Proportion: Interrelations and Meaning in Mathematics

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Every year high school geometry students perform the amazing feat of going through 600 pages of a text, and memorizing hundreds of theorems for tests, though all this may be retained for only a week or two. Similar situations occur regularly in other courses. These accomplishments are impressive, and are therefore taken by many as a sign of excellence. With priority given to covering the ground, students' progress is readily judged by how much material is taken in, and how many tests can be passed. The important considerations for educating, however, are not only the amount of material covered, but what the material means to students; not imposing rote memorizing but awakening creative inquiry; not the number of tests passed, but giving the students a feeling of empowerment over the very processes of learning. Since grasping these meanings and processes of learning are actually the most vital aspect of educating, one wonders why events requiring significant learning and empowerment are so submerged in the mass of material.

We are led to ask: What is our main purpose in the classroom? Are we empowering students to take control of learning and understanding — or primarily only giving formulas? If we produce students who can pass SATs designed to have students mimic rote learning, and to go on and do the same kind of work in college, are we really doing quality educating? Do we seek to awaken and develop intelligence, understanding and creativity, or only to fill minds with facts and formulas? Do students really get a feeling for how to learn? Do they learn how to think about things they never thought about before? Do they learn to mathematize instead of memorizing mounds of mathematical content?

In a paper called "Schooling is not educating," Bob Gowin points out that

> The mere fact that the student is learning math skills is not prima facie evidence that educating is occurring, since it is possible to learn basic skills in ways that hinder the exercise of intellect, emotion, imagination, judgement, and action. [p. 3]

Skills and meaning in mathematics

or why do students always practice scales and never play a piece?

Being able to find the answer to a certain type of problem, such as solving the proportion $x/4 = 7/3$, is certainly an important basic skill. But the repeated solution of dozens of similar problems is not in itself a mark of quality educating. Imparting the ability to solve a number of such problems by a rote method should not be the only goal of a curriculum, nor is the ability to do such a problem an indication that the student has necessarily gotten to the meaning of proportion. Evidence that the curriculum is providing opportunities for deeper understanding would be the presence of questions about the basic concepts involved, such as: What is the significance of proportions? How is a ratio different from a number on the number line? How can we interpret proportions? How is $x/4 = 7/3$ related to $(3)(x) = (4)(7)$? What use can we make of ratios? What historical context can we put these ideas in? Further, how do we actually go about answering mathematical questions? What procedures or methods do we use to "mathematize" and how do we learn how to learn? All these kinds of questions lead us away from a mere rote-oriented mathematics to problems of understanding.

Approaches to proportion

Let us look at the simple example of proportion and some related topics in order to see what might make a math event more meaningful.

Problem: Find A so that $7/4 = A/3$.

The answer can be gotten by students who know how to either "cross multiply" or "multiply by common denominator". Once the method of doing $7 \times 3 = 4 \times A$ is learned, many such problems can be easily solved. Variations of this example can be generated endlessly, and certainly even a computer can be used to provide many examples for students to work on. Probably students have been shown a simple proof of the general case, such as:

$$C/D = E/F \qquad \text{given}$$
$$(DF)(C/D) = (DF)(E/F) \qquad \text{multiply by } DF$$
$$FC = DE$$

If a student knows this proof, it is usually taken as evidence of "understanding" proportion. But there is much more that could be understood about these basic proportion problems. One of the most powerful ways to expand our view of the problem, and concretely illustrate the idea of proportion (and many other algebraic concepts) is to use a geometric interpretation.

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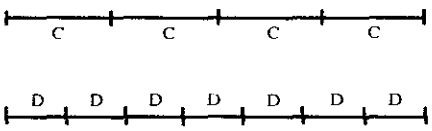
Geometric interpretation of proportion

Let C and D be any two line segments. Since segments are equal if they can be made to coincide (by placing one on top of the other), a natural way to see how unequal segments compare is to try the same idea. Supposing $D < C$, put together enough segments equal to D until the segments coincide with C or exceed it. If the segments of length D exceed C without equalling it, then add another segment to C of the same length as C and continue. If this process ends, i.e. the multiple segments of C and of D eventually coincide at their endpoints, then C and D are said to be "commensurable" or co-measurable. For example, in the picture below, four C-segments and 7 D-segments coincide. We write that the ratio of C/D equals 7/4.



$$4C = 7D$$
$$C/D = 7/4$$

Figure 1

Their coincidence at a common ending point illustrates concretely that the segments C and D are commensurable, and that their ratio is 7:4. In modern language, C/D is said to be rational. The picture also illustrates that two commensurable segments have a "common multiple," i.e. a longer segment (equal to 4 C's or 7 D's) which each original segment fits into an even number of times. The reader may check that having a common multiple (a segment that C and D both fit into) is equivalent to having a common measuring unit (a segment which fits evenly into C and D), called a common divisor in arithmetic. Having a common divisor, having a common multiple, and being commensurable, are all equivalent.

If the process of comparing multiples of C and D does not ever end, then the segments are incommensurable, and C/D is irrational. This ancient notion of incommensurability, which is found in Euclid, also leads directly to the modern idea of Dedekind cuts to define the real numbers!

Notice that geometrically we cannot say whether a given segment's length is integer, rational, or irrational unless we pick units. We are investigating the notion of ratio without relying on units — in fact, it is important to note that ratio is a pure number, for units on C and D "cancel out". We can always pick our unit of measure to be D if we like, and then the length of C becomes rational if C/D is commensurate, and irrational if C/D is not.

Next we consider what it means for two ratios to be equal. In geometric terms, what does it mean for four line segments C, D, E, F to be in a proportion $C/D = E/F$? Or, given segments C, D and E, find F so that $C/D = E/F$.

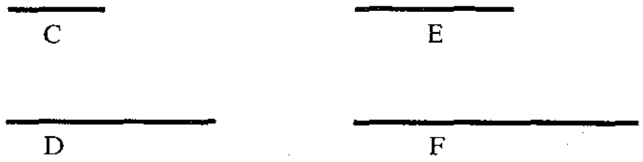
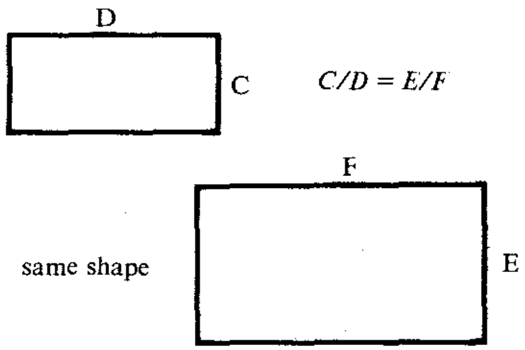


Figure 2

The original proportion $C/D = E/F$, and the resulting products $CF = DE$ can be represented by using rectangles:

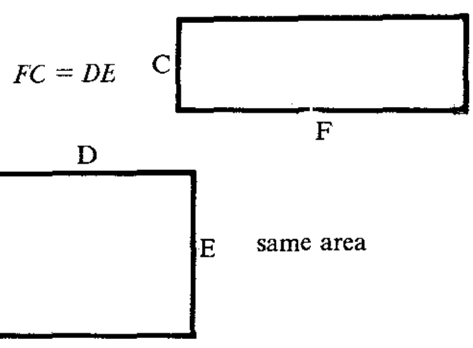
$$C/D = E/F$$

can be represented as two rectangles which have the same shape: C by D, and E by F:



$$FC = DE$$

can be shown by two rectangles which have the same area: F by C, and D by E:



The relation of proportion to rectangles
Figure 3

The ratio C/D gives us some idea of the shape of a rectangle. For emphasis, I will use the notation here "rectangle C/D" to mean a rectangle which has sides C by D. When does another rectangle E/F have the same shape as the rectangle C/D? Euclid's original definition of when four segments are in a proportion is interesting in this context. It seems unduly complicated until we see a picture of what he is saying. We can attempt to compare the rectangles in the same way we compared segments. Put the smaller of the rectangles, in this case C/D, inside the other, at the corner. Now take equal multiples of C and D, magnifying the rectangle C/D by any amount. (It is not necessary to even take integer multiples of C and D, but in Euclid's definition, and in terms of our development, it is better since we know what multiplying by a whole number does to a segment.) Either the whole of the magnified rectangle stays within rectangle E/F or, for a big enough multiple, the whole of E/F will lie completely within the region of the multiple of C/D. The same thing will happen if we blow E/F up by any multiple as well. If there never occurs a case where part of a multiple of the rectangle C/D completely encompasses or is encompassed by every multiple of E/F, then the ratios C/D and E/F are equal, and the rectangles have the same shape.

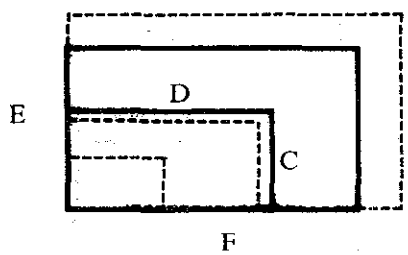


Figure 4

If we think about this process, we can discover an indefinite number of rectangles which have the same shape as E/F (i.e. which have sides proportional to E/F) by taking any points on the diagonal of rectangle E/F (or its extension) and forming new rectangles A/B.

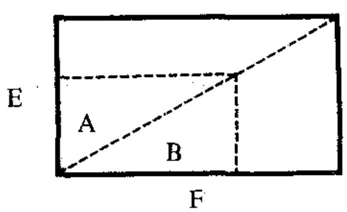


Figure 5

If we return to our two rectangles C/D and E/F we can arrange them vertex to vertex, either internally or externally, so that their diagonals lie on the same line.

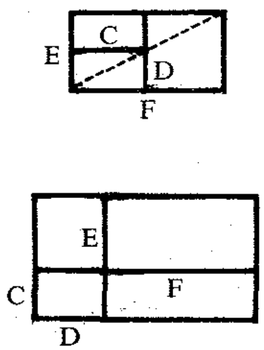


Figure 6

Given three segments, C, D, and E, we can find the fourth segment F making $C/D = E/F$ by using the picture. Construct rectangle C/D and extend side C by segment E. Then draw an extended diagonal of C/D, and complete the rectangle E/F so that its vertex is on the diagonal.

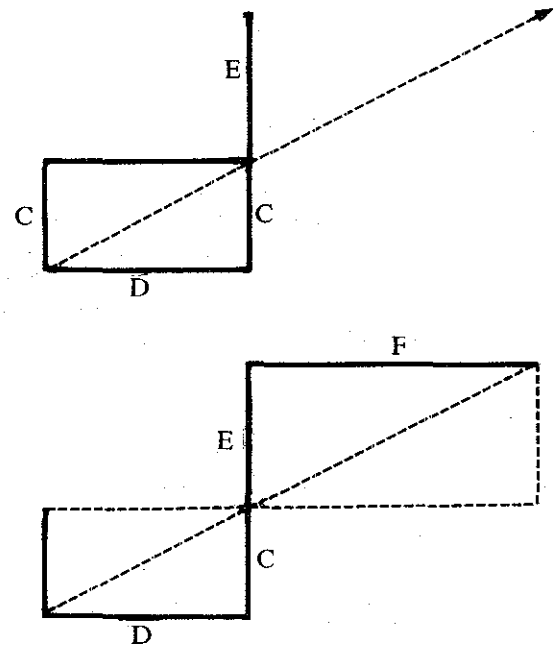


Figure 7

This picture gives us a way of seeing how proportion and cross-product are related to figures. Complete the picture by drawing in the large rectangle with sides $D+F$ and $C+E$. We now have the two same-shape rectangles C/D and E/F, and two other rectangles DE and CF which are easily shown to have the same area (use the fact that the diagonal of a rectangle gives two equal-area triangles). Therefore we see that if $C/D = E/F$ then $CF = DE$. We get a method for finding the fourth proportional, and a resulting remarkable figure relating shape and area:

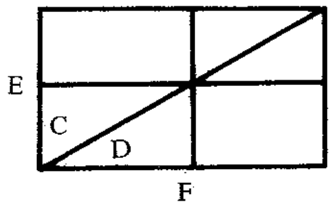
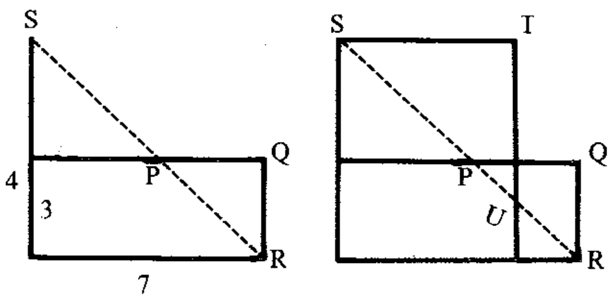


Figure 8

The above figure gives us a beautiful view of a relation between size and shape: the proportion is illustrated by two figures having the same shape, cross-multiplying by two figures having the same size. Ancient philosophers spoke about the shape as quality or form and the size as quantity. The equivalence of proportion and cross-multiplying tells us something about the relation of form and quantity which philosophers have puzzled over for thousands of years.

The reader may complete this demonstration by convincing her/himself that if we started with two rectangles of equal area $DE = CF$ and placed them vertex to vertex, completing the picture again would give two other rectangles C/D and E/F which would have the same shape. It requires an argument that the diagonals of the resulting rectangles C/D and E/F are actually collinear

Since finding a fourth proportional can now be expressed equivalently in terms of an area problem, we can reword the problem as follows. Given a rectangle with a given area DE, and any other segment C, find a rectangle with side C whose area equals DE (i.e. find the other side F of the required rectangle). This can be done using the four rectangle diagrams above, starting with rectangle DE, extending E by segment C, and then drawing the diagonal the other way to complete the picture. If the students have a background in geometry they can show how to actually cut up any given rectangle and put the pieces together to make another rectangle with the same area and with a given segment as its width (A good exercise is to figure out how to do it if the given segment is shorter than either side of the given rectangle.)



Slide PQR up to STU

Figure 9

Pursuing questions in mathematics

After looking at these examples of geometric numbers we find that there is something special about products and ratios, and in the relation of numbers to segments and areas which we normally overlook. What kind of a number *is* a ratio, or an area? How is the number representing the relation of two sides of a figure related to the kind of number representing an area, or a number representing length, or a mark on a number line, for example? Does 7/4 as a ratio of sides have the same meaning as 1.75? Does 3×7 meaning an area have the same significance as the value 21 on a number line?

If we imagine a whole sequence of rectangles — getting larger or smaller in size — with dimensions 7/4, 14/8 and so on, these have different sizes, yet all have a "shape" for which the number 1.75 has some significance. This number is a "pure number", having no units, whereas the sides of the rectangles can be assigned numbers which depend on units.

In terms of relating area and multiplying we have to consider units carefully. Given a rectangle DE, although the area of the rectangle is well defined in size, the product of the segments D and E can come out to any numerical value we like depending on the units we choose. In fact, using the rectangle picture, we can pick our third segment C to be our unit of measure, i.e. to have measure 1. If we then construct the fourth segment F, we get $DE = 1F$, or $F = DE$. Varying the unit of measure makes the segment whose numerical value is equal to the numerical value of the area of rectangle DE vary over the whole range of lengths.

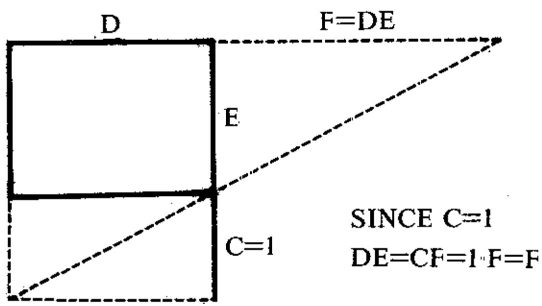


Figure 10

Next consider numbers 3 and 4, and a rectangle 3×4. When we ask for the side of a square with the same area as the rectangle 3×4, we can translate this into solving the proportion $4/A = A/3$.

Taking a rectangle with sides 3″ and 4″, and area 12, we can find a square with the same area. Finding the square root of this area means constructing a side of a square with the same area (12), for which there are several geometric methods. This square figure *will* remain invariable no matter which units we use. That is, the side of the constructed square gives us a well defined "square root" which is independent of the units used

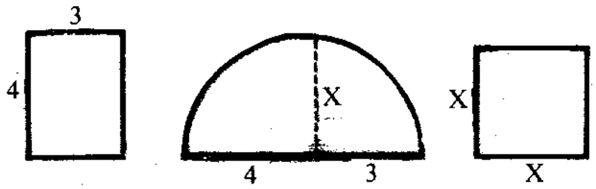


Figure 11

Now represent the numbers as the lengths of segments, and assume we have a segment 12 inches long. The square root of 12 is about 3.4, and a segment of length 3.4 which represents the square root is *less than* the original. But if we change units to 1 foot as the length, then the square root of 1 is 1. The segment which represents the square root is now *the same* as the original.

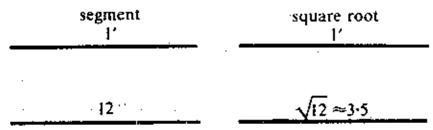


Figure 12

Thus, although we are used to taking numerical square roots, the original significance of square root is to be found in the geometric interpretation. In this example it doesn't make sense to ask which of the two segments we have obtained represents *the* "square root" of the original segment. Taking square roots is not well defined for lengths.

Analysis of meaning

One of the things we can see in the preceding discussion is the possibility of enriching the meaning of a mathematical event. In general, even the most familiar-seeming of concepts may have depths of meaning available to students in grade schools. For example, relating the idea of proportion to its geometric origins expands the context of our inquiry, and brings out certain problems in an operation we normally take for granted.

We can outline three questions of meaning in mathematical events which are important to consider for our curriculum.

1. How can we characterize and make explicit different levels of meaning in mathematical events?
2. How do we focus clearly on the significance of certain "unifying themes" — key ideas which run through many topics and help to unify our understanding of the material — and how do we make these ideas stand out from the morass of material we are conveying?
3. How do the ideas in a mathematical event relate to other areas of thought, and other areas within mathematics?

These notions are closely interrelated, and need further elaboration.

1. Levels of meaning in mathematics

In order to see more clearly the components of a mathematical problem we can identify three levels of meaning. (See Table 1 for a summary.)

A. The level of basic definitions, underlying questions of context and world views

(i) When we ask a question like "What is the area of a circle?" it is a question of calculation and fact; but to ask "What is a circle?" or "What is area?" involves us in a question of meaning. Not every problem has to be questioned in this way, but several should. At least once in a while we can ask, "What are we studying in this course?"

(ii) Underlying questions of context. The meanings of terms and concepts is tied in to the context in which they are used. Whether a line is straight or not may depend upon our usage: are we checking a cue stick, or the side of an ideal polygon? Are we in euclidean geometry or on a sphere? Here we may include thinking about what methods of proof we are using. Is it sufficient to "see" something, or must we write out all the reasons? Also, we may want to consider the elegance or harmony of a result or proof. Perhaps the mathematical content can serve as an illustration or analogy for some other scientific or philosophical principle. Finding an illustration of the principle of complementarity in the dual nature of a line is an example.

(iii) Bringing out implicit world views. What world is mathematics dealing with? In what way are the objects of study in the imagination, and in what way are they sensible or concrete? Do we discover the principles of mathematics, or do we construct them? What is the relation of the person doing the mathematics to the mathematical content? I started my geometry class one semester by asking about the status of mathematical objects such as the circle that we were going to study — what were we doing when studying mathematics? This kind of question set a tone for the course: we occasionally brought into the foreground the larger background of our views of mathematics when we were in the midst of more detailed investigations. In general, this first level of meaning involves us in an investigation of Ideas.

B. The abstract or theoretical level.

The questions here are usually single problems to find, with single answers, and a single formula, trick or method which is used to solve it. "Find A" in the proportion example can be done by "cross multiplying" technique. The examples at this level are the usual ones that are found in long lists of exercises. Even most of the so called practical or concrete examples are really abstractions, such as finding the side of a picture which is reduced in length by 2/3 and so on. This level includes most of the "proofs" we do, for example, the one given above for cross multiplying.

C. The sensible or concrete level

Finding a way to cut up a rectangle and get another with equal area and a given side is a good concrete example. Events which involve the student in actually measuring things (such as the distance across a lake by using proportion) and doing calculations on a calculator can be worthwhile additions. Historical considerations are often very valuable in setting a context for how certain mathematical concepts developed the way they did, how they fit in with other ideas, and why certain ideas are or were useful. Also writing or talking occasionally about how the students *feel* doing the mathematics, and what procedures they actually need to engage in to get different results is valuable. Here we have touched on three essential components of a person — thinking, feeling, and acting or doing — which educating in general, and especially mathematics educating, which is often viewed as the most thinking-oriented, will need to address.

Table 1
A "vertical" or layered view of a mathematical event

A. *Idea level* i. Philosophical questions — Basic definitions — Context of meaning — World views ii. Relation to person — How do I *understand* mathematics? — Aha! experience — seeing the connection — Elegance or harmony of proof iii. Examples: in thought and imagination
B. *Theoretical or abstract level* i. Theoretical questions — Proving a theorem — Finding an answer to equation ii. Expanded relations — Relations to centralizing or integrative ideas, such as proportion, which runs through many topics — Relations to other areas of thought — such as connections between geometry and algebra — Finding other related results, and generalizing conclusions iii. Examples — Abstract examples — such as solving many equations
C. *Concrete level* i. Practical questions — Find facts ii. Relation to person — Useful methods — How do I feel about the problem? iii. Concrete examples — Experience with sensible objects — Historical considerations — as fact, as record, as experience, as a context

Proportion as a unifying idea, and its interrelations with other areas of geometry

This simple idea with many ramifications: that the arithmetic proportion between four numbers can be expressed geometrically as an arrangement of four segments to give either equal areas or equal shapes, is a good example of a unifying idea, providing a continuity between many events at different levels of complexity in most of the topics in geometry. To make clearer to ourselves as educators the implications of a unifying idea such as proportion, and the interrelations of such an idea to other ideas we can make good use of diagrams.

We can, for example, make concept maps for ourselves (and possibly with the students) to show the interrelation of the numerous concepts needed to understand our idea of proportion. A complex diagram, such as the one on the following page (Figure 13) is needed to show even a partial map of the concepts related to finding the fourth proportional.

In order to see more about the third aspect of meaning mentioned above — the relation of the idea of proportion to other areas of geometry — we can start by listing occurrences of the idea of proportion in geometry, including the following:

a) Basic equivalence of proportion to cross-product
b) Expression of proportion as equal shape
c) Expression of cross-product as equal area
d) Idea of equal shape found in parallels cutting proportional segments and, directly, in definition of similar triangles
e) Equal areas used in the fundamental theorem of proportion with a line parallel to the base of a triangle
f) The parallel in a triangle giving the angle-angle similarity theorem
g) The Pythagorean triangle and theorem related to the idea of proportion as equal area, as shown above
h) Similarity showing up in coordinate geometry — in the fundamental properties of straight lines in a coordinate system: slope
i) In the circle, the basic theorems about intersecting chord lengths based on similar triangles

I have listed these instances in a ranked ordering so that we may now enhance the listing by drawing out the interconnections using a concept map type of diagram. Figure 14 shows clearly how much of the whole geometry course is organized under the one, unifying, idea of proportion. Such a map enables the teacher to get a feeling for a larger picture guiding the parts of a course, and helps to give a sense that we are dealing with a continuum of ideas and not with bits and pieces.

The topography of geometry

Putting together the various complex relationships of meaning which we find in a unifying idea, such as proportion is seen to be, requires the aid of several diagrams. These pictures are mainly useful for the teacher, providing a kind of topographical survey of the field of meanings. Having such pictures in mind we can guide the flow of concepts, and we can look out for the key ideas as they arise in discussion.

Finally, we can try to see a larger picture of a course as a series of interconnected events, some of which touch on different levels of meaning. It is not essential that every event touch on every level of meaning, but a few in every topic should touch on various levels. A look at the topography of geometry from the point of view of a unifying idea connecting many topics would reveal how some events reach into different levels of mathematical space. A topography map looks different from a typology map. In showing the contours of the mathematical ideas the topography does not just classify concepts as types, it helps to outline the relations of ideas. An example of such a map might look something like Figure 15.

Implications for the educator

Perhaps the most critical point in this discussion is that once we have expanded our own view of mathematics to recognize levels of meaning, unifying ideas, and interrelationships, we are in a better position to foster a broader view of mathematical learning in the classroom. If we start with a larger context we can follow the threads of discussion through a wider area and yet not lose sight of the central ideas. We enter the classroom more sensitive to the quetions of meaning which students ask, questions which are often an opportunity for a real sharing of understanding and inquiry. We need to develop an attitude that what is important in the classroom is the moment when a student grasps a meaning with an exclamation of "Aha! I didn't see that before", or really sees a problem. We need to become attentive to these moments of seeing, to recognize good questions, and to point out these moments to the students so that they can begin to see what is really essential in mathematical thinking. At times, following out these ideas or moments of "seeing" may have to displace our fixed plans for a lesson. It requires a certain degree of courage, and a familiarity and comfortableness with the material on the part of the educator, to abandon the familiar habits. But if we keep in mind what it is we are really trying to awaken in students, we will find efforts made in the pursuit of meanings well justified.

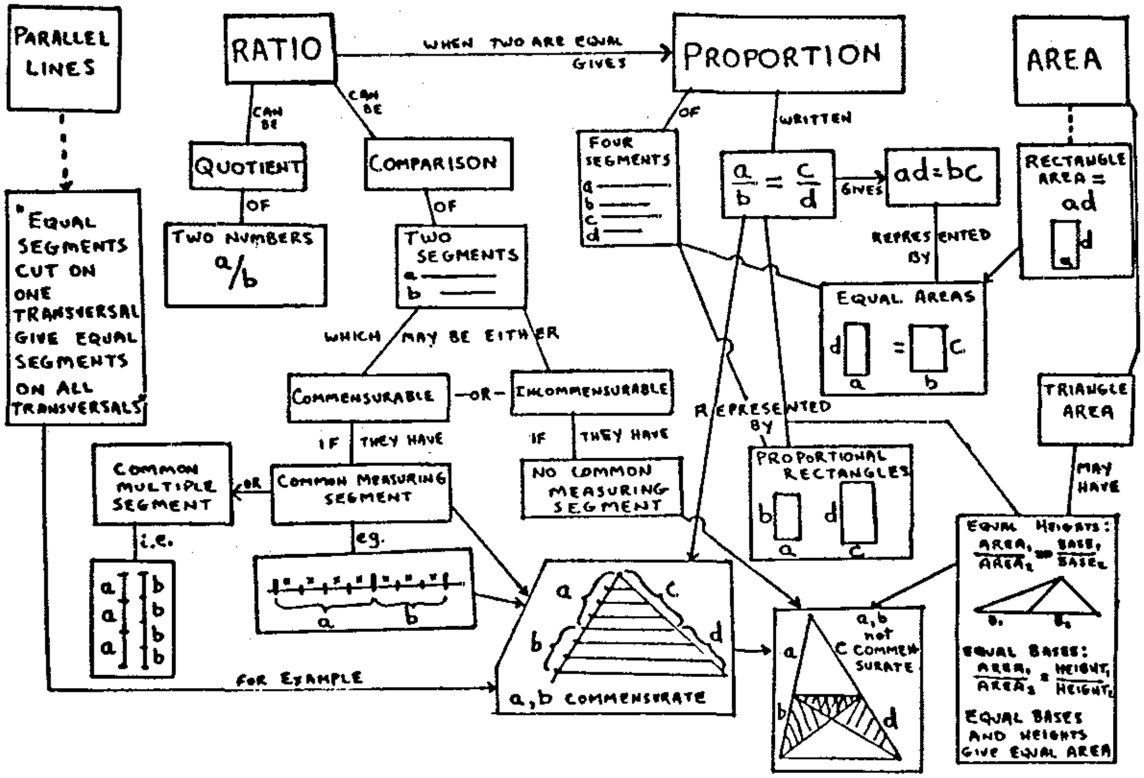


Figure 13
Concept map for proportion.
Even this complex figure is only partial.

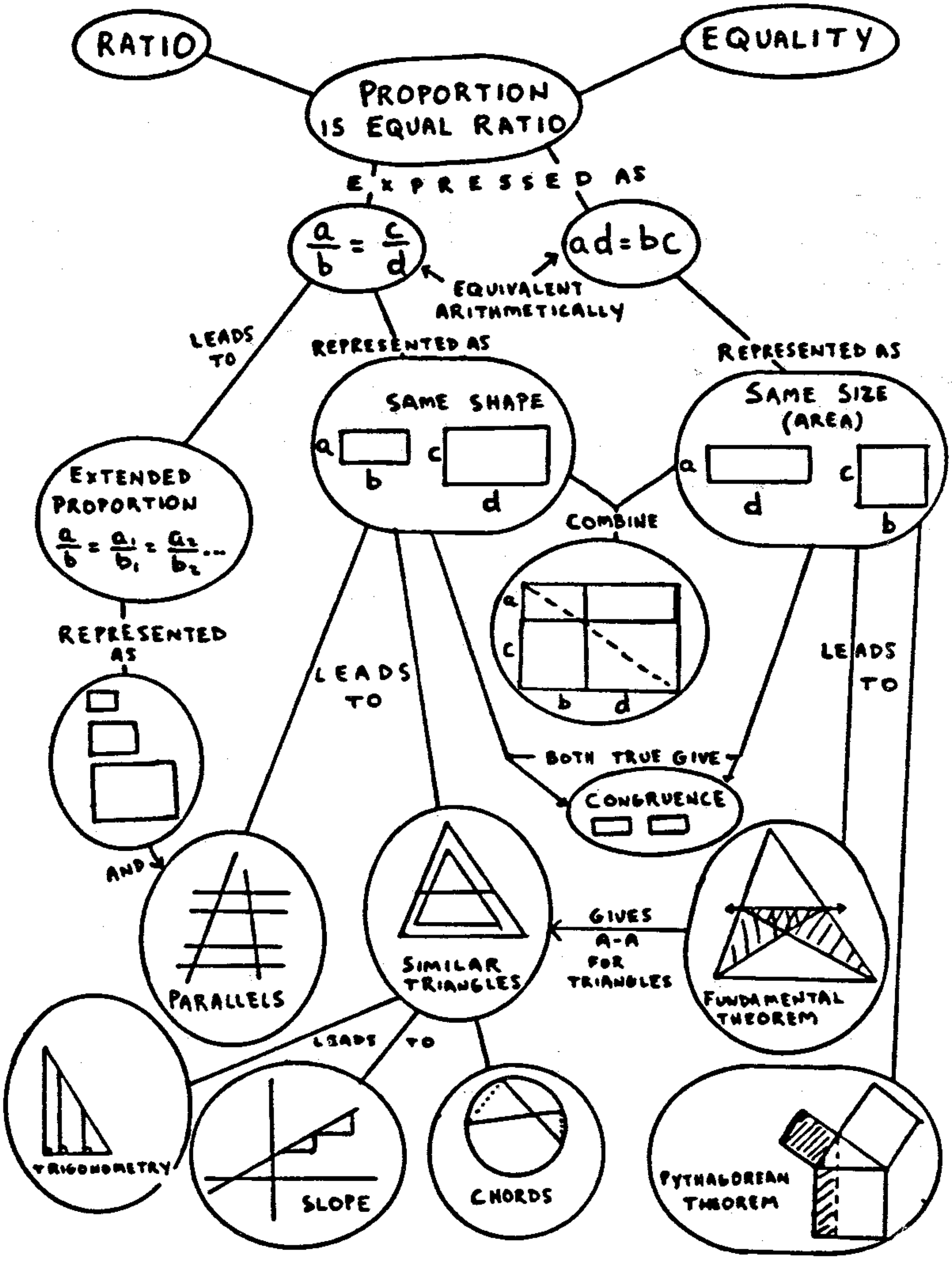


Figure 14
Topography of "proportion" as a unifying idea

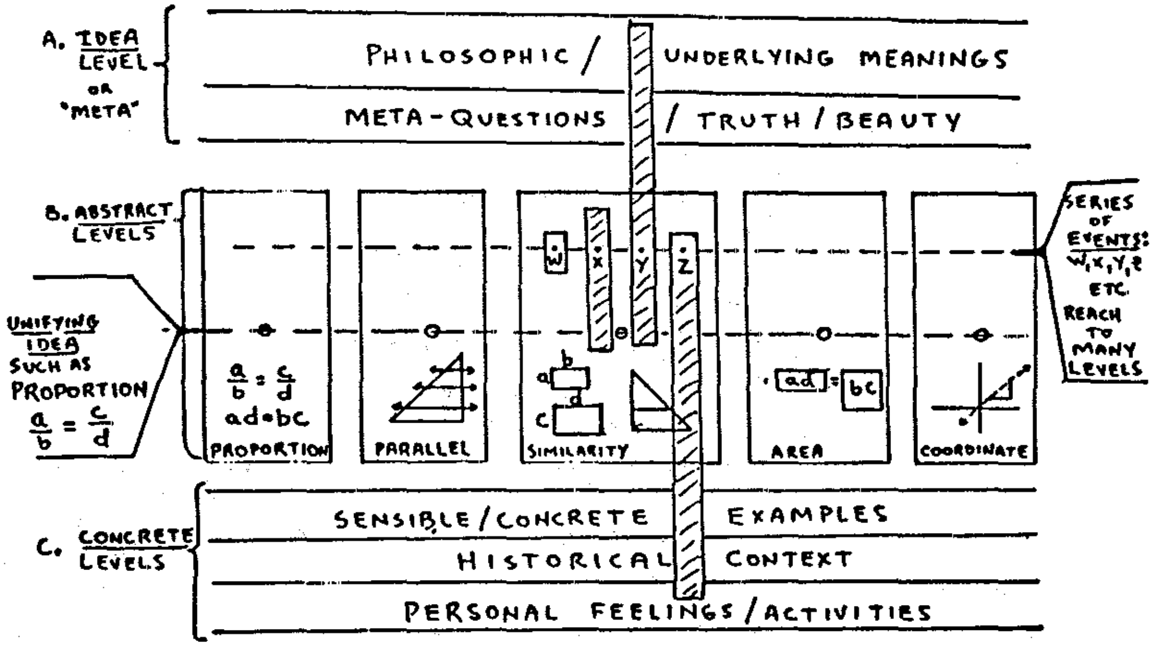


Figure 15
Topography of "proportion" related to many areas of geometry and levels of events

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Sometimes we want a theory when there is none. It is often so with didactics. The teacher has bad results. He is willing to acknowledge that this is his own fault; his teaching was bad but he does not know how to correct it. And there is not much literature that can help him. Of course there are magazines in which teachers tell of their own experiences; such magazines are of great importance, for they tell us something of the practice of teaching. But most of these experiences are very individual; generally the distressed teacher cannot repeat them, and if he could he would have to change them according to his own case. There is an extensive literature of psychology and pedagogy. So it is no wonder that the teacher tries to find help in these sciences. A much better remedy is talking with colleagues. If they are sincere, they will admit that they also have difficulties. Exchanging ideas about the subject will help them all. When I was a teacher we had a weekly meeting at which we exchanged our bad experiences. We called it "weeping ourselves out".

Pierre M. van Hiele
