cherche mathématique appropriée à son niveau et à ses préoccupations et susceptible de l'aider à faire des mathématiques un domaine vivant.

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Mathematics Education as Training for Freedom*

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Our present way of life is democratic, with the accent of personal freedom and personal growth. Yet it is not consistently so, and everywhere there are pockets of authoritarianism — usually of the meritocratic brand; everywhere. This inconsistency is a source of tension and conflict It is most perceptible in our educational system which is essentially mediaeval, with some reforms, mostly patchwork; except for the reform of regimentation which Napoleon introduced into schools rather systematically. His regimentation was, indeed, an attempt to make schools military The teacher is thus caught in a serious conflict and his position is made untenable. The mathematics teacher seems to me to be the focus of the conflict because of both the meritocratic prestige of mathematics as an intellectual discipline, and the increasing importance it has in modern democratic society, whether in industry or in commerce. It seems to me that it is in the personal interest of the mathematics teacher to attempt to reform the educational system, as well as in the interest of the whole of our civilization All we need to do is alter the incentive system — i e. devise a system which enables a teacher to benefit at once from any improvement he may introduce. Towards this end the present talk is a small contribution, but not a humble one. Any attempt, even if totally unsuccessful, to democratize our school system has to be viewed favorably. The claim that mathematics is not democratic is erroneous, since mathematics is not politics and so neither democratic nor non-democratic, just as it is neither red nor green. Yet the claim is used to justify inept elitist mathematics education The foundation of mathematics is in shambles, and the best approach to mathematics is dialectical — as exhibited by Lakatos. And this makes the best mathematics education democratic in the best sense. But in order to democratize mathematics education we need some institutional reforms, especially those which will defend the innovator against unjust punishment. The obstacles to reform are tremendous, but so are also the rewards, since successful reforms will be emulated by all those who do not wish to fall behind. And we have enough evidence from methodology, developmental psychology, learning theory, cybernetics, and more, to make us expect much from democratic mathematics education. My main proposal is that all mathematics teachers air in public their problems, frustrations, fears and hopes.

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Current affairs in mathematics education

That mathematics education is in a sorry state is an acknowledged fact. Anyone who attempts to reform it is guaranteed public sympathy. The New Math was the best known and most esteemed recent project. Its failure is by now officially admitted. The post mortem analysis is a different matter. For my part, I am of the opinion that the reform program did not have a fighting chance — and so no one should be blamed for its failure — because it only intensified the current ills. Thus for us it was not all wasted since it facilitates the analysis of these ills. What the new programs all centered on was the change in the curriculum, the improvement of the didactics, and nothing else. Since instruction theory centers on these two topics — curriculum and didactics — and also on "motivation", let me say at once that we need to replace "motivation" with the student's existing motives, that this will lead to an explosion of the curriculum into myriads of programs made to measure rather than mass produced, and that didactics will thereby completely disappear and be replaced by a teacher-student book of etiquette designed to safeguard and even enhance the autonomy and freedom of both parties.

Let me begin with meritocracy. When I was a beginner student of mathematics, early in the introductory course I was taking as an obligatory part of my science program, the teacher asked whether the statement is true or false which says, "two plus two equals four or five". "Two plus two equals four or five", said he; "true? false? Those who say true, raise your right arm; down. Those who say false, raise your right arm; down. How lucky that mathematics is not democratic", he triumphed; "the vast majority opted for the wrong option: contrary to what most of you have voted, the statement is true. It is quite true that two plus two equals four or five."

End of anecdote, but not end of story. I was never told why the statement is true. I was never told why it looks to so many novices and laymen to be intuitively, obviously, false. In later years I spent much time researching this matter. I found nothing or next to nothing in the literature. There *is* a literature based on the fact that many logical truths appear to laymen intuitively, obviously, false. The most important of these are the paradoxes of entailment, which arise from the fact that any inference with a contradiction as a premise, for example, is logically valid while looking intuitively crazy. There are other instances, of course, such as that any statement entails any disjunction in which it is a component, if I may allude to the anecdote just narrated. For all these paradoxes, there are two or three attitudes which guide all discussion concerning them. There are the commonsense philosophers who attempt to defy logic with reasonable arguments. There are the logicians who make mincemeat of those efforts, which is really shooting sitting ducks. And there are logicians who in response to those efforts defend logic by logical means, which is as easy as pie. My teacher was right and the paradoxes of logic are logically true.

The situation is very disturbing, I think. The fact is that I was not told why I must give up my intuition, why or where my intuition went wrong. I was tacitly invited to perform an act of conversion and prefer my teacher's say-so to mine — since, we remember, mathematics is not democracy.

If mathematics is not democracy it may be blind authority, or intuition, or proof, or simply utility. Blind authority I shall not discuss. Intuition may be the layman's or the mathematician's, how are we to know? It was Lakatos who observed that math professors often treat their students high-handedly with the excuse that there is no time to explain, that a math professor must spend the little time available stuffing his students with information in the hope that mathematical intuition will follow suit. But what if it does not? Then we have proof instead. But is proof intuitive or counterintuitive? Stalemate.

What does it matter? If a math professor does his job well and his student becomes a star mathematician will this not do? Not on your life! The question is at what cost we raise our future geniuses. It turns out to be at the cost of the education of the layman.

Let me go slowly. There are two educational traditions in our midst, sacred and profane. The sacred education was super-elitist: the highest degree of success of an elementary school was that it sent some of its graduates to high school. The success of high school, likewise, that some of its graduates went to university. Universities, then, replenished the hierarchy — the church hierarchy, the university hierarchy (which was originally the same thing), the lay university hierarchy, Royal societies, Nobel laureates, national cultural heroes; the lot. The elitist system had high values to safeguard and peoples' fates only mattered if they were devoted to the sacred mission of maintaining the highest level of our culture and of transmitting it further. Those who dropped out mattered little and may have received consolation prizes. One consolation prize was recruitment into the profane educational system. To begin with the profane system was essentially profunctory; it was the rich man's smattering of education, the preparation of the rich man's daughter for a life of service to her future husband. Later, under the impact of the needs of modern society and under the great influence of the greatest educator of all times, Pestalozzi, vocational training increasingly became the profane substitute for the sacred. Yet, even then, much of the profane or the vocational education innncluded the lowest rungs of sacred education, plus a smattering of vocational training proper.

Let me take two paradigms — music and mathematics as taught in the nineteenth and early twentieth century. The music teacher only sought out the student who could be a world star. He trained him hard, hours of five-finger exercises for years. If he fell short he would join an orchestra as a second fiddle or play a waltz in a café. In fact, most students simply learned to loathe music. Mathematics as far as arithmetic is concerned fared better, and arithmetic swelled enormously to include, as it still does, complex operations of multiplication and division of large numbers and of fractions — all of which really belongs to the algebra of numbers and which is barely useful to anyone and which may be blamed for the stupid, vulgar, popular identification of arithmetic with mathematics proper. Yet at least arithmetic has some use everywhere. Or at least it did before the advent of the pocket computer. Algebra and Euclidean geometry had no use and their study led to the general hatred of the subject — a justifiable hatred. The only justification of the practice I describe, to repeat, is that it acts like a sieve which syphons the real talent, and even the genius, from elementary school straight to the mathematician's hall of fame.

To conclude, the incentive system of education we have is meritocratic and élitist, based on the idea that knowledge is the cultural value to be preserved, not only by having museums and libraries but also by having superior minds who comprehend and appreciate them. Since most educators are not going to become part of the élite, there is a consolation prize in the practical and everyday value of scholarship.

This will not do in the modern world in which both democracy and literacy are fairly common

Yet the system remains almost unaltered.

Before leaving the incentive system I wish to say one thing about disincentives. There is one general disincentive to innovation, more effective in the classroom than elsewhere, but prevalent everywhere. Innovation produces complaints against individuals An individual may best fend off a complaint by showing that his conduct is normal; for to attack the normal is to attack the norm. Yet even when the norm is unacceptable one can hardly blame the majority who live by it. Where there is no incentive for innovation the choice is between stagnation and martyrdom, both unpalatable. Can there be incentives for innovation in the classroom?

Foundations

I often tread on thin ice; today I will not I am not familiar with statistics about the current views and practices amongst mathematics teachers in high schools, academic or vocational, community and technical colleges, or even universities. Research into job attitudes has barely started and has not reached academics. Apart from a few questionnaires which are still being processed, and which are very, very poor anyway, there is nothing. There are, of course, the high school textbooks, the college texts, the philosophy of education common-or-garden rubbish, the columns of gossip magazines like *Science*. But it is unwise to rely on them too much as they are highly unreliable sources of information.

What one can offer are established opinions, both old and new, interesting problems and solutions, both old and new, and impressions as far as they go

Example. My impression is that until recently, if not also today, most high school geometry textbooks are utterly oblivious of non-Euclideanism, of new attitudes to axioms and to axiomatic systems This may lead to the conclusion that high school instructors are ignorant of non-Euclidean geometry. This is false. Question. Why do they not teach non-Euclidean geometry? What do they say to the high school students on the subject? I do not know. Two decades ago I investigated the question and got a blank. In France, as you may know, Bourbaki has conquered the high school textbook The result, I would surmize, is disastrous.

I do not know how much and how early students are into computers, but to the extent that they are, they can conquer a new world. How do they link computer science with mathematics in general? I do not know, but I do know how the best professors in the best mathematics departments in the whole world link computer science with mathematics in general: very, very stupidly I do not wish to blame them: the situation is far from simple, and our whole traditional philosophy is to blame. Let me only briefly say that when tradition falters diverse schools succeed to cover chunks of it — since it falters no tradition covers it all — and so tradition shifts from one school's view to another. On matters of knowledge in general, since knowledge is allegedly about truth, there are two views, for there are two views of truth: truth according to nature and truth according to convention, so-called. I shall have no time to talk of these except in the context of mathematics, and even then only very very briefly. Truth according to nature in mathematics raises the question, What nature? since numbers and shapes and topological spaces and categories dwell not in real physical space, as Plato and Aristotle argued Plato said shapes exist outside real space: their dwelling is traditionally called the Platonic Heaven. Many leading logicians and mathematicians believe in the Platonic Heaven. The anti-Platonists (also called nominalists) who say numbers and shapes dwell in the mind tend to be intuitionists. Others tend to be of the opinion that all mathematical entities are pieces of fiction and view all mathematical truth as truth by convention.

Oddly, most geometricians and number theorists believe in truth by nature, most computer scientists believe in truth by convention The conventionalists reap the advantage of viewing logic as conventional too So, if you want, you can say that "two plus two equals four or five" is false (did you think I had forgotten it?), but you can also say it is true, and it is more convenient to say it is true Believe it or not, one can show that the conventionalist's view is systematic and consistent, though showing this involves decision theoretical considerations, of course, since he *decides* to view "two plus two equals four or five" as true rather than as false Yet decision theory itself presupposes that "two plus two equals four or five" is true. Is this not suspect? It is.

The Platonists began their assault early in the century by the claim, known as logicism, that mathematics is a part of logic Thus the sentence "two plus two equals four or five" is true by virtue of its meaning "two plus two equals four *or* two plus two equals five", and by virtue of the meaning of "*or*" which is defined by the assignment of the truth-value "true" to a disjunction one of whose components is true. This is far from simple. The idea that language exists in the Platonic Heaven has been questioned. Mathematics turns out to be more than logic. Geometry, according to logicism, is less mathematics than arithmetic, since the axioms of arithmetic are universal but the axioms of geometry are not. This has led to the search for an alternative arithmetic in line with a geometry that is an alternative to Euclid's. There was promise here, because Cartesian algebraic space just happens to be Euclidean But things turned out differently when the alternative, i.e. non-standard arithmetic, was found. What Abraham Robinson succeeded in doing was reading the existing axioms and theorems of standard arithmetic in a non-standard way. This is not parallel to the case of non-Euclidean geometry.

All schools of thought about the foundations of mathematics are blocked and we really do not have a clear idea of mathematics. Certainly we cannot present mathematics as referring exclusively either to truth by nature or truth by convention. Nor is it possible to separate the two quite clearly. I hope you all are familiar, at least vaguely, with both Gödel's incompleteness theorem and Henkin's completeness theorem. They seem inconsistent with each other, but are not: the one — Gödel's — starts from the logicist view of mathematics and shows that theorems can be true but not provable. Henkin is a formalist who says every consistent theory has a model — and for that model the theory is evidently complete, and the choice of both theory and model are by convention.

All this overlooks the fact that mathematics serves diverse purposes; that for one purpose it is better to take one system, for the other the other; and that mathematicians, as well as logicians, may wish to develop certain conceptual systems without going too deeply into the foundations.

This is enormously liberating. Consider logic as a tool for debating. Then logic is described by a system of permissible moves which may be used by discussants in their efforts to criticize or to prove a proposition. Then, surely, some moves will be silly and never used, yet be permissible all the same. Some moves in chess may be permissible but not recommended. Now, weakening a true proposition like "two plus two equals four" (did you think I had finished with it by now?) seems silly, and so erroneous or wrong. And when one is asked if the result of an erroneous or a wrong move is true or false, one might easily confuse erroneous or wrong with false — most people do. So most novices will say, "two plus two equals four or five is false". But later they learn to distinguish a silly move from a false statement and are no longer bothered by a clever move which begins with a false proposition, such as in the method of *reductio ad absurdam*. Now I have not yet heard a single mathematician explain all this — I myself learned it from Sir Karl Popper and Imre Lakatos together — because the mathematician believes in leaving mathematical intuition grow and take care of itself.

Here is the place for an experiment: rather than work a student hard and hope he gets the point, learn to articulate the point, explain it to him, and watch his intellect grow to your delight. But I must speak of teaching of mathematics first in some detail.

Mathematics and democracy

Democratic theory is traditionally liberal. Liberalism traditionally respects both truth and individual liberty. These two clash. The Nobel Laureate economist F. A. von Hayek mentions in his book on liberty the fact that Thomas Jefferson, the intellectual leader of the American Revolution, had no use for academic freedom. He was all for the freedom of the press, but not of the professor: the professor is bound by the truth. What, then, is the freedom of the press regarding truth? Here the liberal tradition sides with liberty, not with truth: it is the individual's freedom to err and even to refuse correction and instruction.

The opposite of democracy is authoritarianism. The best kind of authoritarianism is the best kind of illiberalism, namely paternalism. Father knows best, kids do not; so he has the duty and right to impose his authority. It is the parental burden, teacher's burden, clerical burden, White Man's burden. Except that Father does not know best, the teacher is ignorant, the clergy are corrupt, the White Man is greedy.

All teaching is authoritarian and based on paternalism unless performed on request. Is teaching in a Western-style university liberal or paternalist? Allegedly it is liberal: kids need not enter college. If they enter college they need not study; if they study they need not do well. They pay fees and receive instruction in return. Except that this story is a living lie for which society pays highly, and teachers lose everything when they try to play by the book. For on the one hand, the teacher is required not to give cheap grades or his course will be branded a Mickey Mouse or a Gut course. On the other hand he cannot fail too many or he will lose his students, his school will lose fees, etc. There is, therefore, constant bargaining between students and teachers, which destroys the nerves of both. Since a student loses less if he fails a course than a teacher if he loses his job, the game is nerve-wracking for the teacher.

The cause of all this is that the only reason the professor can offer for a high standard of study is a high grade, and the cause of *this* is that he cannot explain the value of his teaching to his students — not because they are kids but because he is ignorant, as the best of us often are, and inarticulate, as mathematics teachers often are. In my own university many courses are taught, in my philosophy department as well as in the mathematics department, in ways which are questionable, which are questioned, and the questions are either not put to the teachers or are dismissed by them with silly answers. Often there are no wise answers; other times there are wise answers but professors have no time to articulate them and the literature, barring Lakatos and his followers, fails them here. What is to be done?

First and foremost, let me say, the vote which my mathematics professor took in my freshman year on the question, (you remember by now) is "two plus two equals four or five true or false?" is not a matter for democracy. Democracy prefers our mistakes to the wise ruler's wisdom — on the assumption that we may attempt to correct our mistakes but the wise rules may turn out to be a fanatic or a fake or a fool. This is why democracy pertains only to public affairs and not to private convictions. If a democracy ever declare that since most of its citizens are Christians, Christianity is to be declared true and admitted by all its citizens as true, then this democracy becomes a democracy no longer. And the same holds for scientific and mathematical truth. The vote in question was taken and came to conceal the absence of a vote on a different question: should we question the content of the syllabus or accept it as obligatory on the department's say-so?

I am particularly sore at the department I studied in because it made me stubborn and thus reduced my slim chance of becoming a scientist. When they taught me matrix algebra I asked for the reason behind the rule of matrix multiplication and was told, like most students who dare ask the question, it is truth by convention. But I had not given my consent to the convention, and having just battled so many conventions in my adolescence I was not going to swallow a new one. I thought I understood the reason when I studied in physics how to rotate coordinate systems. It looked to me clear that such things may serve a purpose. Only after I met Lakatos could I articulate this better and show the usefulness of transformation both in mathematical theory and in practical applications. But by then I was well behind. I am still trying to catch up, but meanwhile I lost my slim chance. Of course I should not complain, my teachers were not always the best; but at times they were of the best and never below average. We simply have to raise the standard.

How?

First and foremost, a teacher who wishes to raise standards should discuss matters with his peers, particularly in conferences. This enhances both his prestige and the clarity of his ideas. Between planning and experimenting much forethought and design and small-scale testing is needed. There is much that can be critically discussed in a manner both profitable and open to a real contribution by any mathematics teacher, from elementary school to graduate level. The aim of mathematics education for one: what is required of a teacher to transmit to what student — in both direction and level, in both content and methods. Remarks like "Geometry improves the mind" which I unfailingly heard from every geometry teacher I talked with, can and deserve to be examined — theoretically and empirically. The question "What mathematical knowledge develops mathematical intuition?" is another. The fact that didactics is still the bible, despite the discovery by psychopathology that perfect didactics creates disturbed minds, by learning theory and by developmental theory and by psycholinguistics that trial and error and active learning surpass any didactic teaching and any passivity, despite cybernetics and ethno-methodology and all the new exciting knowledge about thinking, discovery, memory. This ought to be studied. Most teachers have partial ideas about the nature of their subject, usually ideas inculcated in them and absorbed incidentally and uncritically, not leading to comprehensiveness at all — quite apart from the fact that in most fields of study current comprehensive pictures are under attack and debated by contending parties.

But, and most important, thanks to Popper and to Lakatos we know now that bringing up in the classroom such questions as I have alluded to just now is both exciting and instructive. That by discussing with students of all levels questions of principle — what is mathematics? why should we study it? how? what part of it? etc. — by discussing these we raise the knowledge of the teacher and the student together. What Lakatos has shown in his trail-blazing *Proofs and Refutations* is (1) that everyone is clever enough to do a modest job in mathematics if teachers are patient and tolerant to a reasonable extent and encourage any suggestion, however foolish; (2) that in any group with a minimal grasp of group-dynamics, someone could attempt, however lamely, to criticize some of what is said, perhaps with a little directional aid from the teacher; and (3) most revolutionarily, that by doing so, especially doing so at leisure, is the fastest way to learn mathematics and to develop mathematical intuition.

This is not to say that I recommend that any teacher should attempt this all at once. A teacher's duty is also to secure his job. A teacher with tenure and security may try. A teacher whose department is democratic, or experimental, or simply ambitious, may bring a proposal to the department for approval and then try — perhaps in experimental groups. It is no surprise that most experimental schools have been of juvenile delinquents, of orphans, or of disturbed children — whether of Makarenko in the Soviet Union, of Janusz Korczak in Poland, of Monsignor O'Flanagan in the United States, of A. S. Niell in England. The main reason is that society cared less about neglected children than children from good homes. The reason that they were successful is that children who had a lot of hard experiences are not motivated by teachers: they have enough motives as things are. They crave social mobility and can see that education may be the only lever to that end.

I conclude by observing that the obstacles to educational reform are tremendous and that the power of educational reform is enormous. This is unhealthy, it is keeping an enormous pressure behind an enormous dam which may burst at any time. We — you, really — may prepare for the change by organizing and by study and by pooling experiences, creating publications, study groups, etc. It is in your interest, since democracy is something we are all trained for and it comes to us with ease and pleasure whereas the authority of school, the strains and stresses of school, the conflicts and deceptions and unnecessary stagnation, are painful and all whose like work is in or around school suffer, including teachers and students.

I conclude. There is no need to bully students to learn. It harms them as it blocks them. It harms them as it makes study associated with pressure so that only a few have fun studying, which is a real shame. Hence so many stop learning when they leave school — elementary or graduate — unless they are researchers. This is dangerous to our cultural tradition. Traditionally schools came to transmit and preserve the best in our tradition. But the love of learning is a powerful part of that tradition, and is lost, especially in the average classroom, especially in mathematics. With the advent of the new technologies, of science fiction quality, the advent of computer science, the fear of revolutions that mathematics anticipated from the days of the Bolyais to the days of Robinson and of Lakatos, makes it possible for the most ordinary mathematics teacher to be a pioneer on a new educational frontier and hope to see a major improvement in his culture in a short time and a major improvement in his or her own work and that of the students at once. My main message to you is, try not to hide your problems, difficulties, frustrations, confusions — either mathematical or educational. Air them with both colleagues and students. Just try it for a little while and see for yourselves what will emerge.