ARCHITECTURE OF MATHEMATICAL STRUCTURE

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This article began out of discussions between the first two authors on their linked research and development work in South Africa geared towards building awareness of structure in early number, and the latter two authors' extensive writing on mathematical structure and what it means to develop structural awareness, often drawing on this in designing tasks and pedagogic prompts. In these discussions, we found we were frequently talking at cross-purposes, with each of us using different combinations of terms for similar phenomena and using the word *structure* to refer to substantively different phenomena. It was at this point that we turned to the literature and were comforted to find ourselves in good company in terms of our confusions. The phenomena in Figures 1–3 provide useful contexts for illustrating the kinds of things we were struggling with.

We are interested in these phenomena in relation to differences in the nature and extent of awareness of mathematical structure. But claiming interest in "differences in the nature and extent of awareness of structure" presumes that we know what we mean by the term *structure*. This is the question that guides this article.

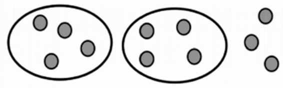
In response to the question
"What is the same and what is different about these two number sentences: 3 x 2 = 6 6 ÷ 3 = 2 ?"
learners are heard to say:
1: "Both number sentences have a three, a two and a six in them"
2: "If I give two sweets each to three people, then I could also say that six sweets shared between three people would give each person two sweets"
3: "I know I can reverse times by three with divide by three"

Figure 1. Phenomenon A.

England assesses learners in mathematics at the end of primary school (11-year-olds). Questions include calculations such as:
725 ÷ 29 and 1320 ÷ 12
(both set out on squared paper as 'long division'),
but also questions such as:
"Given that 5542 ÷ 17 = 326, can you use this fact to show how to work out 326 × 18?"
Performance on the former two calculations is far higher than on the latter question.

Figure 2. Phenomenon B.

Two responses are offered for a division number sentence matching the diagram:
11 ÷ 2 = 4 rem 3
11 ÷ 4 = 2 rem 3



The teacher rejects the first offer on the basis of declaring that the remainder cannot be larger than the divisor, and accepts the second answer.

Figure 3. Phenomenon C.

While structure has been written about widely in mathematics education, it remains a notoriously difficult word to pin down. Recently, Kieran (2018) has noted that:

> Structure is often treated within the mathematics education community as if it were tantamount to an undefined term; it is further assumed that there is universal agreement on its meaning. (p. 80)

Part of the difficulty with pinning structure down is that the term is frequently enmeshed within a set of terms that are sometimes juxtaposed but seen as distinct, and at other times, seen as synonymous. Terms in the 'mix' that particularly interest us are: *structure, relationships, generality/ generalisation* and *properties.*

In this article, we use the examples above as a fertile ground for looking at such terms given the rich networks of relationships between actions and properties that characterise the multiplicative reasoning (MR) conceptual field (Vergnaud, 1983). These MR phenomena provide a base for illustrating what we mean when we talk about the term *structure* and the ways in which it is linked to the other terms. The literature base related to these terms is drawn upon in order to understand key juxtapositions and distinctions made between structure, relationships, generality/ generalisation and properties in different pieces of writing. Through this engagement, we end up with a model that connects the terms of interest to us and highlights their distinctions. Building on this we introduce the notion of *formatting actions* [1] as central to creating, or recreating through editing, structure.

Structure: an overview of the literature

Kieran's comment above on structure as largely undefined, and yet seen as a central feature of mathematics, forms an important backdrop to this overview. In the same writing, Kieran also notes a bias in the mathematics education literature towards a focus on generalising that tends to side-line attention to structure. But the lack of clarity about the notion of structure may be part of the reason for this lack of attention. For example, Mulligan, Vale and Stephens (2009), in their introduction to a special issue on structure of the *Mathematics Education Research Journal,* state that awareness of structure is important for: "mathematical representation, symbolisation, abstraction, generalisation and proof" (p. 1), but leave unspecified what constitutes an awareness of structure. While their formulation suggests abstraction as an outcome of structural awareness, Warren (2005) asserts, in somewhat the opposite direction, that "abstracting patterns is the basis of structural knowledge" (p. 305). Questions about the relationship between structure and abstraction are raised, but not resolved in this debate, and what constitutes structure remains opaque.

For Kieran, structure arises from 'structuring' activity which refers centrally to what she describes as decomposing and recomposing moves in examples such as:

$$989 = 9 \times 109 + 8 \text{ or } 9 \times 110 - 1$$

$$30x^2 - 28x + 6 = (6x - 2)(5x - 3)$$

Such moves produce alternative ways of expressing particular arithmetical or algebraic entities. This resonates with a comment John Mason made in our initial discussions of this article, that, for him, the notion of structure has an architectural quality, a spatial organisation formed by specific relationships that place some element or elements in particular configurations with another element or elements, rather than in random arrangements. This metaphor further resonates with Dörfler's (2016) description of how structure is experienced in the use of diagrammatic representations: "diagrammatic inscriptions have a structure consisting in a specific spatial arrangement of and spatial relationships among their parts and elements" (p. 25).

Across Kieran's examples and Dörfler's description, central to the idea of structure is that it is underpinned by elements arranged in some specific mathematically appropriate relationship to each other. For example, the elements 6, 8 and 48 can be 'structured' into a $6 \times 8 = 48$ arrangement, but not into $6 \times 48 = 8$. Thus, structure hinges on a mathematically appropriate relationship that underpins the spatial architecture, and this represents, a first and useful clarification:

structure ⇔ mathematical relationship between elements

The first response in Phenomenon A ("Both number sentences have a three, a two and a six in them") with no reference to the relationship, either between the quantities within either of the number sentences, or between the multiplication and division sentences, points to an absence of communication of an awareness of structure in this context.

But Kieran's first example and Dörfler's description are dissimilar in the extent to which particular, or more general, relationships are referenced. In Dörfler's description, structure is a characteristic of the *particular* example being worked with and is visible in elements arranged in some mathematically valid relationship to each other. Battista (1999) provides a similarly 'local' attention to relationships in spatial contexts in his definition of spatial structure as involving determinations about an object's form:

> It determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationships between components and the new object. (p. 418)

This orientation to particular examples also links with Mulligan & Mitchelmore's (2009) coding for the extent of structuring, or organising into a mathematical relationship, in children's responses across a range of tasks. Across these descriptions, there is a local 'organising' that is central to Freudenthal's (1991) notion of 'structuring' as 'emphasising form' (p. 10) in which phenomena are organised according either to internal relations or to their relations to other phenomena. Common to all these perspectives is the perceiving or arranging of elements—symbols or images—in some *particular* organisation that serves to stress a mathematically appropriate relationship according to the syntax of the symbols or materials being worked with. This body of work leads to a notion of structure based in noticing, or forming, local relationships that are internal to a specific case, although they may also apply across a larger class. We describe these structures as *emergent*, based on a local relationship that arises, or is represented, in specific cases.

This literature contrasts with a second body of work in which the notion of structure carries a much higher burden of proof. Mason, Stephens and Watson (2009) incorporate the need for awareness not merely of local relationships, but of general properties, properties being defined as the implied behaviour of, and internal relationships in, a named class of mathematical objects, in their description of *mathematical structure:*

> We take mathematical structure to mean the identification of general properties which are instantiated in particular situations as relationships between elements; these elements can be mathematical objects like numbers and triangles, sets with functions between them, relations on sets, even relations between relations in an ongoing hierarchy. (p. 10)

In this formulation, structure continues to be described as made visible in "relationships between elements", but for Mason *et al.,* such relationships alone are insufficient as markers of mathematical structure. Instead, they require the "identification of general properties" within a particular instance of a relationship, suggesting a generalised awareness that characterises, or pre-figures, working with the particular case. Warren (2003) similarly emphasises attention to both relationships *and* mathematical properties (those characteristics that are immutable within a class and/or define a class) within her focus on mathematical structure in the context of arithmetic, in order to support the transition to algebraic thinking:

> In particular, mathematical structure is concerned with the (i) relationships between quantities (for example, are the quantities equivalent, is one less than or greater than the other); (ii) group properties of operations (for example, is the operation associative and/or commutative, do inverses and identities exist); (iii) relationships between the operations (for example, does one operation distribute over the other); and (iv) relationships across the quantities (for example, transitivity of equality and inequality). (pp. 123–124)

Freudenthal (1983) suggests a particularly high bar for the term mathematical structure, using it to refer in totality to the overall network of basic and derived properties and actions that can be associated with an initial relationship like $a \times b = c$.

The stipulation that mathematical structure involves attention to specific relationships as instantiations of general properties contrasts with the writing overviewed earlier focused on what we called emergent structure. Emergent structure expresses a focus on the nature of, or the organising into, a local mathematical relationship between elements—where awareness of any general properties of a class is yet to surface. In contrast, in the second body of writing, structure has a *general* flavour, with focus on the properties of relationships within some class of examples. We distinguish the local nature of relationships that we noted in 'emergent' structures from the general relationships that underpin what we term as 'mathematical' structure. So, for instance, there are general features of the multiplicative relationship that do not depend on particular examples (as Freudenthal notes, and as seen in response 3 in Phenomenon A: "I know I can reverse times by three with divide by three"). Kieren's juxtaposition of two decompositions of 989 as $9 \times 109 + 8$ and $9 \times 110 - 1$ is of interest here. For some, these two decompositions may be interpreted as two emergent structures—two ways of breaking down 989 in relation to a multiple of 9. However, the selection of a relatively large number (989) in relation to 9 and the juxtaposition of the multiple of 9 just below 989 and the multiple of 9 just above 989, with their associated remainders, points to possibilities for seeing a range of more general ideas—that all numbers lie between two consecutive multiples of a given factor, and that—in the case of 9 as the factor, a number being n above the preceding multiple of 9 also means being $(9 - n)$ below the next multiple of 9. Creating or noticing some of these more general relationships indicates moves into the terrain of mathematical structure. A similar distinction marks responses 2 and 3 in Phenomenon A—the statement "I know I can reverse times by three with divide by three" is offered in general, rather than the particular terms seen in response B: "If I give two sweets each to three people, then I could also say that six sweets shared between three people would give each person two sweets".

This leads to a second clarification of the notion of structure: distinguishing the formatting of elements into a local, or emergent relationship (creating an emergent structure) from the formatting of elements into, or on the basis of, a more general relationship that is understood to hold across some broader class of examples.

Emergent structure
(involving analyzing/forming/
seeing local relationships)

vs

Mathematical structure
(involving analyzing/forming/
seeing general relationships)

Empirically, Mason & Pimm (1984) have noted that examples can be worked with as specific ('the even number 6'), the generic ('an even number like 6'), and the general ('any even number'). This points to the importance of listening for the extent of generality in language as one route to exploring the basis of offers in emergent or mathematical structure. While formatting elements into a relationship is common across both emergent and mathematical structures, we make the distinction that emergent structures arise in a discourse of particularity, whilst mathematical structures arise in a discourse of generic/general relationships, applicable within some class of examples.

Properties, in the formal sense, are brought into play in Phenomenon C, where the first learner offer is rejected on the basis of a convention that when dividing, the remainder in division usually has to be non-negative and strictly less than the divisor. There are openings here to consider relationships between grouping and sharing models of division, as well as the range of reformatting possibilities that mark processes of division by grouping, and lead to a final, singular format, as the conventional outcome:

$$11 \div 2 = 1 \text{ rem } 9 = 2 \text{ rem } 7 = 3 \text{ rem } 5 = 4 \text{ rem } 3 = 5 \text{ rem } 1$$

All of these expressions have the form $a \div b = c \; rem \; d$ because they all satisfy the property that: $a = bc + d$ where a is the dividend, b the divisor, and d is a remainder (rather than the remainder). These algebraic expressions are general, and hence, they all give perspectives on the multiplicative mathematical structure of number.

But structure can also be in focus within task design and within materials. The tasks set in Phenomenon B provide useful contrasts here. The setting out on squared paper in traditional long division algorithm form of the first two examples: $725 \div 29$ and $1320 \div 12$, directs attention towards calculation, using an algorithm that can be completed with an entire focus on 'digits' rather than on numerical relationships within the dividend or between the dividend and divisor. This 'formatting' of the problem-setting is particularly interesting in the case of the 1320 dividend problem, where 1320 can be re-formatted as $1200 + 120$, rendering the problem open to solving efficiently through awareness of numerical relationships between this partitioning and 12 as divisor. Davis' (2014) advocacy of enhancing attention to number by focusing more on its spatial *form* (or more accurately from the vantage of re-formatting, forms) is apposite here in terms of building attunement to flexible ways of decomposing numbers for the purposes of the problem at hand. In contrast, the final division task formulation demands more direct attention to the inverse relationship between multiplication and division in $5542 \div 17 = 326$, and to comparing the outcomes of 326×18 and 326×17. The lower response rates for this last task suggest more limited awareness of structure alongside much higher levels of computational fluency displayed on the other items.

Number relationships and form are similarly emphasised in Wing's (2009) descriptions of 'structured' materials as materials providing "representational affordances [that] emphasise either number notation conventions (such as bead strings, abacus and base-ten blocks) or number relations (such as Cuisenaire and Numicon)" (p. 11). In contrast, cubes and counters are described as unstructured materials, as no mathematical relation inheres in their design, though they may become structured in use through formatting actions. It is also worth noting the caveat that any 'in-building' of structure into resources does not guarantee that the relevant structure will be noticed or appropriated by the materials user.

This overview leads us into a summary and schematic model that helps us to clarify our thinking about the notion of structure in mathematics—presented in the next section.

Structure: a summary and a schematic model

We use the term 'formatting' to describe the highlighting of a relationship between elements, whether through spatial arrangement or other notational means. Formatting can be applied to specific examples or to more general example spaces. In the first case, formatting can give rise to an emergent formation, or noticing, of a relationship within a particular case. In the latter case, formatting can arise from an anticipated structure based on previously encountered examples (that can be explicitly present or implicitly invoked), that pre-figure and direct the seeing or forming of a relationship between elements. Formatting for mathematical structure, more generally, is formatting undertaken with awareness that goes beyond the specific relationship that is produced.

This summary discussion distinguishes structure in the context of particular cases (emergent structure) from structure in the context of generic and general cases (mathematical structure). Connecting these two strands from the particular to the generic/general are generalising actions, which, as Kieran points out, have already received attention in the literature. Moves in the reverse direction are specialising actions.

This discussion leads to an exploratory clarification of terms in Figure 4.

In this model, we stop short of Freudenthal's (1983) high bar requirement for the totality of the network of basic and derived properties and actions in our formulation of mathematical structure. Instead, we follow the more pedagogical line that reflects our interests and view mathematical structure as coming increasingly into view as awareness of a network of basic and derived general relationships expands over time. In taking this line, we follow the approach taken by Brown (2011) in her discussion of concepts as structures that emerge and expand through experience across tasks that afford different possibilities for noticing similarities and distinctions and the reasons that might underlie these patterns of relationships.

The diagram provides a lens with which to look at phenomena with attention to a difference between perceiving relationships within particular (local) cases, which is the beginning of apprehending and perhaps conjecturing an emergent structure, and thinking about general relationships. General relationships can be seen via generalising activity across a class of examples, or through working with a particular case that is viewed as generic of the class.

Concluding comments and reflections

We conclude with a comment on the role of format and formatting as key pedagogic actions that accomplish a focus on structure in mathematics teaching. Our prior work points to the importance of patterns of variation and invariance in format in rendering format changes more accessible to learners. Attention to format allows formatting/re-formatting to move from a local relationship in an initial form to an awareness and/or expression of the same relationship in a different presentation. Thinking about the domain of applicability of these formats/re-formats similarly supports connections between emergent structure and mathematical structure. An example of how these considerations can inform practice, in our chosen context of MR, can be found in what we have come to call the Drakensberg Grid [2]. This grid presents several formats of symbolic, iconic, graphical representations of multiplicative relationships. It also enables the user to juxtapose these to produce formats for encountering some of the structure of multiplication in the field of real number (Mason, Watson, Askew & Venkat, 2018).

The commentary and clarification presented in this article allows us to distinguish *formatting* moves from *generalising* moves. Formatting can emphasise, intentionally or other-

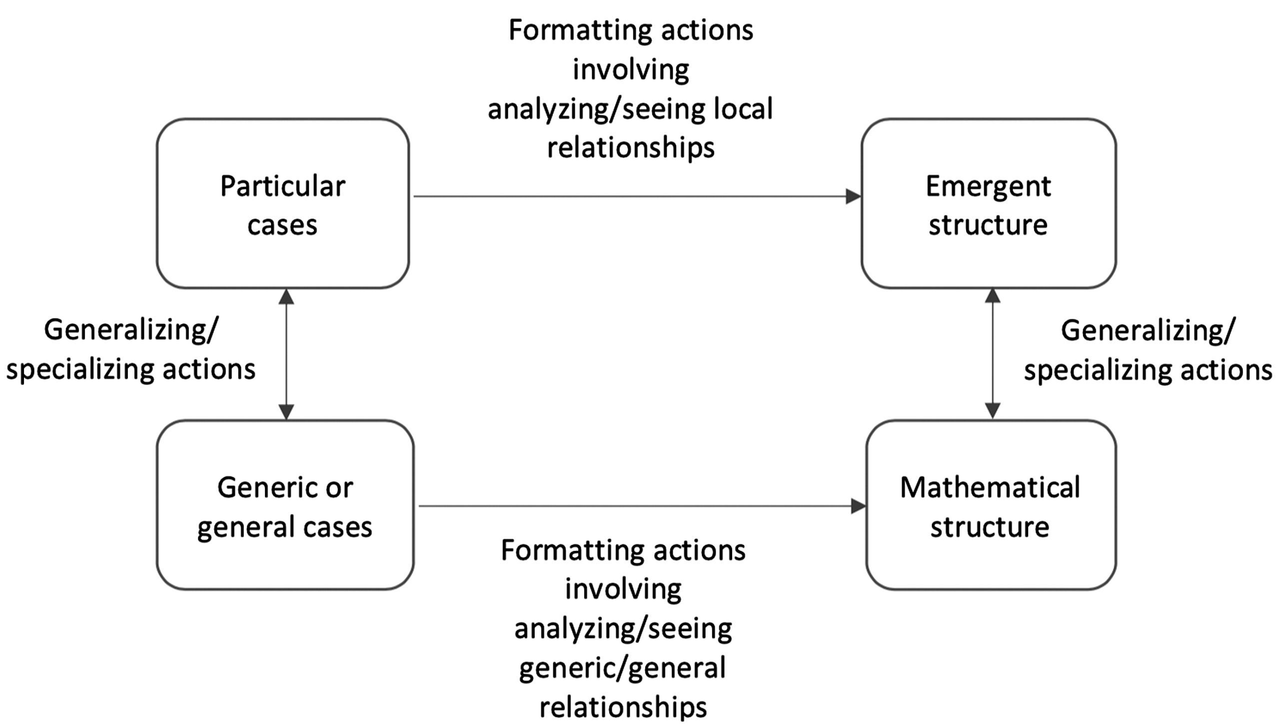


Figure 4. Clarifying terms and their links.

wise, structural properties. Emergent structure arises through formatting by decomposing or recomposing particular examples; awareness of mathematical structure can arise through formatting by decomposing and recomposing general properties of a class. Generalising moves are required to appreciate mathematical structure and can be seen via work with generic or multiple examples.

Notes

[1] Not to be confused with Bruner's *formats of interaction* discussed by Anna Sierpinska in issue **17**(2) or the *formatting power of mathematics* discussed by Borba and Skovsmose in issue **17**(3).

[2] See http://www.pmtheta.com/reasoning-about-numbers.html#Drakensberg

References

Battista, M.T. (1999) Fifth graders' enumeration of cubes in 3D arrays: conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education* **30**(4), 417–448.

Brown, L. (2011) What is a concept? *For the Learning of Mathematics* **31**(2), 14–15.

Davis, B. (2014) Toward a more power-full school mathematics. *For the Learning of Mathematics* **34**(1), 12–17.

Dörfler, W. (2016) Signs and their use: Pierce and Wittgenstein. In Bikner-Ahsbahs, A., Vohns, A., Bruder, R., Schmitt, O. & Dörfler, W. (Eds.) *Theories in and of Mathematics Education: Theory Strands in German Speaking Countries,* pp. 21–31. Dordrecht: Springer.

Freudenthal, H. (1983) *Didactical Phenomenology of Mathematical Structures.* Dordrecht: Reidel.

Freudenthal, H. (1991) *Revisiting Mathematics Education.* Dordrecht: Kluwer.

Kieran, C. (2018) Seeking, using, and expressing structure in numbers and numerical operations: A fundamental path to developing early algebraic thinking. In Kieran, C. (Ed.) *Teaching and Learning Algebraic Thinking with 5- to 12-Year-Olds: The Global Evolution of an Emerging Field of Research and Practice,* pp. 79–105. Dordrecht: Springer.

Mason, J. & Pimm, D. (1984) Generic examples: seeing the general in the particular. *Educational Studies in Mathematics* **15**(3), 277–289.

Mason, J., Stephens, M. & Watson, A. (2009) Appreciating mathematical structure for all. *Mathematics Education Research Journal* **21**(2), 10–32.

Mason, J., Watson, A., Askew, M. & Venkat, H. (2018) Multiple formats for multiplication. *Mathematics Teaching* **262**, 21–22.

Mulligan, J. & Mitchelmore, M. (2009) Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal* **21**(2), 33–49.

Mulligan, J., Vale, C. & Stephens, M. (2009) Understanding and developing structure—its importance for mathematics learning: Editorial. *Mathematics Education Research Journal* **21**(2), 1–4.

Vergnaud, G. (1983) Multiplicative structures. In Lesh, R. & Landau, M.(Eds.) *Acquisition of Mathematics Concepts and Processes,* pp. 128–175. London: Academic Press.

Warren, E. (2003) The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Education Research Journal* **15**(2), 122–137.

Warren, E. (2005) Young children's ability to generalise the pattern rule for growing patterns. In Chick, H. & Vincent, J. (Eds.) *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 4, pp. 305–312. Melbourne: University of Melbourne.

Wing, T. (2009) The post-Piagetian child: early mathematical developments and a role for structured materials. *For the Learning of Mathematics* **29**(2), 8–13.