USING THE NUMBER LINE TO INVESTIGATE THE SOLVING OF LINEAR EQUATIONS

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The curriculum (DfEE, 2001) for eleven-year old students in the UK, currently adopted by most schools, includes solving linear equations with the unknown on one side only before moving onto those with the unknown on both sides in later years. The suggestion in the *Framework for Teaching Mathematics* is that the former kind are solved:

> by using inverse operations (p. 122)

and the latter by:

> beginning to understand that an equation can be thought of as a balance where, provided the same operation is performed on both sides, the resulting equation remains true. (p. 125)

Implicit in these recommendations is the belief that there is a significant shift in the level of complexity when moving from one kind of equation to the other, and hence a need for a different set of solution strategies. The following classroom exchange (see Figure 1), recently observed by the authors, would appear to substantiate this.

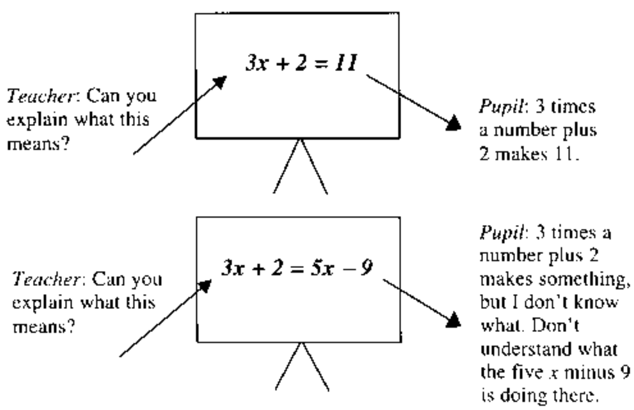

Figure 1: Classroom exchange discussing linear equations.

This shift in complexity is well documented and is sometimes referred to as the "didactic cut" (Filloy and Rojano, 1985, 1989; Herscovics and Linchevski, 1991, 1996). The basis for this "cut" would seem to be the interpretation of the equals sign as a "do something signal" (Behr *et al.*, 1976; Kieran, 1981) rather than as suggesting the "quantitative sameness" (Boulton-Lewis *et al.*, 1997; Saenz-Ludlow and Waldgrave, 1998) of the two sides. That is, the expression on the left-hand side of the equation is seen as a process, and consequently the right-hand side must show the (arithmetic) result of this process. So, in the above example, $3x + 2 = 11$ can be solved by inverting or 'undoing' the given operations and there is no requirement to work directly on or with the unknown. To solve $3x + 2 = 5x - 9$, however, 'undoing' is not enough and it is now required to operate directly on the unknown quantity. Sfard (1991) describes this distinction as one between conceiving of the algebraic symbols operationally (as processes) or structurally (as objects) and suggests "*a deep ontological gap*" (p. 4, emphasis in original) between the two.

While many teachers grapple with bridging this gap and attaching meaning to such equations and their solution strategies, for example, through the use of the balance metaphor (see Vlassis, 2002 for a recent evaluation of this), evidence suggests that for many students solving equations remains a matter of learning rules and performing blind manipulations. At best, they develop "cover stories" (Pimm, 1995, p. 89), such as "take it over the other side and change its sign", to deal with this.

Similarly, school textbooks struggle with the balance between developing algebraic understanding and training algebraic skills (Wijers, 2001).

In this article we describe an attempt to encourage students to exploit an already familiar image, the number line, in order to address the difficulties described above. What should also be stressed, however, is that we do not consider what follows to be a new *method* for solving equations, nor a new form of representation. It is simply the exploitation of an image with which the students are already comfortable, to support their developing understandings of linear equations through access to the solution strategies available. This is a crucial distinction and will be referred to a number of times during the course of the article.

Beginnings

A couple of years ago we attended a seminar on the 'model' approach used in Singaporean schools to help primary students solve 'higher-order' algebraic problems (Fong and Chong, 1995). Figure 2 shows an example of this.

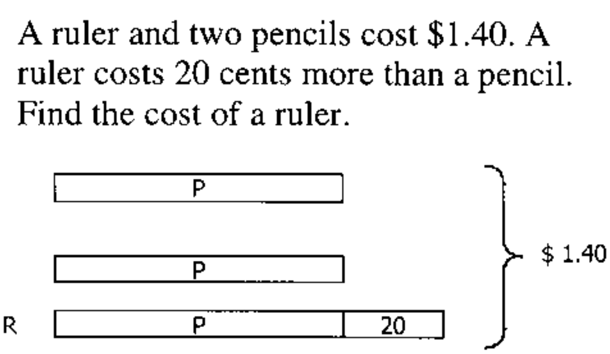

Figure 2: The use of the Singaporean model approach to solve a problem (Fong and Chong, 1995, p. 34).

As we attempted to solve more complex problems we met with the difficulty of needing to 'drop' our existing algebraic knowledge if we were to explore this new approach fully. At this stage, some people in the seminar group began to feel more comfortable using an empty number line instead of a series of blocks (see Figure 3 for the new representation).

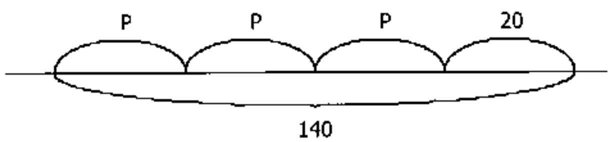

Figure 3: Using a number line to solve the problem.

From here, it seemed a relatively small step to begin representing equations such as:

$3x + 12 = 5x + 6$

on a double number line (see Figure 4).

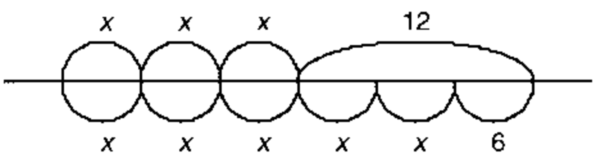

Figure 4: Using the number line to solve equations with the unknown on both sides.

As this happened, we were struck by the immediacy (for us) of seeing $2x + 6 = 12$, and decided to present this representation to a group of teachers as a topic for research. The research group consisted of about ten mathematics teachers drawn from a number of schools local to the university where we work as teacher educators, who were already meeting on a regular basis to discuss their classroom practice. These meetings include reflection on relevant academic writing, as well as discussion on ways of working within the classroom. This article describes the initial work of the group in relation to this topic.

Each teacher was asked to introduce these ideas in a similar way and to keep notes on the lessons and retain copies of the students' work. Some lessons were also videotaped as a further means of gathering data. The results reported here are the amalgamated outcomes of the experiences in all the classrooms.

Early lessons

Familiarity with the number line seemed to be important for students. So, when the idea was initially taken into classrooms, teachers were asked to do some preparatory work (often a series of mental starters) on using the number line to solve addition and subtraction problems. For example, the problem 153 – 68 might be represented as shown in Figures 5 or 6.

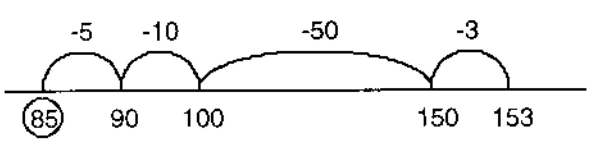

Figure 5: Subtraction jumps to solve 153 – 68.

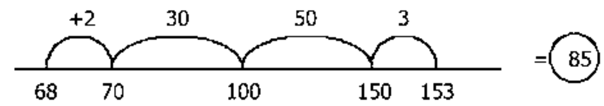

Figure 6: Addition jumps to solve 153 – 68.

Although not initially apparent, it became useful later on for students to be willing to drop the notion of jumps on the line being representative of actual sizes. Early on it was noticeable that jumps of, say, twenty were regularly drawn twice the size of jumps of size ten. However, students eventually seemed happy to draw diagrams such as Figure 7.

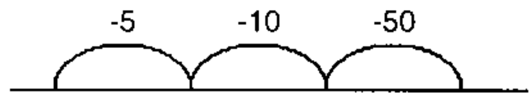

Figure 7: Moving from seeing jumps as a 'model of' something to a 'model for'.

This would seem to be an early indication that pupils were beginning to make the transition from seeing the number line jumps as a 'model of' something to a 'model for' that thing, a transition originally identified by Streefland (1991). Although this shift may not necessarily be permanent, as discussed later in this article, it is crucial if students are to move successfully from informal methods to more formal mathematical knowledge (Gravemeijer 1990).

Following these starter activities, the teachers introduced equations simply by putting $3x + 4 = 19$ on the board and then (slowly) drawing the representation in Figure 8.

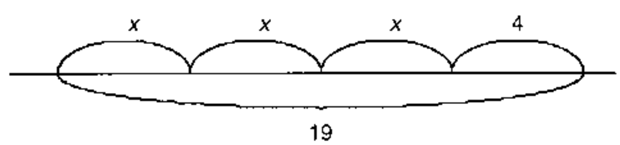

Figure 8: Representation for $3x + 4 = 19$.

Students were initially asked to tell the teacher what they could see. One immediate issue was the number of familiar difficulties that this question helped to highlight. In particular, a number of pupils either stated that all the *x*s were the same, or asked if this had to be the case. Other responses included "3 times a number plus four is equal to nineteen", "3 *x*s must be fifteen", and "*x* is five". The latter was a very common response, which, although obviously useful, actually worked against us at times, as will become clearer below.

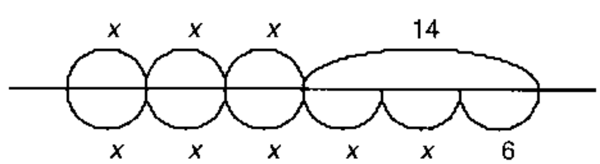

Figure 9: Representation for $3x + 14 = 5x + 6$.

The apparent accessibility of this model allowed classes to move quite quickly into problems involving the unknown on both sides and as early as the second lesson the teachers introduced the problem in Figure 9 to students who were again challenged to say what they could see.

It was noticeable in one classroom that some students wrote $3x + 14 =$ and $5x + 6 =$, a response similar in many ways to the student comment at the beginning of this article:

> 3 times a number plus 2 makes something, but I don't know what. Don't understand what the five *x* minus 9 is doing there.

A response like this could also indicate that the student is perhaps suggesting a search for two different values of x to satisfy the equation (see Sfard and Linchevski, 1994). Many other students, however, clearly did recognise the equality inherent in the drawing, sometimes referring to "equal length". What was also noticeable was the number of students who identified $2x + 6 = 14$ as being something they could see, along with many others who went straight to "x is 4". This latter statement was checked by using the jumps along the line, which again seemed accessible to the vast majority of students. It was also noticeable even at this stage that a number of students, when asked for amplification of their ideas, were beginning to cover up or cross out *x*s so as to simplify the problem. It was quite common to see representations such as that in Figure 10 in students' exercise books.

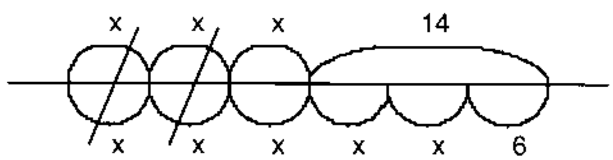

Figure 10: Crossing out xs to simplify the problem.

Statements such as $2x + 6 = 14$ and, when pushed by teachers, $x + 14 = 3x + 6$ were particularly interesting as these were seen as encouraging signs of students starting to see the algebraic terms as *objects* instead of merely *processes* (see, for example, Crowley *et al.* (1994) for a discussion about the importance of this distinction). We will come back to this point in more detail later in the article.

The apparent 'accessibility' for many students also created dangers, however, and when they first taught with the images, many teachers reported feeling that they were in fact moving *too* quickly onto more sophisticated problems. When teaching the topic a second time, teachers tried to ensure that much more discussion focused on the equivalence of the two sides of the line, with students' attention being drawn to this equivalence rather than to the solution of the equations. This proved to be a tension in some classrooms, with students showing reluctance to engage with issues once they knew "the answer" to the equation. This, of course, is nothing new and is even more prevalent when trying to introduce some students to more sophisticated solution strategies (*e.g.* balancing) to equations that for many of them can be solved mentally (with most 'arithmetic equations' coming into this category).

Mathematical development

Seeing eleven-year-old students of around average ability 'solving' equations such as:

$$3x + 17 = 8x + 7,$$

and being able to explain what they were doing, goes some way to confirming the initial accessibility of this model, but little else. We stress again the importance of regarding the model as a means of granting access to solution strategies, and not as a strategy in its own right. At this stage of the development it would be too easy (and unfortunately too common) to either stay with many similar examples simply to practise the 'method', or to drop the model and move onto 'harder equations'. The aim, however, is neither of these. It is to use the ideas that the students have developed, and to encourage them to formalise these ideas.

The belief that students can do this, coupled with the conviction that we are not here teaching a new method, but simply allowing students to explore possible strategies, is crucial. This has significant implications for how teachers proceed from this point. We believe that Treffers's (1987) distinction between horizontal and vertical mathematisation is important here, and in this case we interpret 'horizontal' as being the ability to represent an equation on a number line and solve it, and 'vertical' as the development and formalisation of solution strategies that will ultimately be generalisable. The questions asked of students were now of paramount importance, and teachers who in earlier lessons had managed to divert students' attention away from the actual solution now experienced more success. For example, with reference to:

$$3x + 14 = 5x + 6,$$

a common approach was to ask students for a number of statements that they knew to be true, and to justify these. Some students came up with $x + 1 = 5$ "because we know x is 4", and when this happened a lot, some teachers found it necessary to resort to equations where the solution was a non-integer, in order to be able to proceed.

The next stage was to give students a series of statements, and ask them to justify whether or not they could be deduced from the original equation. These were of the form (using $3x + 14 = 5x + 6$ as the original equation)

$14 + 3x = 5x + 6$

$3x + 20 = 5x + 12$

$x + 14 = 3x + 6$

$5x + 14 = 7x + 6$ and more challengingly,

$6x + 28 = 10x + 12$ and, importantly,

$14 = 8x + 6$,

$6x + 14 = 10x + 6$ *etc.*

Classrooms were soon full of many equivalent expressions, with students being challenged to find 'different' ones and always to justify these. Such 'free productions' revealed a lot about the kinds of strategies currently being adopted by the students (see Streefland, 1990, 1991) for further discussion of the value of asking students to produce such work). While students may at this stage have been mainly engaged with examining the procedural operations associated with solving equations, their justifications invariably hinted at more structural aspects. This, in our experience, is quite rare, and is discussed further in the conclusion to this article.

For example, adding $2x$ to each side was dealt with quite confidently, with students regularly making comments such as "as long as you add the same number of xs to both sides the lengths (on the line) will still be the same". Also, the symmetrical properties of equality appeared to be self evident to many students. In fact it was rare at this stage to find students disputing or having difficulty with the notion of equality.

From such experiences discussion was common about strategies that were 'allowed'. It also became clear that students were now beginning to generate such strategies by referring to the equation rather than to the number line (the algebraic form of each equation was always present along with the model in all lessons). Students beginning to act on the symbols as objects was clearly a major step, though it was still important that teachers regularly checked that students could use the original image to continue to justify their strategies. Indeed, it was central to the entire investigation that the representation should remain available throughout, and that pupils could refer back to it at any time.

Another strategy adopted at this time was to ask questions of the form:

$$48x + 24 = 27x + 87.$$

This form of equation provided valuable information concerning the amount of progress being made by students. Responses varied from those who wanted to draw 48 individual jumps of x, through some who represented 48 as four tens and an eight, to those who (impressively) drew the representation in Figure 11.

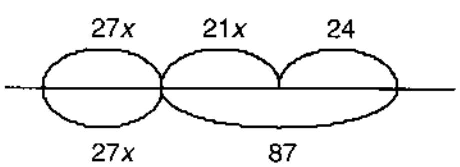

Figure 11: Students' representation of 48x + 24 = 27x + 87.

It was also noticeable, however, that many students asked if they had to draw a number line, claiming that they "could see" that $21x + 24 = 87$. When pushed further, some stated that they "imagined" the number line "in their heads", whilst others were simply operating on the equation.

We believe that the vast majority of the above work is analogous to the idea of 'doing the same to both sides', and some students began to articulate their strategies in these terms, sometimes referring to "two sides of the line", and sometimes to "two sides of the equation". The significance here is that the students are developing their own procedures and explanations for them, rather than just being told by a teacher to use mechanical procedures. They have an underlying image to return to if they need to be reassured or need to re-evaluate a particular strategy. The final outcome may be the same, but the level of understanding is very different. This idea of learning mathematics through proceeding from your own informal mathematical constructions to more formal mathematics has also been very successfully used within the Realistic Mathematics Education (RME) approach developed in The Netherlands (see Treffers, 1991).

Using negative numbers

The issue of how and when to introduce negative numbers is one that is being looked at in future research, but the actual representation of equations including them is interesting. In the time that this model has been used in the classrooms, a number of possible representations have emerged. Four representations for the equation $2x - 3 = 5$ are discussed below.

1. A lot of students initially wanted to use the notation in Figure 12. It was discounted as not following the accepted convention on the number line, although it does allow for some equations to be solved. Having done early number line work for addition and subtraction questions did appear to help students here.

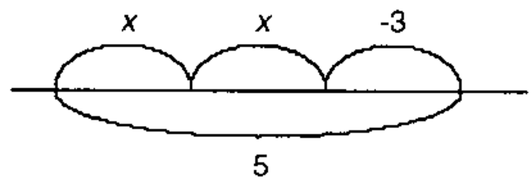

Figure 12: First representation for 2x – 3 = 5.

2. and 3. Figures 13 and 14 were used by many students, and seem to reflect how most schools represent subtraction on a number line.

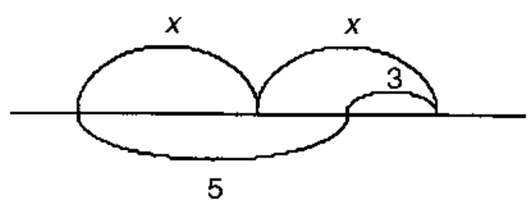

Figure 13: Second representation for 2x – 3 = 5.

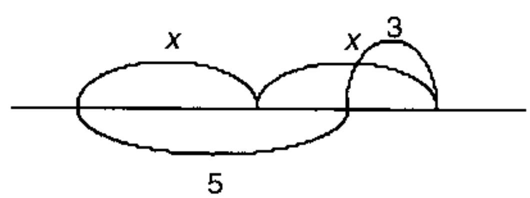

Figure 14: Third representation for 2x – 3 = 5.

They also continue to allow students access to solution strategies. For example,

$$3x + 7 = 5x - 12.$$

is now seen as shown in Figure 15, from which similar questions to before can be posed, or a 'solution' can be found through $2x - 12 = 7$, or by seeing $2x = 19$ directly.
Similarly, $13 - 2x = x + 1$ can be represented as shown in Figure 16.

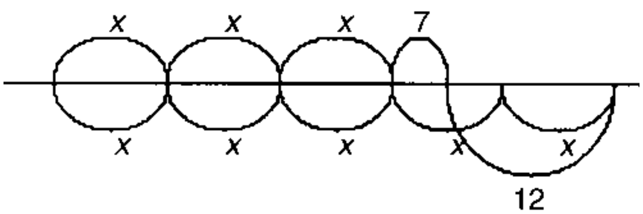

Figure 15: Representation of an equation with an unknown on both sides and a negative number.

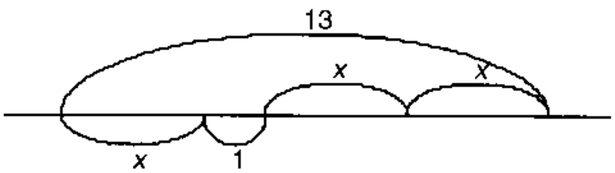

Figure 16: Representation for 13 – 2x = x + 1.

And $2x - 7 = 5x - 12$ as shown in Figure 17.

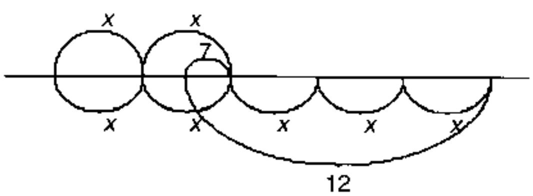

Figure 17: Representation for 2x – 7 = 5x – 12.

One possible obstacle with this representation was the re-emergence of the issue of the size of jumps on the line. For example, in the question above, students were worried about "how far back to go" for – 7, and whether or not a jump of 7 was greater or larger than a jump of *x*. Interestingly, this was a concern even for some more high-achieving students who, in previous questions, had clearly come to view the model simply as a representation 'for' the equation. This apparently retrograde step may be seen as a particular example of a phenomenon detected in a similar situation by Filloy and Rojano (1989) and described as "[t]he temporary loss of previous abilities, coupled with behaviours fixated on the models" (p. 21). This obstacle is clearly not insurmountable, however, and the discussion that it generates can be useful in itself.

4. The first time Figure 18 was seen was when a student drew it as part of a homework. The teacher was initially unsure whether to accept it or not, but was persuaded partially because many other students immediately wanted to adopt it.

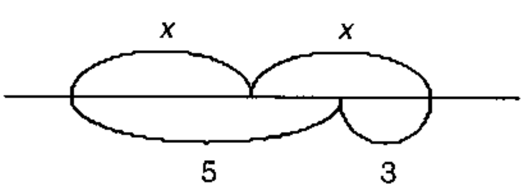

Figure 18: Fourth representation for 2x – 3 = 5.

So, for example, this representation would yield $3x + 7 = 5x - 12$ as shown in Figure 19.

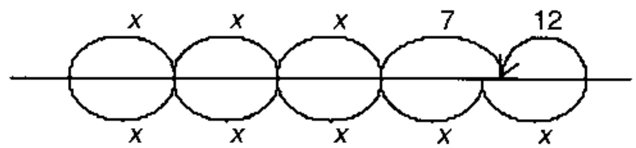

Figure 19: Representation for 3x + 7 = 5x – 12.

And $13 - 2x = x + 1$ would be represented as shown in Figure 20.

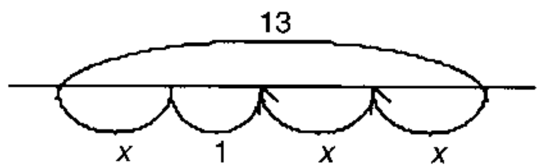

Figure 20: Representation for 13 – 2x = x + 1.

Interestingly, in the school where this first emerged, students now began describing other transformations of the equations that had not been considered before. For example, $13 - 2x = x + 1$ is clearly equivalent to $13 = x + 1 + 2x$ and suddenly the image of 'moving terms over the equals sign' begins to emerge.

As this happened, it felt important to return to previous questions. The representation of $3x + 14 = 5x + 6$ shown in Figure 21 also yielded expressions such as:

$3x + 14 - 6 = 5x$

$3x + 14 - 6 - 2x = 3x$

$5x + 6 - 14 - 3x = 0.$

These explorations led to a lot of discussion about how what was now happening compared to the notion of 'both sides' that had been around in previous lessons.

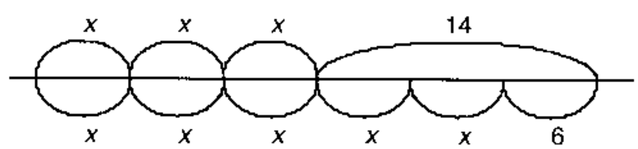

Figure 21: Returning to an earlier representation, 3x +14 = 5x + 6.

Results and limitations of the model

Some of our students have now taken in-school tests and also the national tests required at ages fourteen and sixteen. Initial results are encouraging. Certainly the image of the line seems to be around for quite a lot of students, and we regularly saw over 50% of a class drawing a diagram to help to solve an equation. This does feel different to our experience of working with other images such as the balance, where students would rarely evoke the model once it had been 'dropped' (see Boulton-Lewis *et al.*, 1997 for further evidence of this).

It is also true that the teachers appeared to stay with the number line much longer than might usually be the case, attempting to create examples that encouraged students to move away from the line, but allowing others to remain with it if they so wished. Many of the teachers, when questioned on this issue, cited the accessibility and sustainability of the model as the reason for this change in approach.

The model seems to allow quite natural movement to dealing with different forms of an equation. For example, $1 + 2x = 5$ seemed as accessible as $2x + 1 = 5$ (and $5 = 2x + 1$), $3x + 2 = 5x$ was seen as the same kind of equation as, say, $3x + 2 = 5x + 1$, and when this was written as $2 + 3x = 5x + 1$, students appeared comfortable with having to 'rearrange' the equation for drawing purposes. This contrasts with some other models where a rearrangement, or the move to negative coefficients, causes severe obstacles for learners or even the complete breakdown of the model (Filloy and Rojano, 1989). In this respect, it may be that the number line can offer greater insight into the properties of equality (for example symmetry and transitivity), and hence create valuable opportunities to involve students in discussing such properties. We will return to this point in our conclusion.

We consider the strength of this model to be the fact that it gives access to possible solution strategies and gives students some means of deciding which of these are viable. It may even begin to attach some meaning to what for many students were previously meaningless procedures. It does not model all equations successfully, nor is it intended to do so. For example, when x has a negative solution in an equation such as $3x + 10 = 5$, we have as possibilities the representations shown in Figure 22.

The first of these feels unsatisfactory mathematically; the second is likely to prove a difficult extension of the idea for many students. Some teachers are still working on this, but while it may be of academic interest, it is not the crucial issue. The contention of this article is that, through using the number line model, by the time students need to be solving equations with negative solutions, they will have developed suitable strategies for dealing with all simple equations. If they are still totally reliant on the line, one could argue that they are not ready to move onto such equations.

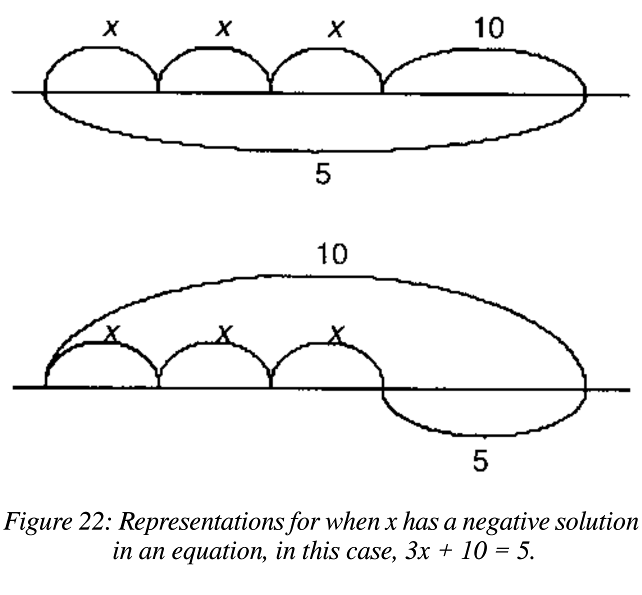

Figure 22: Representations for when x has a negative solution in an equation, in this case, 3x + 10 = 5.

Discussion

What we have described here are some early trials in a small number of schools. We have restricted ourselves to a brief description of the kind of strategies that teachers used, and initial student responses to these strategies. We will be working on more detailed case studies of student learning in the future.

However, students in the initial trials appeared to enjoy working with equations in this way, and certainly made progress. Some more able students could very quickly begin to develop effective strategies for solving a whole range of equations, and were clearly beginning to formulate the more abstract procedures necessary for solving higher-level problems. Perhaps, most striking of all, was the access given to middle-and lower-achieving students who, experience suggests, would usually struggle with such problems.

One feature of our work was the way that teachers could relate to what others were saying because of the similarity of their experiences and outcomes in different classrooms and different schools. This allowed for much productive discussion whenever our teachers met, and undoubtedly helped them to reflect on and refine their teaching strategies. This collective approach to classroom research was a valuable aspect of our work, and adds weight to our contention that this approach to the teaching of equations can be useful in a wide range of classrooms.

The main feature of the model that came out of all our classrooms, however, was its initial accessibility to students, and in this respect the model may serve to limit the impact of the 'didactic cut' referred to earlier. Sfard (1991) stressed the need for a lengthy period of experience before procedural conceptions could be transformed into structural ones. We tentatively suggest that when using the number line, students may be regularly interchanging between the two. For example, when a student draws a number line representation of, say, $3x + 14 = 5x + 6$, it is likely that this represents an operational ('procedural') view. When, however, they describe seeing that $14 = 2x + 6$, or are asked to justify why $7x + 14 = 9x + 6$, then we believe this to be working within the structural domain. We also see comfort with 'different' forms of the same equation as further evidence of students beginning to see algebraic statements as objects as well as processes. For example, the move from $2x + 1 = 5$ to $1 + 2x = 5$ represents an increasing complexity and sometimes a serious difficulty for students. The flexibility to be able to interpret $2x + 1$ as both a process (to be evaluated) and an object (to be manipulated) is crucial for algebraic progress and seemed to be developing in our students.

However, the role of the teacher and the questions they pose are crucial here, and this, in many ways, is the crux of the matter. We do like the number line for both its accessibility and its sustainability, and for the fact that, for many of our students it is an existing, familiar image. In this respect it may serve a similar purpose to contexts and models used in the RME approach which not only provide initial access *into* the mathematics but also support development *through* the mathematics (Gravemeijer, 1990). To think about being in favour or against this approach, however, or to begin to compare its relative merits to other models would be to miss the point. There are simply too many variables to attempt such a comparison and there is no doubt that, in unskilled hands, the number line model remains firmly in the 'cute new ideas' file. What we are really interested in are the strategies that may be employed to help students to become more effective algebraic thinkers, and the role that the number line may play in this. If the number line has a role it is likely to be because the accessibility of this model allows equations with the unknown on both sides to be introduced much earlier than is the case in our current curriculum structure. In turn there will be more time for the development of important concepts and skills without the pressure of frameworks and tests that so often force upon teachers a fixed endpoint (often an algorithm or meaningless procedure). Beginning with eleven-year-old students (and it would seem possible to use the model even before this) gives students of average achievement at least three years to work towards more formal, abstract methods, and to attach meaning to standard solution strategies.

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