Cognitive versus Situational Analysis of Problem-Solving Behaviors

NICOLAS BALACHEFF

Let us start with a banal remark: to solve a problem consists in finding a solution. But how are we sure that what we have found is a solution? Or, to take this question in the field of mathematics education: how are the students convinced that they hold a *correct* solution? Do they have any criteria? What is the basis of their conviction?

It could happen that the best way to be sure of a solution is to set down a proof But as we know, this is not necessarily the case: more often than not, the basis for certainty rests in the problem-solving process itself

Following Poincaré [1908], proof is very often presented as something like a verification after a problem has been solved. Such a point of view is contradicted by both clinical and historical analysis. For example, Schoenfeld [1984] notices in his analysis of a problem-solving protocol of a mathematician that "rather than being an afterthought or a means of post facto verification, mathematical argumentation ("proof") is a *means of discovery* for him" [p. 25] Lakatos [1976] shows, in his well-known essay, that the basic components of a proof do not come afterwards but stem from the problem-solving process, within which proof is deep-rooted.

By proof we not only mean a strict mathematical proof (what we call in French: *la démonstration*) but rather the various means of control (such as semantic control, logical control ...) within the problem-solving process. We call these *procedures for validation.*

Thus we propose to examine the following question: how are the students sure that what they obtain in the course of the resolution of a problem is free of contradictions? Another form of this question would be: what are the procedures for validation used by students while they solve a problem?

Most of the researches on problem solving are centered on *heuristics*. Thus they forget both the mathematical content—in terms of its psychological complexity—and the situation in which the activity takes place. The aim of the present paper is to contribute to a discussion on this point with special focus on the dialectical relations between situational components and cognitive (mathematical) ones.

As a *chercheur en didactique des mathématiques* we develop this analysis with reference to mathematics teaching and learning.

The problem of contradiction

A contradiction does not exist by itself but relative to someone who notices it. It has a witness So it may happen that it exists for one person and not for another one. In the classroom, a contradiction may be obvious to the teacher but not to some pupil, and vice-versa:

> Some students have shown that the sequence $\{u_n\}_{n \in N}$ has l as a limit, and that $l < 1$
>
> Then they write: $\lim\{u_n^{n+1} + u_n^n + u_n^2 + u_n + 1\} = \lim\{u_n^2 + u_n + 1\} = l^2 + l + 1$
> [Robert, 1982]
>
> Pupils had to evaluate the volume of a parallelepiped with small cubes. Having counted in one direction, some of them decided to omit one small cube when counting them in the other two directions, because of the corner cube that should not be counted twice
> [Vergnaud, Rouchier, et al, 1983]

In the first case, as teachers we see a contradiction in using two different treatments for *n*, whether or not it is an exponent. These students are not aware of the contradiction. On the contrary, in the second case a contradiction exists only for the pupils.

These examples provide evidence that to notice a contradiction depends on the *background knowledge* of the problem solver Thus, to analyse the problem-solving behaviour of a pupil, we need to make a conceptual analysis. A procedural analysis—pointing out the heuristics patterns—is not sufficient to understand and to explain what is going on. We have to know how the pupils' conceptions are related to the mathematical notions engaged in the problem space

We would like to assert that it is those conceptions which give meaning to the heuristics used by pupils and which allow them to have semantic control over what is produced.

But when the students become aware of some contradiction in the course of the resolution of a problem, to overcome it may need very much more than a simple logical treatment or a local adjustment. The case of the "corner cube" contradiction requires a fundamental change in the pupils' conceptions of volume. Contradictions in the pupils' activity are in general considered as errors, but "an error is not only the consequence of ignorance, or of uncertainty, or of an accident, as seems to be believed by empiricist or behaviorist theories. An error may be the consequence of previous knowledge which had its own interest, its own

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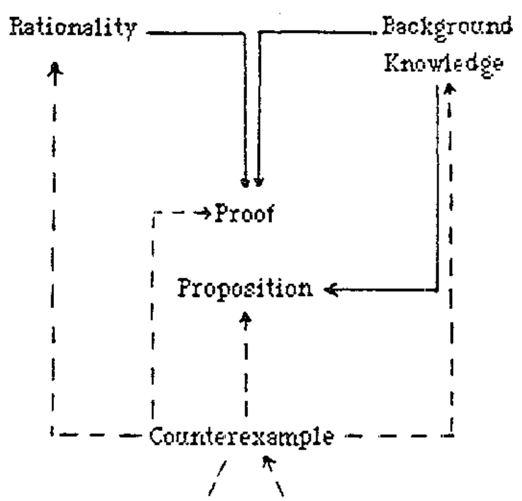
successes, but which at the moment appears to be false, or more simply, not yet adapted." [Brousseau, 1976, p 171] Thus to overcome a contradiction may need a dramatic change in the conceptions related to the problem. It could be the starting point of an evolution of the pupil's knowledge.

This is in fact the basic hypothesis underlying the use of problems in the French *approche didactique* of researches on mathematics teaching and learning.

About the non-deterministic effect of counterexamples

A contradiction can be analysed as a conflict between a conjecture—not necessarily explicitly stated, like an expectation—and a counterexample—a denial from the facts or from an opponent.

When faced with such a counterexample, there are many alternative ways to overcome (or to escape) the contradiction. Lakatos [1976] in "Proofs and refutations" discusses this point in detail, referring to an epistemological analysis. Rather than recalling the categories of effects Lakatos has shown, we sum up the main types in the following schema:



Rationality and background knowledge constitute the two foundations of a proposition (a candidate for becoming a theorem) and its proof. Here "rationality" means the fundamental basis for truth and the set of logical rules.

Lakatos considers the effect of a counterexample on the proof (**P**), the proposition (**L**) to be proved, or the background knowledge (**K**)—i.e. the consequences for the concepts themselves. Throughout Lakatos' work he also discusses the effects on rationality (**R**), although his discussion assumes a mature scientific community which has rejected empiricism.

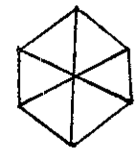
An experimental setting* designed to study the students' proving processes shows that most of the Lakatos' categories appear. But from this analysis rise two questions which constitute open problems for our researches on problem solving:

☐ What determines whether the counterexample should modify **P**, **L**, **K**, **R** or should be rejected?

☐ The best decision might be to reexamine the rational basis of the proof, or the set of logical laws which underly it. What would be the cause of such a critical decision?

The following examples point out the very meaning of those questions and underline their relevance for teachers and for research on mathematics education.

Having solved the problem of giving a formula for the number of diagonals of a polygon, the pupils—13-14 years old—were faced with some counterexamples. We must mention that for them a polygon is a cultural object far more than a mathematical one; it is something, say, like a "solid" in the 18th century...

> To find the number of diagonals of a polygon we just have to divide by twice the number of its vertices.
> Polygon with 6 vertices
> 6 : 2 = 3 diagonals
> example
>
> 
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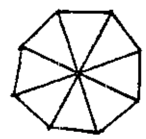
> Polygon with 8 vertices
> 8 : 2 = 4
>
> 
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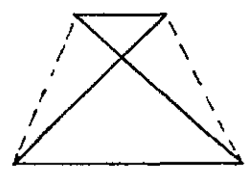
> ---
>
> polygons are shapes
> *with an even number of vertices
> *with all edges of the same length
> *edges having to be on a circle
> *from one vertex starts only one line
> [Naïma and Valérie]

In this example pupils oppose an *ad hoc* definition to each counterexample, although the relevant behavior would probably be to reconsider both the empiricist foundation of their conviction and their conception of "polygon" and "diagonal." *But how can they know?*

> Martine and Laura explain in a text too long to be reproduced here, that the number of diagonals is $(n-3) + \Sigma(n-p)$. Faced with some counterexamples they add to their text:
>
> ☐ it could happen that a diagonal meets the polygon in more than two vertices,
>
> ☐ when the shape represents the diagonals of a parallelogram or a trapezium, there are only two diagonals (see the schema)
>
> 
>
> [Martine and Laura]

In this last case the first counterexample is not considered as such, but as a contingent production of drawing. Where-

*A research report has been published

as the second counterexample is "explained" and not rejected for not being a polygon But *why would this choice not be relevant?* Is there any means—except *to know* that . . —to decide that the best decision here is to throw away such a polygon? We must realize that our opinion as teachers about these questions might be the result of an *a posteriori* point of view.

When we meet a contradiction while solving a problem, it does not mean that what has been done is to be thrown away . . but that there is something to change. Problem solving processes are dialectical, as we have already mentioned; to understand them we need a conceptual analysis But cognitive analysis alone is not sufficient to explain the orientation of this dialectic

Toward a situational analysis of problem-solving behaviors

When faced with a problem, pupils are in a situation which has more than a purely cognitive determination

As pupils they encounter problems in the classroom, in a situation which has been organised by the teacher And for teachers problems are means in the teaching process: problems specific to a mathematical content, problems to train pupils, problems for the evaluation of pupil learning, and so on. Each of these functions of a problem leads to a social interaction between pupils and the teacher We call the product of this interaction a *didactical contract*. [Brousseau, 1981] This contract is a set of rules, most of them implicit, which determine the *rôle* of both the teacher and the pupil; which determine the place of the problem in the teaching-learning situation; which guarantee the relevance of the problem to the didactical situation.

For example, this didactical contract allows the pupil to conjecture that the mathematical content needed to solve the problem has not been taught, or not too recently—if it is a *problem*—nevertheless it has something to do with the mathematical content of the moment . . Thus this contract is one of the elements which determine the problem space, together with the background knowledge of the pupil and the mathematical location of the problem

The didactical contract is one of the components which determine the meaning of a problem. This meaning is not definitely fixed by the problem text itself. One of the well known effects of this contract is to transform the problem of establishing the validity of a proposition (i.e. to "show that") into the problem of establishing that the pupil knows something about mathematics. This may modify very deeply problem-solving behaviors: the pupil does not search for *a* solution of the problem, but for a solution which *looks* mathematical [Balacheff, 1982]

There is no doubt that there exists a similar *experimental contract* [Balacheff and Laborde, 1982] in the situations designed to study problem-solving behaviors under laboratory conditions In such situations we have to take into account "the pressure of being recorded, [the subject's] belief about the nature of the experimental setting, and the subject's belief about the nature of the discipline itself." [Schoenfeld, 1984, p. 14]

In fact we have to realize that most of the time the pupil does not act as a *theoretician* but as a *practical man.* [Balacheff, 1984] His job is to give a solution to the problem the teacher has given to him, a solution that will be acceptable with respect to the classroom situation. In such a context the most important thing is to be effective. The problem of the practical man is *to be efficient,* not *to be rigorous.* It is to produce a *solution,* not to produce *knowledge.* Thus the problem solver does not feel the need to call on more logic than is necessary for practice; *de facto* he applies a principle of *economy of logic.* [Bourdieu, 1980] The procedures for validation used by the problem-solver do not depend only on his cognitive skills but also on the pressure for certainty in the situation. [Popper, 1972] Thus to be an empiricist may be a matter of circumstances and not exclusively a matter of intrinsic competency.

Then, when the problem-solver meets a counterexample his decision takes into account the proper economy of the situation. To consider it as an exception might be the most efficient thing to do, for in practice it appears to be very seldom that such a contradiction happens.

So it appears that when we study the problem solving behaviors of pupils we have to consider the situational background, and especially its social determination. It is in the light of the interaction between cognitive processes and social processes that may be found the key to understanding the behaviors of a pupil faced with contradictions and choices in a problem situation

As a final remark, let us stress that from this point of view the mathematics classroom is a very specific situation because of its goal-directedness, the nature of the knowledge concerned, and the relationship of the partners.

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