, which includes ineffability and which may be less demanding than understanding. So a representation which may be complete judged by the knower's standard becomes incomplete for some writers, who prefer a better matching that will bring to the surface the details to be characterised as an act of understanding.

In the past, proof was central and important, just as rules and examples were, and geometry, algebra and arithmetic were vehicles for promoting the manipulation of these symbols in teaching mathematics. It meant and still means to some teachers the reproduction of a solid and constructive mathematical argument supported by axioms and definitions. The stability implied by this reproduction of symbols was apparent. For as soon as such activities were followed by questions of how and why things work the way they did, then the reproduced rules or proofs of theorems became unstable. This perturbed state of the representation created a corresponding perturbation of the mathematical ideas for which the rules or proofs stood.

Present-day mathematics teaching requires the use of symbols and models which are structurally stable, and some of these models do not reserve a special space for proofs and rules, as was done in the past. There are, on the one hand, activities such as explorations, experimentation, arguments and discussions designed to give learners an opportunity to create their own symbols which will represent mathematical ideas for them. On the other hand, another aspect of the work focuses on activities that will lead learners to appreciate a structural approach to mathematics, and it is expected that these symbols will represent mathematics as a unified subject. The difficulty with this approach is that if the exploration of the constituents of mathematics receives more attention than the texts suggest or require, we may not be able to articulate mathematics as a unified whole.

SECOND LIMITATION

Polanyi [1973, pp. 88-89] gives another example of a limitation in articulation which will be useful to us. It has to do with our attempt to grasp the relationship of the particulars of our knowledge which jointly form a whole. In this example, which he takes from the subject matter of topographic anatomy, the ineffability applies to the relationship and whole object and not to the particulars which, in this case, may be explicitly specifiable, but we may not be able to tell explicitly what the relationship is between all these particulars which we explicitly know Here is the example.

> The major difficulty in the understanding, and hence in the teaching of anatomy, arises in respect of the intricate three-dimensional network of organs closely packed inside the body, of which no diagram can give an adequate representation Even dissection, which lays bare a region and its organs by removing the parts overlaying it, does not demonstrate more than one aspect of that region. It is left to the imagination to reconstruct from such experience the three-dimensional picture of the exposed area as it existed in the unopened body, and to explore mentally its connections with adjoining unexposed areas around it and below it

In trying to give further illustration of this difficulty, he argues that the limitation cannot be removed by assuming that our bodies are all identical and proceed from that assumption to map out the body by reducing it to several thin slices which can be memorised precisely by the student He observes that this mapping of the body will provide the student with a set of data which will fully determine how the organs are spatially arranged in the body. Yet the spatial arrangement itself will remain unknown to the student The cross-sections which ke knows will be useless to him if he cannot use them and interpret them to yield an arrangement not yet known to him.

> while, on the other hand, had he achieved this topographic understanding, he could derive an infinite amount of further new and significant information from his understanding, just as one reads off itineraries from a map. Such processes of inference, which may involve sustained efforts of intelligence, are ineffable thoughts. [Polanyi, 1973, p. 89]

This example and that of riding a bicycle will suggest a model for describing the problems of representing mathematical knowledge. With either the system M of mathematical ideas, or the system S of symbols corresponding to them, we associate the following: we shall use the letters P for the set of particulars, R for the set of relationships between the particulars P in M or S Then from Polanyi's example, in which M corresponds to the human body and S to its sections, his argument suggests that the difficulty in understanding is in the assumption that we can reduce the human body (M) to a form (S) in which our focus of attention is on P. The relationship, R, we get leading to this form will never be the same for both M and S It is the reverse in the example of riding a bicycle. In this case the cyclist can ride the bicycle; he says it and knows it. So M and S agree, but he cannot tell you the particulars, P, of this is S

While R is the source of the difficulty in the first example, P is the source of difficulty in the second example. It would seem that a difficulty with either of these will present some problem and make it almost impossible for M to be accessible through S It is not always possible to have a one-one correspondence between P's, R's, M and S. A one-one correspondence between S and M would imply a perfect match.

The stability of symbols

The question about representing mathematical ideas in textbooks as a unified subject is no different from the problems faced by teachers in trying to give the textual presentation some form believed to be suitable for teaching. We can think of stability in different ways and with respect to different persons. For instance, when the form a teacher believes is suitable for teaching is very much the same as the textual presentation we may then say that the teacher's representation is stable with respect to that of the writer, but this does not necessarily imply stability of either the mathematical ideas or stability with respect to the student's experiences. The mathematical content which the two representations carry may not be the same, although this may appear to be the case. If the systems M do not correspond for both teacher and writer, then they must be focusing attention on different aspects of M, although S is the same for both. This means that there is a mismatch between M's for both. If the teacher is also the writer this kind of mismatch is unlikely, and our concern is with what goes on when the writer acts as a stabilising factor. This may help, but the possibility of mismatch still exists. A match or mismatch between M and S or even between symbol systems which are available to all — students, teachers and writers — depends on how the teacher manipulates P and R to interpret M through S. Just as P and R exist for the symbol system S, so do they exist for the objects the symbols represent Stability must not be looked upon only in terms of symbols and relationships of parts to a whole, but also in terms of corresponding objects and their relationships to the whole. The stability internal to the system S of symbols, or system M of mathematical ideas, we may refer to as internal stability, and that which exists between M and S we may call external stability.

The problem for the teacher using a unified text is that of working on all these kinds of stabilities so that teaching provides a space for these and those known to the student only in terms of his previous experience, which has been provided by his society. The teacher certainly requires skill, but he will always fall below the expectation of the writer if he is required to provide a perfect match. The result will be a mismatch of some kind between the writer and the teacher, and the writer will frown on the teacher's distortion of the symbols he has provided. This may be an advantage, for it may mean a minor perturbation of the relationships between symbols and objects known to the student. So while at one extreme we have an unstable situation between teacher and writer, at the other we have a stable situation between teacher and student. Internal correspondences not only show up, they support the external correspondences for both M and S, as perceived by both teacher and student.

The extreme kind of distortion may be irreversible and creates a situation in which symbols become independent of objects and even fail to show external stability. Mapping or matching symbols to mathematical ideas may have serious defects, and the examples we have used to illustrate this may seem to be extreme cases, but they give us an idea of what is really involved As soon as we pass from the mapping of objects in a system of mathematical ideas onto symbols or objects in a plane to mapping them onto objects on a curved surface, we begin to realise that the image is a distortion of the object. This distortion may be violent in some cases and we may not be able to perceive what properties of the objects are preserved or any relationship between object and image. Often all we depend on is the image, for the object itself is not accessible to us, but this is really an advantage rather than a disadvantage. Polanyi [1973, p. 89] has expressed the situation as follows:

> We can map the whole surface of the earth on a flat sheet of paper only in the form of a distorted projection, while its representation by a globe is clumsy and shows only one hemisphere at a time. This inadequacy is increased to the level of impossibility when we come to an intricate three-dimensional arrangement of closely packed opaque objects. Diagrams and demonstrations of instructive aspects of the aggregate will now merely offer clues to its understanding, while understanding itself must be achieved by a difficult act of personal insight, the result of which must remain inarticulate

Implications for teacher training.

We are using examples here to highlight the problems of representing objects in general. These can be seen not only in the teaching and learning of mathematics, but also in the writing of textbooks or even the training of teachers. It can be observed, bearing in mind the examples we have taken, that it is not possible, or even helpful, to claim that the articulated rules and formats we give teachers will fully disclose the already known or tacit parts of the art of teaching. Whatever form the rules and formats take they are likely to ignore important particulars or state relationships that are not truly representative of teaching. Difficulties may not exist if all concerned are interested in the particulars and these enable the teacher to be in control of every aspect of what is done or known. But a new limitation may arise when he does not understand or know how to derive the relationships from the particulars of which he is in control

One reason for this difficulty is that the final task of determining how things fit together in a classroom situation, or of developing a relationship between the format which has been offered and actual teaching, is left for the teacher to struggle with. Formal guidance when offered may not be very helpful; it may interfere with the work of the teacher.

In the light of these limitations and inadequacies, it is reasonable to ask how helpful, useful and appropriate are books and formats for teaching and for representing mathematical knowledge. It seems that their usefulness and appropriateness depend entirely on our ability to adjust them to the various things to which they refer. This implies that the objects being mapped, the conception which the mapping suggests and the experience on which it bears must all be able to withstand (or adjust to) perturbations and distortions which are introduced by the mapping and those using it.

If a symbolic representation is going to be stable with respect to what it represents, then two other factors must always be involved: the conception which the symbol suggests and the experience on which it bears. A constant adjustment between these three is needed if the symbol is to be stable with respect to the object it represents.

Concluding remarks

We must not be tempted by the limitations I have pointed out into rejecting the use of symbols for representing mathematical ideas. There are gains to be made by transferring our thoughts to some mode of representation and these seem to outweigh the difficulties that accompany such transfer if we constantly adjust the symbols to our conception of mathematical knowledge and the experience bearing on this knowledge.

Unintended perturbations of our symbol systems will always accompany our operations with them, and some of these perturbations can be very severe, especially when we have not created the symbols, or when we have to handle some symbols for which our conception is blurred and there is no experience to refer them to

These are human difficulties arising from the complex systems of symbols we have constructed to enable us to represent and interpret experience. Since we cannot do away with such systems, as other animals can afford to do, we must be capable of taking our symbol systems through a process of adjustments. We sometimes free ourselves from one system of symbols and transfer our activities to a new system, so that our practice of mathematics and mathematics teaching is never completely free from perturbations The new system is always an extension of some old system It evolves as a result of re-interpreting mathematics in the old system in terms of a new system, entailing an adjustment of our conception of it and our experience with it. The stability of the system depends on such adjustments as we operate with its symbols.

It is hard to conceive of mathematics or the teaching of mathematics without symbols and operations with these symbols. If teachers cannot do without these, as I believe we cannot, we should be prepared to run the risk that accompanies the use of symbols

> The mind which entrusts itself to the operations of symbols acquires an intellectual tool of boundless power; but its use makes the mind liable to perils the range of which seems unlimited [Polanyi, 1973, p. 94]

It seems to me that the use of symbols allows the mind to operate in two domains. One of them is unstable and risks are associated with operations in such a domain. Our task is to reduce this domain of instability and extend the domain of stability so that we create in students an awareness of the power of symbols and the way in which they function

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