IMPACT OF LANGUAGE ON CONCEPTUALIZATION OF THE VECTOR

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In teaching Euclidian geometry to the *collège* (students from 10 to 15 years old), drawings are generally used to support reasoning. This geometric setting serves to introduce students to vector geometry, where a vector $\vec{AB}$ is defined between two points A and B. In this context, the graphical representation of vectors is a central element for the solution of the first problems studied by students. Progressively, these graphical representations are no longer necessary for the solution of (nor required in the working of the students for) the majority of problems set, especially for proofs of vector equalities.

The object, vector, appears in the setting of Euclidian geometry because of its representation by a drawing: an arrow joining two points. This representation reduces the idea of vector to this arrow. Its frequent use stabilizes as a conception for the students – especially as the majority of the exercises that exist in the textbooks are based on 'vector as arrow'. Besides, the semiotic representation system (Duval, 1993) adopted in the texts of the textbooks reinforces this reduction.

To justify what is in the meantime a hypothesis, it seems to be adequate to look for how the vector object is presented and what conception is constructed from its presentation in the textbooks. According to Vergnaud (1995, p. 174), conceptualization is linked to transmission processes and appropriation of knowledge. This conceptualization underlies expertise and expertise is linked to conceptions. The relation between conceptions and expertise is created by the concept of "scheme". I am interested in the conceptions of the vector object and how its conceptualization is constructed over time through the mediation of the activities proposed to the students.

Vergnaud (1995, p. 159) argues that natural language has a double function, communication and representation, alongside one of supporting thought. In fact, all information passes through the intermediary of language, in the broad sense of the verbal language, symbolic or representational, as specific representations of information. According to Duval (1995, p. 1), the mental representations are expressed by the means of signs (natural language; formulas; graphs; geometric figures). Students, however, do not recognize the same object through representations in various semiotic systems. In this analysis, I am going to distinguish between different registers presented in textbooks to clarify the emergent conceptions of students.

Types of semiotic representations and the conceptions constructed

I will distinguish between two types of problem where the system of representation permits what I call either the conception *free vector* or the conception *tied vector*. There is, of course, sometimes interaction between the two conceptions.

Type 1: free vector

Construct the vector $\vec{v}$ such that $\vec{v} = \vec{u} + \vec{w}$ where $\vec{u}$ and $\vec{w}$ are vectors represented by two separate arrows.

In this type of exercise, the student is taught to choose a point, name it O for example, and draw a vector equal to $\vec{u}$ with O as origin, naming the endpoint of the arrow A, for example. A vector equal to $\vec{w}$, also with origin O, is then drawn with endpoint named B, say. The vector sum is then constructed according to the parallelogram rule ($\vec{EI} + \vec{IF} = \vec{EF}$, attributed to Chasles in French textbooks). Finally, the student has to declare that the vector sum is the vector $\vec{v}$.

Commentary: The representation of every vector by only one letter helps students to see the arrow they have drawn as an instance of the corresponding vector and, thereafter, this vector does not depend on the point's extremities. On the other hand, to choose a point (O) and to draw a vector equivalent to $\vec{u}$, shows that this vector is not only the arrow that carries its name, it represents all arrows that have the same direction, the same sense and the same length. This type of exercise is related to the conception free vector.

Type 2: tied vector

Construct the vector $\vec{AM}$ such that $\vec{AM} = \vec{AC} + \vec{AD}$. A, C and D are three non-collinear points.

In this type of exercise, the points are in the plane and the vectors $\vec{AC}$ and $\vec{AD}$ are visually associated with the segments $\vec{AC}$ and $\vec{AD}$, respectively. The first point of the first vector is the same as that of the second vector. Here, the procedure is to draw the segments AC and AD, followed by completing the parallelogram ACMD. The segment AM is then drawn in, putting a small arrow on it.

Commentary: The representation of the vectors from determined points in plane geometry, allows the association of every vector to its points' extremities. The next step strengthens this association. So, the conception *tied vector* is revealed, that is, given that a vector is bound to its extremities, then it is unique by its graphical representation.

In Type 2, consider the following variation, *Construct the vector $\vec{AM}$ so that $\vec{AM} = \vec{BC} + \vec{BD}$, A, B, C and D being non-collinear points in the plane*. In the vector relation, the origin of the vector, $\vec{AM}$ is different from the origin of the vectors, $\vec{BC}$ and $\vec{BD}$. Two techniques are possible:

First technique: Draw BC and BD, completing the parallelogram BCND. $\vec{BN}$ will be the sum, in the conception *tied vector*. Then, through A, draw a vector $\vec{AM}$ equal to $\vec{BN}$.

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We can make the hypothesis here that the last action is based on the rule of equality of two vectors and not on the representation of the same vector from different points, because of the equality sign being present in the relation that characterizes the vector $\vec{AM}$. Thereafter, the two stages of the construction of the vector $\vec{AM}$ seem to be the result of the realization of action consistent with the conception *tied vector*. This type of exercise prepares the way for those of type 1 above but does not identify itself with it.

Second technique: From A draw a vector $\vec{AL}$ equal to $\vec{BC}$, and a vector $\vec{AK}$ equal to $\vec{BD}$. Complete the parallelogram ALMK, drawing in the vector $\vec{AM}$. In this procedure, the technique that permits these actions seems to be consistent with the conception *free vector*. Indeed, the equal sign is between a vector ($\vec{AM}$) and the sum of two vectors ($\vec{BC}$ and $\vec{BD}$). Therefore, the correspondence, entity by entity through equality, does not exist.

The realization of the principle of the equality between two vectors is therefore not achieved here. The hypothesis is that the schemes realised are related to the conception of *free vector*. Indeed, to draw $\vec{AM}$ of which A is a determined point, it is sufficient to look for the point M. Given that the origin is A, draw from A the representative of the vector $\vec{BC}$, called, say, $\vec{AL}$ and the representative of the vector $\vec{BD}$, called, say, $\vec{AK}$. Until this point, the procedures appear to us to be consequences of the conception *free vector*. However, at this stage the situation is brought back to a context implementing the conception of *tied vector*. The procedure is recognized as "routine" in the sense of Chevallard (1998).

Another variation for Type 2 is in the case of the following situation, *Construct the vector $\vec{AM}$ such that $\vec{AM} = \vec{BC} + \vec{DE}$. A, B, C, D and E are non-collinear points in the plane geometry and every vector of the vector equality given has a different first point.* The second technique, above, is the one that is adequate, unless a calculation is made using the rule of Chasles in order to get one of the types above.

Semiotic representations in textbooks

Different registers are used in textbooks to represent, to define or to manipulate the mathematical object 'vector', classified as:

- *verbal register* for a text in the literary language;
- *geometric register* when there is a geometric drawing as a carrier of information;
- *algebraic register* when a vector relation exists containing the operators' addition and/or subtraction;
- *analytic register* when coordinates are used to manipulate the vectors.

Impact of the verbal register

1) To define vector equality, the authors of the Lebanese textbooks [2] say: *two vectors are equal if they have the same direction, same sense and same length.* [3] The words "two" and "equal" can lead to two erroneous conceptions about vectors:

a. *Two*: In general, the word "two" indicates two different objects. In the practice of the students in the *collège* and in the domain of geometry, the word "equal" is used like a relation between two different objects, for example, *Two triangles are equal if…*; *Two segments are equal if….* So, the presentation of the equality of the vectors gives the conception of *tied vector*. I think that there would be the conception of *free vector* by saying: *Two ordered pairs are equivalent if they represent the same vector, therefore, if they have same direction, same sense and same length.* [4], showing the importance of the notion of *ordered pairs* [5] in making the connection between the geometric object and the concept object.

b. *Two vectors are* equal *if…* the vectors that meet the conditions are, on the one hand, two different objects, and are, on the other hand, equal objects. However, the word "equal" can mean equal measures. Indeed, in the geometric setting, all cases of equality met previously by the students, expresses the respective equality of measures, such as, two equal triangles; two equal segments; two equal angles. In her research, Lê Thi [6] noticed that the equalities of type $\vec{AB} = \vec{BC} = \vec{CD} = \vec{AD}$, making ABCD a square, frequently appear in students' written work.

The current language influences the conceptualization of the notions in all domains, scientific or not. Here, for vector geometry, "sense" and "direction" do not mean the same thing, although they do in the French language, so the student risks identifying the "sense" with the "direction" whilst manipulatiing the vectors (Lê Thi, 1997).

The same form of language is used to designate two equal vectors in the analytic setting. Indeed, texbooks mention that *two vectors that have the same coordinates are equal vectors* and, also, that *two equal vectors have the same coordinates*. However, the use of the verbal register reveals that these are two vectors and not representatives of the same vector. In fact, when a reference mark is chosen, the *free vector* appears (for the expert) to be its analytic representation in R^2 (a couple of real numbers). The analysis above suggests that, for the student, this representation is not a carrier of the *free vector* notion.

2) In the textbooks, especially in the statements of problems, the language used infers the uniqueness of a direction vector and a normal vector for a line in the plane. In fact, the textbooks use the following formulations:

> Find the equation of the line (D) passing through A(7, 5), with direction vector (2, -1).
>
> Find the equation of the line (P) passing through A(7, 5), with normal vector (2, -1).

In 9th grade (15 years old), students solved several exercises of the type: *Find the equation of the line passing through A(5, 2), with gradient (direction coefficient [7]) 5*. Yet, the gradient (direction coefficient) of a line is unique, whilst the direction vector of a line is not. Thereafter, the language used risks the inference by students of the strong uniqueness of the direction vector and the normal vector of a line.

Impact of the geometric register and geometric habits

3) In the textbooks, there are different ways of presenting the object 'vector'. In the national textbook, the vector is presented like an abstract object, represented by congruent arrows in the plane. In addition, it is written in the textbook that *in every point of the plane there exists only one representative of a vector which is determined by its direction, its sense and its length*. The definition and the corollary make allusion to the conception *free vector* without naming it explicitly, of course. But, this vector is only made concrete by its geometrical representation, naming it either by one letter or by the two letters of its extremities. Thereafter, the geometrical representation of this object 'vector' is a figure, an arrow. Effectively, at this level of teaching, a *free vector* is represented geometrically only by a *tied vector*, its extremities named or no. This is an inescapable constraint.

On the other hand, the implementation of the conception *free vector* proves to be necessary, in general in exercises calling for the construction of a point or a vector from a vector relation. The types of situations evoked above illustrate this and reveal that the names chosen for the vectors can be a *didactic variable* in the sense of Brousseau (1989).

4) In geometry there is a regular practice that the letter written next to an object on a drawing is the name of that object only; an object that is equal [8] to it must have another name and it is another object but with the same measurements, equal measures. This training is a condition of the geometric practice and its regular practice makes a difficulty for the concept *free vector*. Indeed, the name of a vector $\vec{AB}$ or $\vec{u}$ written next to an arrow indicates the name of this vector (arrow). A vector that is equal to this vector has another name and, therefore, following the usual geometric practice, it represents another object. It is right to notice that with the notation $\vec{AB}$, the name of this vector must change when the position of the arrow changes. The notation $\vec{u}$ permits a change in the position of the arrow whilst keeping the same name. In the autonomous practice of the students there can be a renaming every time the position of the arrow changes.

Impact of the analytic register and the analytic treatment

5) The analytic treatment strengthens the association of a vector with the points at its extremities, reinforcing the conception *tied vector*. Indeed, the components of a vector named by two letters are calculated through the intermediary of the coordinates of its extremities. If the vector is named by only one letter, then the situation requires a vector relation in which the existing vectors can be named either by only one letter or by two. If they are named by only one letter, then it is necessary to use their components directly. If they are named by two letters, it is necessary to use the coordinates of the extremities. Thus, if the vector is named by two letters, even the algebraic procedures in the analytic setting implicitly solicit the conception of the *tied vector*. Nevertheless, if the vectors in a vector relation are named by only one letter, the calculation of the components solicits, in general, a practice of calculation, not necessarily an aspect of the vector. So the treatments according to the conception *tied vector* remain valid in this type of analytic problem, and the conception is reinforced.

6) On the other hand, there is a risk that the correct technique when looking for the components of a vector from a vector relation might be at the root of an erroneous technique related to finding the length of a vector in the same situation, that is from a vector equality. In fact, having a vector equality, it is possible to put the two following questions:

a. To calculate the abscissa of a vector given by this vector equality …

b. To calculate the length of a vector given by this vector equality …

The same verb is used and the same basis for the reasoning, the given vector equality. But, *to find the abscissa of a vector, one replaces every vector by its abscissa*. Therefore, to calculate the length of a vector, first of all, construct it from the vector equality given to convert the data to the graphical geometric register and then make the treatment in this same register. But, either by economy of work and writing by analogy, students risk applying the erroneous theorem, *to find the length of a vector, replace every vector by its length* in the vector relation that contains it.

Impact of the algebraic register

7) In the textbooks, the following relations, worked a great deal with numbers, stay valid when vectors are substituted for a and b:

$a + a = 2a$

if $a + b = 0$, then $a = - b$ and a is the opposite of b

if $ka = eb$, then $a = (e/k)b$ when k and e are real numbers

$(a + b)^2 = a^2 + b^2 + 2ab$

$(a - b)^2 = a^2 + b^2 - 2ab$

$(a - b)(a + b) = a^2 - b^2$.

These manipulations of vectors as numbers risk the identification of a vector with its number characteristic, length.

8) Chasles's algebraic rule ($\vec{AB} + \vec{BC} = \vec{AC}$) is a specific rule for vectors. However, this rule is only permitted with the *tied vector*. So, this rule contributes to the reinforcement of the tied aspect of the vector.

The previous discussion leads to the formulation of an hypothesis in the following way:

> The system of semiotic representation of vector notion adopted in Lebanese teaching induced the conception of *tied vector*.

The register and the practices used can lead students to treat a vector as a number identified to its length.

Support for the previous analysis

In order to evaluate this hypothesis, I'll discuss the responses to a test given to 82 10th grade (15-16 year old) [9] students in three different schools.

Question 1: *EFG is a triangle and I the middle of the segment FG. Construct the point A such that* $\vec{IA}$ *= ?* $\vec{EF}$ *– ?* $\vec{EG}$*.*

This exercise is of type 2, first variation, as stated above. In addition, every vector after the equal sign is multiplied by a scalar.

There were only 15 successful solutions. Many students had difficulty in constructing the vectors equal to 1/2$\vec{EF}$ and 3/4$\vec{EG}$ respectively using I, or to construct the vector sum through E. A majority (including many of those that succeeded in the task) tried, by calculation, to create vectors having the same origin I, whilst evidently using Chasles's rule.

Commentary: Having a majority of students who did a calculation to get that all vectors in the vector relation had the same origin, I made the connection with regular practice in geometry (see impact 4). Thus, in such a situation as this question, some students have difficulty moving to the conception of *free vector*. As before, it seems like the strength of the conceptualisation of the *tied vector* is crucial.

Question 2: *ABCD is a square of side 5cm. K is a point such that* $\vec{AK}$ *= 1/5*$\vec{AB}$ *– 4/5*$\vec{BC}$*. What is the length of* $\vec{AM}$*?*

There were 9 successful solutions. 24 other students replaced every vector by its length in the vector equality. Yet, we chose values that gave an impossible answer: a negative result for the length. In fact, 1/5 × 5 – 4/5 × 5 = –3. It could be that 11 students took the absolute value of this result, but 13 others applied this erroneous theorem without thinking about its validity. 18 other pupils made mistakes in algebraic calculation while trying to work with a vector relation where all vectors had the same origin.

Commentary: For the 24 students who replaced every vector by its modulus, their schemes can be found in the explanation given in impact 6 or the algebraic impact 7, or the impact of the sign "–" as showing objects of equal measure according to previous geometric practices in 1b.

On the other hand for the 18 students who attempted to proceed by an algebraic calculation, having all the vectors with the same first point, they had difficulty implementing the conception of the *free vector*.

Question 3: *D is a line given in a given plane P. How many direction vectors of D, of length 5cm, exist? One only, two, more, it does not exist? Justify your choice.*

Question 4: *d is a line with equation 3x – 4y + 1 = 0. Find all direction vectors of d of length 3.*

The two exercises belong to the same type of task but the semiotic representation system chosen in each question does not lead to the same reasoning. In fact, the two questions are about finding all leading vectors of the line with a fixed length, specified in the statement, which are indeed two opposite vectors. But, in the first exercise the representation of the line by its name (D) and of a plane by its name (P) focused attention on the geometric setting; whilst in the second exercise the representation of the line by its equation led students to the analytic setting. The verb used in Question 4 seems not to lead to reasoning in the geometric setting but rather to a calculation since the register used in the statement (the representation of the line by its equation) permits it.

For Question 3, 27 students answered "there are several" and 14 students decided that "there is only one pair of coordinates (–v, u) because these are the coordinates of the direction vector of a line". On the other hand, for Question 4, 39 students gave the vector with coordinates (4, 3). 8 of these students calculated the vectors length and decided "not some". 3 students drew the graph, of which 2 from this said "several". 7 other students mentioned the relation that must exist between the coordinates without obtaining a solution.

Commentary: For the majority of the students who answered "several" for Question 3, there was a small drawing near their working, consisting of a line above and some arrows below. We think these pupils implemented the conception *tied vector* represented by an arrow.

Yet, for the 39 students who gave the vector like in textbooks, as (–v, u), the coordinates of a direction vector when the line equation is in the form, ux + vy + w = 0, they implemented knowledge relative to the notion of the direction vector of a line for its analytic form, the register used in the statement.

19 students answered "several" for Question 3 and "only one" for Question 4. So, it is possible to see the impact of register on the responses. The results of these two questions show well the impact of register and the strength of the conception of *tied vector*.

Conclusion

This analysis of textbooks and the work of students shows the impact of the language [10] used in teaching vectors on its conceptualization as an object. This language use strongly infers the reduction of the notion of vector to its tied aspect, which is on the level of the naming of a vector with two letters or one letter, or its graphical representation or even in the language used to define some rules on the vectors. The constraints of transposition play a central role in this reduction. It is known that the language of the question links to theoretical references for experts, but this same language links to visual references, or situations met or worked with before, for the students (Balacheff, 2002; Harel, 1990; Fischbein, 1993; Noss *et al.*, 1997).

Besides, in the practice of the students, the vector as drawing amounts to a segment at the extremity of which one puts an arrow: the direction is decided by the drawing and the sense decided by the arrow. Therefore, the two notions direction and sense are decided visually on the drawing, but length asks for a calculation and reasoning so the student tries to take advantage of information given to work it out.

Otherwise, at the time of the resolution of problems using vectors other registers are used. For some, the resolution is made in the algebraic register (addition) accompanied often by the special vector technique (the rule of Chasles). For others, the working is carried out in the register of Euclidean geometry (being based on drawing) and for others in the analytic register (with the help of coordinates). The texts and graphics defining the question (input register) and the form of the answer wanted (register of output) can be in the same register, in two different registers or even a mixture of registers. In general, the register or language of the statement of the problem leads to the register of the reasoning. Otherwise, if the choice of this last is in charge of the student, there can be difficulties with resolution, because the tasks of conversion are, in the majority, handled by the texbook. Questions 2, 3 and 4 are examples of this.

It seems that the teaching of the notion of vector in these classes can limit students' ideas to the conception of *tied vector*. I now want to study situations in which students can demonstrate the conception *free vector*. The main question that persists is: *Is it possible to find, and thereafter to use, a more suitable language that permits an adequate conceptualization of the vector?*

Notes

[1] I owe a great debt in developing these ideas to my two doctoral supervisors, George Nahas and Claude Tisseron.

[2] The two textbooks referred to are:
CRDP libanais (1998-2001) *Construire les mathématiques*, Classe EB8; EB9; 1ES; 2ES; 3ES (SV), Ministére de l'éducation libanaise.
Al-Ahlia (1998-2002) *Mathématiques. Collection puissance*, Classe EB8; EB9; 1ES; 2ES; 3ES, G. Karroum, N. Badr, A. Moarbes, C. Merheb, K. Atieh, M. El Asmar, H. Nassar.

[3] *Deux vecteurs sont égaux s'ils ont la même direction, le même sens et le même module.* In English, it seems that *same direction* is used instead of the distinction, transliterated from the French, between "same sense and same direction".

[4] *Deux bipoints sont équipollent s'ils représentent le même vecteur, donc s'ils ont même direction, même sens et même module.*

[5] *Bipoint* in French.

[6] Lê Thi, H. (1997) *Etude didactique etépistémologique sur l'enseignement du vecteur dans deux institutions: la classe de dixième au Viêt-nam et la classe de seconde en France*, Thèse de doctorat de l'université Joseph Fourier, Grenoble.

[7] *Coefficient directeur* in French means *slope* in English.

[8] Meaning *superimposable* – we abandon, for the moment, the incorrect use of the word "equal" in set language.

[9] In Lebanon, vectors are first introduced in the 8th grade (13 to 14 years old) to characterize translations. This study was done in the 10th grade (15 to 16 years old), because by then students have knowledge about vectors as objects.

[10] The word "language" has three senses here: graphical representations (figures, drawings); algebraic (operations) and sentences used in the statements of definitions, theorems, properties and problems.

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Conclusion

The introduction of multicultural, interdisciplinary perspectives into the mathematics curriculum has many points in its favor:

- Students become aware of the role of mathematics in all societies. They realize that mathematical practices arose out of people's real needs and interests.
- Students learn to appreciate the contributions of cultures different from their own, and to take pride in their own heritage.
- By linking the study of mathematics with history, language arts, fine arts and other subjects, all the disciplines take on more meaning.
- The infusion into the curriculum of the cultural heritage of "minority" students builds their self-esteem and encourages them to become more interested in mathematics. As one eleven-year-old boy wrote in his evaluation of a classroom activity based on African culture: "As you probably don't know I feel very strongly and am in deep thrust with my black people, and the math has made me feel better." There is little that one can add to this heart-felt comment!

(Claudia Zaslavsky (1991) 'World cultures in the mathematics class', FLM **11**(2), p. 36.)