Conjecture and Verification in Research and Teaching: Conversations with Young Mathematicians

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> Intuitive representations will not disappear from mathematical endeavors merely because one decides that such representations do harm to the rigor of a formal reasoning process. They will remain because they are an integral part of any intellectually productive activity. (Fischbein, 1987, p. 21)

The working lives of mathematicians are important to mathematics educators for many reasons. Mathematicians produce new mathematics, write curriculum and teach future teachers of mathematics. Their teaching generates potent images of mathematics and mathematical inquiry for all college students, including prospective teachers. Some mathematics educators (e.g. Lampert, 1990; Schoenfeld, 1992) have further suggested that the goals and methods of mathematics education should be organized more around mathematicians' *practices* than the accumulated results of the discipline. If this focus on professional mathematical practice makes sense, it is crucial to analyze and come to understand mathematicians' individual and collective experience and working practices.

Recent analyses of mathematical practice have focused on its informal, heuristic, material and social aspects (Restivo, 1993: Rotman, 1993; Schoenfeld, 1991), as complementary to the logical, deductive and symbolic qualities that serve to dominate 'public' mathematics (e.g. journal articles and textbooks). These analyses have highlighted the human face of mathematics in communities of practice. But detailed accounts of individual practices are still relatively rare (Davis and Hersh (1981), Hadamard (1945), Polya (1945/1973) and Thurston (1995) being notable exceptions) [1] What are the origins of mathematicians' professional interests? How do they go about their work? How social and interactive is it? How are initial ideas, insights and conjectures transformed into established results? More generally, how do the processes of discovery and verification interact? Two psychological notions -- intuition and understanding - have oriented analyses of the generative, informal stages of mathematical practice.

Ideas in practice: intuition, understanding and mathematical objects

Following Kant, most conceptualizations of intuition have centered on some process of 'informed seeing', on grasping physical or mental phenomena in a conceptually coherent way (e.g. Noddings, 1984). Intuition brings structure and coherence to perception. But Kant also asserted that intuition was structured *a priori* by some 'necessary' mathematical notions (e.g. a Euclidean view of space). When these notions no longer appeared necessary, modern mathematicians became skeptical of the role of intuitive judgement in mathematics. Hahn (1956) cited many examples from modern mathematics where accepted intuitions proved faulty.

More recently, however, some mathematicians have cast intuition in a more productive light. Wilder (1967) argued that both individual and collective intuitions have been productive in the development of mathematics, even when they were faulty. [2] Davis and Hersh (1981) emphasized the central role of intuition in the 'mathematical experience', linking it to interaction with mathematical objects.

> We have intuitions because we have mental representations of mathematical objects. We acquire these representations, not by memorizing verbal formulas, but by repeated experiences (on the elementary level, experience of manipulating physical objects; on the advanced level, experience of doing problems and discovering things for ourselves) (p. 398)

Intuitions are born from experience with mathematical objects, both ordinary physical objects and mental objects with 'reproducible properties'. Fischbein (1987) also placed intuition at the center of productive mathematical reasoning and characterized its general qualities. Intuitions are immediate and self-evident, generative (extending 'beyond' the given facts), compelling (difficult to doubt or change) and non-analytic (not subject to decomposition).

Noddings (1984), a philosopher, has explicitly linked intuition in mathematics to understanding as well as to acquaintance with objects. Intuition is a mental faculty that seeks direct contact with physical or mental (mathematical) objects to develop meaning and understanding. It is most productive in arenas of rich and extensive experience and produces affective as well as cognitive responses - feelings of excitement and/or anticipation. The products of intuitive thought, representations of objects, are then 'given over to' the faculty of reason for statement, manipulation and transformation by explicit definition and logical argument. Her analysis squares well with Fischbein's: intuitions are non-analytic, involve direct contact with objects of thought and produce a sense of certainty, but may be faulty.

The term 'understanding' appears even more frequently in discussions of mathematical education (e.g. Sierpinska, 1994). In most accounts, understanding differs from the simple recall of results or procedures. It involves the personal grasp of some aspect of mathematical reality. It

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engages and connects to prior knowledge; is generative in supporting further conjecture and inference; and involves gradual progress forward from a state of non- or weaker understanding. Thurston (1995) has argued that the main task of mathematicians is to help others *understand* more mathematics. Teaching and research are distinct but related efforts to communicate complex ideas. While acknowledging the centrality of proof, he emphasized that understanding is not a logical, formal process. It involves ideas and how they relate - the 'basic infrastructure' that supports but is not represented in formal arguments.

If intuition and understanding play important roles in creating and communicating mathematics, and if productive links can be drawn between professional practice and classroom mathematics, then there is work to be done. Analyses such as those discussed above take us only so far: for example, the vague generality of Davis and Hersh's descriptions of mathematicians' experiences as "doing problems and discovering things ourselves". Accounts of mathematicians' actual practices are few and limited to self-report.

This study explored the mathematical practices of three young mathematicians in an extended interview setting. We focused on the interaction of discovery and verification, the role of conjecture in discovery and the place of intuition and understanding in reasearch. To establish some specific and (for us) accessible mathematical content, we presented some examples of college students' intuitive leaps of reasoning for our participants to evaluate. Their reactions allowed us to compare their stance toward conjectural and non-deductive thinking in their research work with teaching situations. We found an interesting mismatch between how they valued their own guesses and how they reacted to guesses produced by students - a result with important implications for teaching and learning, especially at the college level. Given the depth and complexity of these issues and our small sample, our results can be only suggestive. We hope our analysis encourages others to explore and report on the content and diversity of mathematicians' views and practices.

The Study

The participants and the interviews

We chose advanced graduate students because they were actively engaged in both research and teaching, new to both tasks and open to participation. [3] Three male graduate students [4] were interviewed by the second author. Allan was currently completing on his doctoral thesis in complex analysis. Brett had recently completed his dissertation in topology and was working on new problems and preparing for his first faculty job. Charles, four years past his Masters degree, was searching for a thesis topic in real analysis. Allan and Brett's interviews were audiotaped and videotaped for analysis; Charles' interview was audiotaped only, at his request.

In structuring the discussions, we chose not to focus solely on the mathematicians' own research work. Neither of us had advanced training in mathematics, so communication in their area of specialization would have been difficult. Instead, we framed our interviews around two main themes:

(i) the participants' views of their discipline and their work in it (both research and teaching);

(ii) their ways of thinking about some simple problems of fraction order and equivalence.

The fraction problems provided a context for discussion where both interviewer and particpants were knowledgeable and allowed us to present some examples of pre-college and college student's intuitive reasoning documented in our prior research (Smith, 1990, 1994; Smith and Hungwe, 1994). [5]

Each participant was comfortable with the interviewer and became quickly engaged in the issues discussed. Charles asked to extend his interview to a second day. The resulting dialogues - 1.75, 2, and 2.5 hours in length respectively - were rich and detailed. [6] Collectively, they generated a corpus of 140 pages single-spaced text. Figure 1 summarizes the issues explored in the interviews.

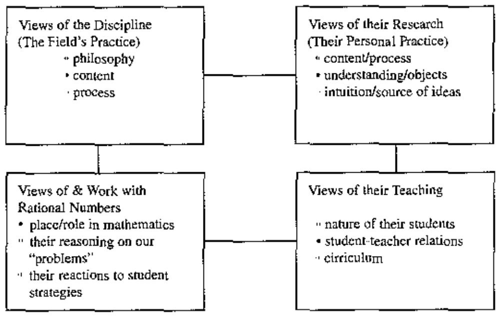

Figure 1 The interview topics

The student strategies

Most of the fractions problems were simple comparisons. We asked our participants to judge whether one fraction in a given pair was greater or whether they were equivalent and to explain their approach. On one pair, they were also asked about numbers between the given fractions.

Two student strategies were presented for their evaluation. We had observed some college students using an *Increasing Sequence Strategy* to 'solve' difficult comparisons (e.g. 4/11 and 5/12 or 3/5 and 5/7). [7] They nested the given pair in an infinite sequence (e.g. 4/11, 5/12, 6/13, 7/14, ...) claimed that the sequence increased to 1 and concluded that prior terms were less than subsequent ones. They did not argue rigorously that the sequences were strictly increasing, though they knew enough calculus to do so.

Similarly, some pre-college and college students asserted a *Betweenness Strategy*, when we asked them about fractions that were between a given pair: namely, a fraction whose numerator and denominator were between the respective numerators and denominators of the two given fractions was itself between them.

(This strategy is mathematically valid at least under the condition that both given fractions fit the form $(D - n)/D$, where n is a fixed integer less than D. In the above example, 6/8 is between 5/7 and 7/9, because 6 is between 5 and 7, 8 is between 7 and 9, and $n = 2$ for both given fractions.)

We presented these strategies as the students had expressed them. We intentionally did not 'clean up' or complete the students' reasoning We considered these assertions representative of the kind of student thinking that mathematicians see in their classrooms and offices and wanted to compare their reactions to them with their views of conjecture and intuition in their own work

Participants' evaluations of the student strategies

Allan's reaction to the Increasing Sequence Strategy to compare 3/5 and 5/7 was quite negative:

> this is completely wrong. I mean, this [3/5] is certainly less than this [5/7]. That argument [appeal to the sequence to infer the local order] is not correct.

He noted that the student's argument that the sequence 3/5, 4/6, 5/7, 6/8, ... 99/101 approached 1 failed to show that the sequence increased "continuously". Why should they not just compare them in pairs, he asked, by cross-multiplication? "Why make your life more difficult?"

Since Allan emphasized the importance of stating and formulating the right propositions in describing his practice (see below), the interviewer pressed him to consider the proposition that 3/5 was less than 5/7, if the sequence converged to 1. Allan replied: "It's certainly a proposition, but it's a wrong propositon". Only after he compared consecutive general terms of the sequence (n/n+2 vs. n+1/n+3) by cross multiplication did he admit that it strictly increased to 1. His evaluation suggested he wanted students to keep their reasoning simple and avoid generalizing without first proving their results Allan was not presented with the Betweenness Strategy because of time limitations.

Charles was also dissatisfied with the Increasing Sequence Strategy for comparing 4/11 and 5/12

> certainly that reasoning is not valid ... [I: so what [*sic*] would you describe that kind of kind of strategy?] Fallacious The reasoning is [fallacious] because, because you're not wielding any tools You don't have a tool at your, you need to wield tools, which you, which are, which are true. You might speculate [... ,] but having speculated, now you have to show that it is true.

The conjecture without justification was worth little to Charles. Proof was only possible by applying 'tools', here the definitions of order and equivalence among the rationals. His evaluation changed, however, when he found that the sequence did in fact strictly increase to 1. He graphed the function f(n) = n/n+2 and verified that its derivative was always positive. The next day he used "that theorem" to compare 6/9 and 8/12 and also 7/11 and 7/13. He also found the Betweeness Strategy (that 6/8 was between 5/7 and 7/9) lacking. "Those observations they made weren't, again they weren't powerful mathematical truths. They're just, they're guessing, which is good " [8]

Brett's reaction to the same strategies was quite different, in content and tone He was less dismissive, more open to the possibility that the students' conjectures could be true When the interviewer presented the Betweenness Strategy (for 5/7 and 7/9) and asked if this was "valid reasoning", [9] Brett replied:

> No. [...] Well it is, I don't see now why. It doesn't sound right. [ ] It might be true though, let's check, it's interesting.

Then he verified that 6/8 was indeed between by converting to a common denominator and went on to state (but not prove) the proposition in general terms.

> I don't see right away whether it is always true or not This is something I never thought of it [*sic*], so, but it might be the case.

He was also skeptical of but open to the Increasing Sequence Strategy.

> I would believe it if I see [*sic*] its proof. It might be the case all the time [...] I won't object this [*sic*] right away The reason is, see, this [3/5] is like 1/2.

He noted that the other terms in the sequence 3/5, 5/7, 7/9, 9/11, ... were getting "closer and closer" to each other and to 1. That fact did not guarantee that the series was strictly increasing, "but at least it shows that it might be the case".

Taken as a group, these mathematicians' reactions to students' intuitive reasonings emphasized verification over discovery Each mathematician noted the absence of proof in what the interviewer presented. Allan and Charles were sharply critical. Neither appreciated the content of students' conjectures or the generalizations they embodied (in the case of the Increasing Sequence Strategy). Instead, this leap was seen, prior to their own verifications, as "incorrect" and "fallacious". Only Brett was interested in the strategies prior to verification. His reaction balanced a skeptical stance toward their truth with a more open, 'wait and see' attitude. With these reactions in mind, we turn to their views of the discipline of mathematics and their own emerging research practice within it

Three portraits of mathematical practice

Charles: powerful tools and rigorous foundations

Of the three, Charles' perspective most closely matched the view, commonly held by non-mathematicians, that mathematics is an objective, rational and deductive enterprise. He was articulate and expressed his views with deep conviction The core of mathematical practice was formal proof The edifice of mathematics stood on the foundation of clear definitions and axioms.

> it's critical in math to understand what's a definition and what's not What's a definition and what follows from the definition, what's an interpretation of a consequence. [...] the foundation really rests, rests on your axioms, which are, the axioms are really the foundation, you assume the axioms and then you proceed.

Definitions created mathematical concepts: "as soon as you make a definition and say what it is, then it exists" Proven propositions were objectively true; "truth is a separate entity". Standards for the rigor in proofs existed, though they were not explicit. If a proof was difficult to follow, it was because it was written for readers who possessed more sophisticated mathematical tools.

He saw his own work in analysis as grounded in set theory. From that foundation, he built up a 'tool kit' of definitions and important theorems, mastered by careful study. 'Tools' were general propositions with wide utility. Each was internalized until it became "part of you", like your arms. Mastering theorems required working through each step in their proof, "digesting" them, giving them "personalities" and placing them in the "bigger picture" of other powerful theorems. His interest in logical foundations had been present since adolescence.

> in junior high [...] I spent a lot of time thinking about the logical foundations, you know, at least questioning the logical foundations. Um, when you question the logical foundations all the time, eventually you're, you're, you get the back, what it takes to be, to do this discipline. Um, a sense that A implies B, if A implies B, it may imply C, then A implies C, that makes perfect sense to me.

But his objective foundationalism was complemented by a process of discovery structured by cycles of conjecture and refutation (Lakatos, 1976).

> Well, that's what a math, that's what a math student does, is try to make sense of things. [...] Um, when we don't understand something, we stop and think about it. Try to come up with examples, or counter-examples, or. That's how we proceed, really, by, by, we try to underst[and]. [pause] So, there's a certain amount of guessing and checking that goes on. When we don't understand something you look at examples that you're familiar with, and ask questions, and resolve those questions. And if you proce[ed], if you do that, that's how you make progress, and then you look at all your guesses and checks, all the data that you've come up with and try to guess again. [...] You ask yourself, what, what could, what could conceivably be true? You know, and then see if it is. Prove whether, see if you can prove whether it is or not. Maybe, probably it's not completely true, but if you assume something else, perhaps it's true then. And that's the way research is done in math.

But Charles had very little to say about the origins of guesses, despite many probes from the interviewer. When asked where his guesses came from, he replied only that "it takes courage to guess 'cause you might be wrong". If he had ideas about the role of intuition in generating conjectures, he did not express them.

Charles frequently used the term 'understanding', but, like the others, did not clarify its meaning. His use of the term, however, indicated that to 'understand' a result was to 'know' it deductively and completely, so that it had 'tool' status. Other forms of knowing, such as memorized statements and isolated facts, did not count. It was not possible to work through a proof and still not understand the result: "How well can you follow the logic if you don't get it?" A 'reasonable result' was a proven one. Partial states of understanding were simply cases where he had been wrong. His mathematical objects were the standard mathematical entities created by formal definition: set, order, integers, rational numbers (from integers), equivalence, real numbers (from rationals). Physical objects and models were instructional techniques to engage students in before introducing formal definitions. He mentioned making drawings to help him think about surfaces but did not elaborate.

Charles referred frequently to his students' experiences with mathematics. While he appreciated how their time and engagement with the subject differed from his, he focused on what they were unable or unwilling to do. He traced their struggles to failures to learn concepts via formal definitions:

> if I give them the definition, if they're not willing to think about it to see that it, it makes sense, but they don't want to, they don't put forth that kind of thought. So I would say that's one of the reasons they have trouble. The student who had trouble is the student who is not willing to really, um, to draw on the ideas.

His initial approach to comparing fractions reflected his definitional approach to mathematics. Charles noted that the first pair (2/14 and 3/14) was easy to think about but nonetheless introduced the definition of order for the rationals as a 'general rule'. He then applied the rule to two more difficult pairs, 4/11 and 5/12 and then 10/13 and 99/102. After proving and using the Increasing Sequence Strategy, he reasoned less formally, arguing that $7/13 < 7/11$ because "it has a bigger denominator" and $8/11 > 7/15$ because 1/2 lay between them. These numerically specific strategies match those used by college and pre-college students.

Allan: tricks and inspiration in complex analysis

Allan had more to say than Charles about the intuitive origins of conjectures. His portrayal of mathematical practice gave equal weight to the processes of discovery and verification. To make good guesses (those you could eventually prove), one needed both solid insights into the underlying structure ("what is going on") and the ability to formulate propositions that clearly captured those insights.

> you always need to make propositions. You always need to think about your choices, what they are, and hopefully you have a limited number and you can try all of them [both Allan and interviewer laugh]. Sometimes it [making propositions] can be more difficult than to check the propositions. [I: OK.] Because formulating the proposition is just realizing what's going on.

Proving one's propositions depended on deploying "tricks", specific procedures to deal with obstacles in arguments. Serious students of mathematics must first learn all the known tricks in their field and then invent their own. Generating a new trick represented a substantive contribution: "you can get your Ph.D.". Tricks were logical, "but things that people didn't see before". Developing a new trick took "inspirations". For Allan personally, this inspiration came via "help from the Holy Spirit".

When the interviewer suggested that intuition could generate the insights that led to new tricks, Allan disagreed:

> Intuition, I think doesn't mean that you invented a new trick. [It] means that you understood what is going on and you can make a good statement. There are two

things To have a good intuition of something, to intuitively understand what is going on, and that means to get the right statement. But then you need a trick to prove it.

Where 'tricks' turned on inspiration, mathematicians were responsible for generating their own intuitions of "what is going on". Though intuition and understanding were tightly linked in generating promising new conjectures, there were accepted results Allan did not understand.

> I had a friend who used to say you cannot understand this [result], you can just get used to it. And some mathematical notions are like this. And maybe many are like this. But as soon as you get used to it, you understood it There is a certain overlapping of this kind of notion, of getting used, and understanding. [I: So how do you get used to them?] Just [pause] by using them [laughs]

For Allan, like Heidegger, understanding an idea was achieved more through its use than by conscious reflective analysis. He was bothered by failing to understand but hoped eventually to understand all the mathematics he knew to be true

Allan's comments on the nature of numbers provided some sense of his view of mathematical reality Though his specialization was complex analysis, he was not as strict a foundationalist as Charles. Axiomatic systems only partially explained the existence of numbers; they remained a "certain abstract nonsense" for mathematicians. Their nature was revealed by use, without ever completely understanding what they are: "it is important to use them, and not think about them" In explaining the power of numbers (and mathematics more generally), he felt that the fit between the physical world and mathematical objects was key and nature controlled it The structure of both the world and mathematics was intricate and complex. The history of mathematics was filled with examples where the physical world rejected overly simplistic mathematical models (e.g. all lengths were ratios of integral lengths) Only the mathematical theories that generated useful and adequate application survived.

Unfortunately, partially due to time constraints, Allan spoke less than the others about the specifics of his own research work. He affirmed that he deployed models in his own work but did not elaborate or offer examples. His comments on mathematical objects centered on matters of teaching. Physical-like objects (pies, graphs) were useful in introducing concepts to students (rational numbers, functions) because they were visually accessible and manipulable. [10] 'Mathematical objects' were the standard concepts he understood very well and used to gain access to more complex ideas, as, for example, the limit concept provided access to the derivative of a function.

Allan formulated the first few fraction comparisons as propositions, applying the general rule of cross-multiplication when numerators and denominators were different. In those cases, "mathematical manipulations" were needed. But he left this formal pattern on 10/13 and 99/102.

> But this can be seen that this, this is less [10/13], without multiplication [...] Well this is not the proof. That's maybe just an explanation [... A] quick way of seeing it, this [99/102] is very close to 1, right? [...] So after manipulating a lot these fractions [...], you can observe certain things, that if they are very, very big, both of them, even though the difference between them is 3 here and is 3 there [...] But the value of this one [10/13] is a lot less than the value of this one [99/102].

This was the only context where Allan acknowledged a role for finding and "explaining" patterns as informal counterparts to "propositions" and proof.

Brett: objects, abstraction and understanding in topology
Brett offered the most extensive commentary of the three on the informal aspects of his practice. Though he never used the term 'intuition', he gave an interesting account of understanding and the role of different sorts of mathematical objects in its development In contrast to both Charles and Allan, his view of mathematical practice emphasized its human and problematic nature Objects, for Brett, were not limited to concepts stipulated by definitions.

Mathematicians were drawn to open questions. When they started thinking about them, ideas emerged spontaneously either from an internal process, talks with other mathematicians ("things you have heard") or published work ("something you have seen"). Whatever their source, the emergence of ideas could not be explained; they simply appeared in your thoughts "Most of the time, 90%, one's initial ideas were faulty: your guess doesn't work". The next step was to attempt minor repairs, often by adding additional assumptions. "If it is a small problem, you try to go around it." But if the problems were major, the idea got discarded and you looked for another. Time and immersion in the content were key; months and even years were required to understand and make progress on problems. He explicitly denied that 'flashes of insight' were possible without extended periods of deep immersion in the content. For Brett personally, ideas were always present, even in his dreams, once he became engaged.

Abstraction was central to his work in algebraic geometry. Though he initially found topology's highly abstract objects a "kind of nonsense" that was difficult to grasp, he eventually came to see how they simplified inquiry and analysis. Defining more abstract objects from a collection of more specific physical or mathematical objects simplified the process of keeping track of their properties. Abstraction aided both discovery and verification of important relationships. He usually worked on intermediate abstractions, models or examples which he "understood very well" When the ideas worked there:

> then you modify your, you know, ideas and [...] you start to understand how your ideas will work for the general case

For the much of the interview, Brett used 'understand' as a synonym for 'know'. Understanding developed gradually and more or less continuously as a function of engagement, effort and time with a problem. Echoing Allan, understanding was at the heart of his view of the profession: "I guess I decided to be a mathematician just you know, for the career, to understand what is going on". Mostly, he felt he developed statisfactory levels of understanding.

But the time and engagement necessary to achieve understanding made communication mathematics problematic. When Brett first presented the main ideas for his thesis, his advisor rejected them, "well this, this is not something, eh, that is good on [*sic*], to work on, it doesn't work". Some weeks later his advisor's reaction was quite different: "he said, oh, this is really nice ... this is very good". Brett felt his advisor's initial failure to appreciate his ideas was "mostly accidental", paralleling the accidental emergence of his own ideas. [11] In the field more generally, understanding rested with a small group of experts in each subfield: "there are always some people who understand". Other mathematicians believed - but did not understand the result - if the small group working on the problem did.

More importantly, understanding did not always emerge from Brett's focused attention on important results. Unlike Charles, logical conclusions sometimes defied his understanding when key procedures ("machinery") were opaque and obscured the argument. For example, he complained about the difficulty of understanding the general method for taking derivatives on manifolds. He found he could not relate the argument to the result. In these situations, he developed his own personal alternatives to the ideas and approach in the proof. These provisional mathematical objects were similar but not identical to the ideas in the accepted proof. He recognized that they were flawed.

> I think I do understand rationals, so I just use the general rules. I don't have to create something that I do understand. But there are things that I don't understand the way the people usually does [*sic*]. In high math there are several things that I don't really understand. So what I do, I create some way that I can understand. Maybe it is not the right way, maybe the way I am thinking might have some gaps in it, it's not correct, but at least it makes that object something that I can imagine.

Given his specialty, Brett's diverse references to 'objects' were not surprising. In addition to his intriguing provisional objects, he discussed objects of everyday experience in ideal conceptual form, e.g. spheres and tori. His solutions of the rational number problems referenced physical-like objects like "pies" and divided "quantities". He also referred to the standard concepts of the field, from elementary objects such as quotients and their terms (e.g. natural numbers and polynomials) to the more abstract objects topologists defined and studied.

Lessons about practice and education

These dialogues taught us two main lessons about the views and practices of novice mathematicians. First, it is a mistake to consider "mathematicians' practice" as unified and singular. Beyond a shared commitment to public, deductive proof, these young mathematicians held quite different views of the nature of mathematical knowledge, how it is generated, the role of deduction and intuition in understanding, and how initial ideas develop into arguments. Their chosen fields of specialization may have been one important factor underlying these differences. Charles' mathematical experience in real analysis closely paralleled published arguments (the 'definition - theorem - proof model of mathmatics' in Thurston's (1995) terms). His ideas were created by definition; his understandings were embodied in written arguments. Brett, the topologist, constructed ideas prior to their clear definition, some of which he knew were flawed. His understanding did not always follow the path of accepted proof.

The three portraits suggest that there is as much diversity as commonality in mathematicians' working practice and further work is needed to explore and map this diversity. Some focus should be given to the informal dimensions of that practice, including the intuitive origins of conjectures and the potentially problematic nature of understanding. While mathematicians may be no more gifted than other people in clarifying their personal meanings for notions like 'intuition' and 'understanding', skillful questioning in interview settings - with the appropriate use of examples and contrasting positions - can draw out their views. Preliminary studies such as this provide such examples and positions.

The second result concerns the interesting tension around the role of conjecture in mathematics. Though all three participants indicated that guessing played an essential role in their research, Charles' and Allan's reactions to the students' conjectures about order among fractions were strongly negative, as if justification completely overshadowed discovery. (Charles' single instance of positive evaluation, "they're guessing, which is good", was the only exception to this pattern.) They focused on the absence of rigorous argument to support the conjectures. Charles was dismissive because the definitional 'tools' he valued were not used. Allan's reaction was so strong that he mistakenly claimed that the statement of the Increasing Sequence Strategy was false. Only Brett balanced his doubts about the logic of the arguments with genuine interest in the claims themselves. If these three beginners are at all representative of their profession, mathematicians' stances toward conjectures can be highly context-dependent: guessing by mathematicians may be viewed quite differently from guessing by students. Two questions arise about this mismatch:

(i) what is important about it?

(ii) what might explain it?

We take the second issue first.

The difference in reaction becomes sensible if we presume that two types of guessing were discussed in these dialogues. What participants described in their accounts of their practice was *disciplined guessing*. It was both part of the broader practice of doing mathematics and a specific result of extended immersion in their own questions and ideas. Most important, making good guesses was only the entrée to the main cycle of attempting to prove the result, revising and attempting again. In contrast, the students' strategies were taken as *undisciplined guessing*. There were leaps of intuitive insight that did not indicate deep engagement with the content and did not necessarily lead to rigorous examination and verification. They did not suggest an informed respect for the particular nature of mathematical claims. Charles and Allan's reactions may have been negative because they felt they needed to assert their discipline's focus on proof. Undisciplined guessing was simply not mathematical.

This distinction, if it can account for the different stances toward guessing in these interviews, may also illuminate one important way that university teachers of mathematics (which all participants were) and their students fail to connect in classrooms. Most college teaching involves presenting content that mathematicians have understood very well for some time. The cycle of conjecture, proof attempt and revision goes on elsewhere, in very different mathematical content, but not in calculus. Students in introductory courses neither see nor are encouraged into the practice of disciplined guessing. When they find the courage to guess themselves, they may not be encouraged - as the students who generated the Increasing Sequence and Betweeness Strategies would not have been - because their guesses look (and likely are) undisciplined. If guessing and the resulting cycle of inquiry does not become visible to students, they are left with only public mathematics - the carefully crafted propositions and polished arguments they see in their texts. They miss entirely the stumbling human process that created those results in the first place. When guessing is not exemplified and supported, most students find it more difficult to understand and enter the practice of doing mathematics.

This is not to suggest that undisciplined guesses should be accepted as disciplined. To do so would be to misrepresent radically the nature of mathematics. Conjecturing is only the first step in the cycle of inquiry and verification, not a substitute for it. But since few will become serious students of mathematics, our goal as educators should be to draw all students closer to these mathematical practices and experiences, at least for a time. This goal is better served by encouraging and building on students' conjectures, thereby drawing them into the need for proof, rather than rejecting their guesses if they fail to provide rigorous argument. Undisciplined guessing can become the foundation for disciplined conjecture, inquiry, and proof.

Notes

[1] There are numerous biographical and autobiographical 'portraits' of mathematicans (e.g. Albers and Alexanderson, 1985; Halmos, 1986; Weil, 1992), but this work generally does not explore the psychology of mathematical work, expecially the generation of new ideas and results

[2] 'Collective intuitions' are developed by and shared among mathematicians working in a common subfield

[3] Our preliminary work indicated that experienced mathematics faculty were harder to interview, in part because of the demand on their time

[4] The names of participants are all pseudonyms

[5] The reasoning involved generative leaps beyond the problem as given that were not analytic, relatively compelling to students who proposed them and quite different from the standard solution methods

[6] The length of Allan's interview was constrained by his time commitments.

[7] The authors generated these names for the strategies

[8] The positive element in Charles' evaluation was not elaborated. The balance of his assessment was negative in tone, emphasizing what students were not doing

[9] The interviewer could have asked how Allan thought about the conjecture itself, before turning to the issue of its validity, but did not. His question nearly assured a negative response. Yet Allan still found a way to engage with the substance of the conjecture.

[10] We use the term 'physical-like' here to distinguish graphs and divided quantities from the everyday physical objects such as tables and chairs. By 'divided quanties', we mean common geometrical shapes (circles, rectangles and squares) that are partitioned into specific numbers of equal-sized parts and are seen to represent other everyday objects that may be partitioned.

[11] Though Brett did not make the connection explicitly, his advisor's inability to see the promise in his ideas was consistent with Brett's view of the time and immersion required for work on specific problems

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