PROOF PROBLEMS WITH DIAGRAMS: AN OPPORTUNITY FOR EXPERIENCING PROOFS AND REFUTATIONS

KOTARO KOMATSU, YOSUKE TSUJIYAMA, ARUTA SAKAMAKI, NORIO KOIKE

In many countries, students encounter proof and proving within geometry at junior or senior high schools. Proof problems with diagrams are standard tasks in the sense that most proof problems in school geometry include diagrams that indicate the meanings of the problems. A proof problem with diagrams is a problem in which a statement is described with reference to particular diagrams with symbols (one diagram in most cases) and solvers are required to prove the statement. An example of such a problem is as follows:

> As shown in Figure 1, in isosceles triangle ABC (AB = AC), we draw perpendicular lines BD and CE to sides AC and AB from points B and C, respectively. Prove that AD = AE.
>
> Proof (summary):
>
> In ΔABD and ΔACE, $\angle$ADB = $\angle$AEC = 90˚, AB = AC, $\angle$BAD = $\angle$CAE (common).
>
> Hence, ΔABD $\equiv$ ΔACE and therefore AD = AE.

Lakatos insisted that "informal, quasi-empirical, mathematics […] grow[s] through […] the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations" (Lakatos, 1976, p. 5). He described processes of mathematical development through the rational reconstruction of actual histories of the Descartes-Euler conjecture on polyhedra and uniform convergence, discussed more fully later in the article. Lakatos's ideas can form a basis for authentic learning that mirrors a process of mathematical progress through conjectures, proofs, and refutations (Lampert, 1990). In this article, we show how a specific type of proof problem with diagrams functions as an opportunity for students to experience certain aspects of proofs and refutations.

Proof problems with diagrams

Contrary to general propositions in mathematics, a proof problem with diagram shows a statement according to the attached diagram. In Euclid's *Elements*, for example, propositions are stated in general and abstract words without any symbols or diagrams, and diagrams only appear in the phase of proving the propositions. On the other hand, in school geometry, students are usually given proof problems for which diagrams and symbols have already been prepared (Herbst & Brach, 2006). This is generally because statements can be described more simply and clearly with a combination of diagrams and symbols. Diagrams also make it possible to visually grasp the meanings of problem statements (Herbst, 2002).

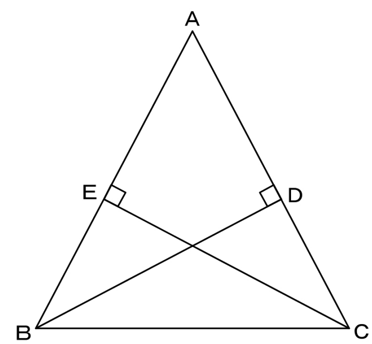

Figure 1. A diagram attached to a proof problem.

There are two interpretations of a proof problem with a diagram. The first is that the problem questions whether the statement is true only in the attached diagram. The second is that the problem deals with a certain general class to which the attached diagram belongs. In the above example, the first interpretation asks for proof that AD = AE only for the triangle in Figure 1; the second interpretation considers many possible shapes of isosceles triangle ABC, and asks whether segment AD is equal to segment AE for these shapes (see Figure 2).

In this article, we focus on the second interpretation because it enables us to achieve a process of proofs and refutations. For example, in the above problem, if we draw various shapes of isosceles triangle ABC to verify whether AD = AE in all of these isosceles triangles, we can find a case in which perpendicular lines from points B and C do not intersect with sides AC and AB, respectively (Figure 3). In a mathematical sense, this case is not a counterexample, but a non-example. The above problem assumes, though implicitly, that these perpendicular lines and sides intersect with each other and segments AD and AE can be constructed (thus, its class of objects is acute isosceles triangles). Under this assumption, the problem statement claims that these two segments are equal. A counterexample satisfies the assumption of a proposition but not the conclusion, while a non-example does not satisfy the assumption. Therefore, Figure 3 should be regarded as a non-example rather than a

For the Learning of Mathematics **34**, 1 (March, 2014)
FLM Publishing Association, Fredericton, New Brunswick, Canada

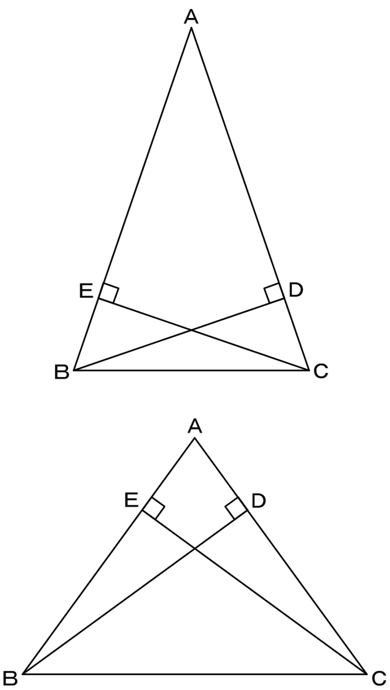

Figure 2. *Different shapes of isosceles triangle ABC.*

counterexample because this diagram does not satisfy the above assumption of the problem statement.

Thus, if the second interpretation of a proof problem with diagrams is adopted, a class of objects in a problem may become vague as it is sometimes inevitable that the attached diagram includes an implicit assumption. This situation is quite different from general propositions in mathematics, in which it is required to make explicit the class of objects in the proposition by specifying its assumption (Jahnke, 2007). However, this vagueness may enable students to find counterexamples and non-examples by transforming the diagram, to make hidden assumptions explicit, and to look for new statements that hold for these counterexamples and non-examples. In the following, we discuss such mathematical activity with reference to a specific action in *Proofs and Refutations* (Lakatos, 1976).

Increasing content by deductive guessing

Lakatos's research is well known in the mathematics education research community, and some researchers explicitly refer to the mathematical actions in *Proofs and Refutations* (Lakatos, 1976). For instance, Balacheff (1991), Reid (2002), and Yim, Song and Kim (2008) deal with various students' processes regarding conjectures, proofs, and refutations, and parts of the students' responses to counterexamples are described as "monster barring" or "exception barring". Larsen and Zandieh (2008) construct a framework that consists of monster barring, exception barring, and "proof analysis" (lemma incorporation), and they argue that this framework can not only serve as a description and explanation of students' mathematical activity, but also as a tool for designing instruction to support guided reinvention. Monster barring, exception barring, and lemma incorporation relate to the restriction of a conjecture's domain to exclude its counterexamples.

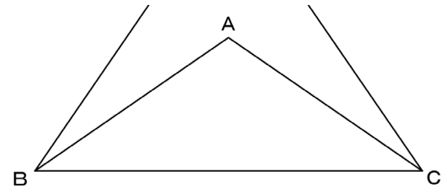

Figure 3. *A non-example.*

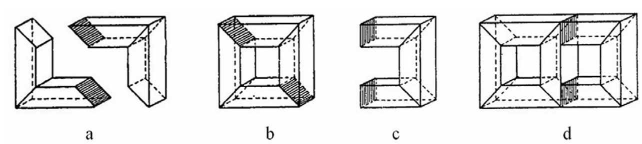

Figure 4. *Increasing content by deductive guessing (slightly modified from Lakatos, 1976, p. 77).*

In contrast, we address Lakatos's (1976) idea of "increasing content by deductive guessing" (p. 76) with which mathematics education researchers have not yet explicitly dealt. Lakatos meant that, after proving a conjecture and facing its counterexamples or non-examples, one deductively invents a more general conjecture that holds even for these counterexamples or non-examples. This action is quite different from monster barring, exception barring, and lemma incorporation because it is a method for extending the domain of a conjecture. Our research places more emphasis on this action, although we align with Larsen and Zandieh's research in that we approach instructional design from a viewpoint of task design [1].

Lakatos illustrated the idea of increasing content by deductive guessing with the Descartes-Euler conjecture on polyhedra, expressed as $V - E + F = 2$, where V, E, and F are the numbers of vertices, edges, and faces of polyhedra, respectively. First, a certain polyhedron "picture frame" (Figure 4b) was proposed as a counterexample that refuted this conjecture, as in this case, $V - E + F = 0$. However, we can use two polyhedra (Figure 4a) to construct the picture frame by pasting them together (Figure 4b). Because each value of $V - E + F$ of the two polyhedra in Figure 4a is 2, the sum of the values is 4. However, the above pasting results in the four pasted faces vanishing, and decreases the number of vertices, so that it is the same as the number of edges. This examination explains why $V - E + F = 0$ for the picture frame. Furthermore, by pasting the polyhedron shown in Figure 4c to the picture frame, and considering the increase and decrease in the numbers of vertices, edges, and faces, we can similarly reason that $V - E + F = -2$ for polyhedra with two holes (Figure 4d). Continuing this examination leads to a more general conjecture that $V - E + F = 2 - 2n$ for polyhedra, with n holes and with all their faces simply-connected [2].

The notion of increasing content by deductive guessing is applicable not only to counterexamples, but also to non-examples. In fact, in the story of *Proofs and Refutations,* although the picture frame was at first a counterexample of the primitive conjecture that $V - E + F = 2$ for all polyhedra, the primitive conjecture was then restricted to simple polyhedra through lemma incorporation. Through this restriction, the picture frame became a non-example of the restricted conjecture because it was not a simple polyhedron. After that, a more general conjecture was invented by deductive guessing, so that the invented conjecture could include the picture frame as an example.

There are two characteristics of increasing content by deductive guessing. The first is related to "increasing content", that is, the product of invention. A new invented conjecture is more general than the previous one in that it holds even for counterexamples or non-examples of the previous one. In the above example, the new conjecture is true for the picture frame that was considered as a counterexample of the primitive conjecture, and it also generally explains polyhedra with *n* holes that were not initially considered. The second characteristic is related to "by deductive guessing", that is, the method of invention. A new conjecture is produced by a deductive method, though there are different types of mathematical reasoning, such as induction and analogy. In the same example, the new conjecture was not produced inductively by counting the numbers of vertices, edges, and faces in various polyhedra, observing these numbers, and finding their regularity. Instead, it was invented deductively by reasoning from polyhedra whose values of $V - E + F$ were already known, pasting together these polyhedra, and considering the change in the numbers of vertices, edges, and faces. When Lakatos considered increasing content by deductive guessing, he thought about producing statements that were difficult to find non-deductively.

Increasing content by deductive guessing can be seen as an important action for mathematical progress. In fact, Lakatos formulated this action as one of his *heuristic rules.* A heuristic is both evaluative and normative for Lakatos, in that "the heuristic-methodology looks *backward* to identify the rules that made such a growth possible in the past, and at the same time it looks *forward* to advise on how to obtain progress in the future" (Motterlini, 2002, p. 34, emphasis in original). Thus, we can see increasing content by deductive guessing as a method that has enabled the development of mathematics in the past, and as a compass indicating a direction for expanding mathematics in the future. In particular, Lakatos considered this action to be a leitmotiv of proofs and refutations because it represents humans' brave attempts to overcome counterexamples by inventing a more general conjecture (Stöltzner, 2002).

It may not be appropriate to directly introduce the notion of increasing content by deductive guessing into school mathematics because Lakatos's main interest lay in describing a process of growth in the discipline of mathematics and there is a great difference between school mathematics and mathematical research. For example, in school mathematics, the domain of geometry functions as the main opportunity for students to learn about proof and proving. Due to such educational contexts, some characteristics of increasing content by deductive guessing might not match with school geometry. In such cases, it may be necessary to elaborate a modified version of this action for school geometry. In the next section, we discuss such modification with reference to proof problems with diagrams.

Extending statements in proof problems with diagrams

Proof problems with diagrams are similar to the Descartes-Euler conjecture on polyhedra in *Proofs and Refutations* (Lakatos, 1976), in which the obscure meanings of polyhedra led to a variety of counterexamples, opening up mathematical possibilities for refinement of the conjecture through continuous interactions between proofs and refutations. A class of objects in a proof problem with a diagram may also be vague because, as illustrated earlier, the attached diagram may include an implicit assumption. We wish to capitalize on this vagueness of proof problems with diagrams to provide students with an opportunity to experience proofs and refutations.

It may not, however, be appropriate to directly adopt the second characteristic of Lakatos's idea of increasing content by deductive guessing as mentioned in the previous section. In the problem shown at the start of this article, for example, some students would first extend sides AC and AB to produce intersection points D and E with the perpendicular lines from points B and C, respectively (Figure 5). They then would conjecture that AD = AE even in this case. At this point, they may make this conjecture not deductively by using the previous proof [3], but perceptually and inductively from the appearance of Figure 5, or analogically from the case of acute isosceles triangles (Figure 1). Considering that deductive reasoning is abstract and difficult for many students, it is a merit of proof problems in geometry that students are able to visually grasp the meaning of the statement and the deductive process. Thus, by changing "sides AC and AB" in the problem statement to "lines AC and AB" or "rays CA and BA", it is possible to invent a more general statement that holds even for obtuse isosceles triangles. However, this invention can be performed not only by a deductive method, but also by inductive or analogical methods.

For this reason, we focus on the first characteristic of increasing content by deductive guessing in order to achieve proofs and refutations in proof problems with diagrams. In other words, we address an action to extend a statement so that a new statement can include counterexamples and non-examples of the primitive statement without restricting how this extension is performed. In the next section, we describe an episode from an eighth grade (13–14 years old) classroom to demonstrate that proof problems with diagrams can function as an opportunity for students to experience this extension.

An episode: parallelograms, diagonals and perpendiculars

The episode is taken from our larger study that aims to develop, through design experiments, a set of tasks and associated teachers' guidance to prompt students to engage in proofs and refutations (Komatsu & Tsujiyama, 2013). The third author taught a sequence of three lessons (50 minutes per lesson) to 35 eighth graders in a junior high school in Japan (all names are pseudonyms). We were all involved in the lesson design, and the first author observed all of the lessons.

The students had learnt to prove geometric statements related to various properties of triangles and quadrilaterals, using conditions for congruent triangles. They had also learnt the hierarchical classification of quadrilaterals where

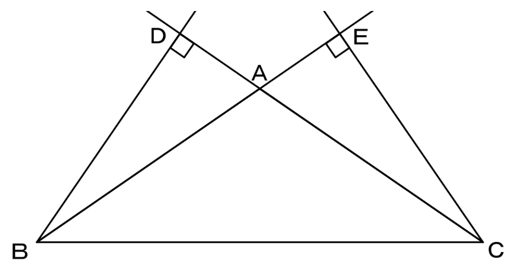

Figure 5. Extension of the statement.

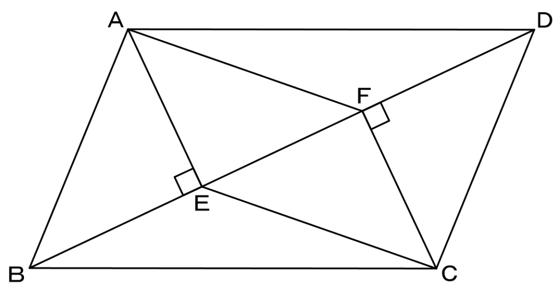

Figure 6. A diagram attached to a proof problem in the episode.

a square, rhombus, and rectangle are special cases of a parallelogram.

Primitive statement and its proofs

The first lesson treated a primitive statement and its proofs. We describe the first lesson only briefly, because the focus of this article is on the students' processes after proof construction. The teacher first proposed the following problem:

> In parallelogram ABCD (Figure 6), we draw perpendicular lines AE and CF to diagonal BD from points A and C, respectively. Prove that quadrilateral AECF is a parallelogram.

As discussed earlier, we interpret this problem to deal with not only the particular shape drawn in Figure 6, but also other shapes of parallelogram ABCD.

After planning how to solve this problem, the students worked individually or cooperatively for about twenty minutes. Meanwhile, the teacher had a student, Emi, write her proof on the blackboard, and then explain it to other students.

> Emi's Proof (summary [4]):
>
> In ΔABE and ΔCDF,
>
> $\angle AEB = \angle CFD = 90^\circ \ldots (1)$
>
> $AB = CD \ldots (2)$
>
> $\angle ABE = \angle CDF \ldots (3)$
>
> From (1)–(3), $\Delta ABE \equiv \Delta CDF$, and therefore $AE = CF \ldots (4)$
>
> From (1), $\angle AEF = \angle CFE = 90^\circ$ and therefore $AE // CF \ldots (5)$
>
> From (4) and (5), quadrilateral AECF is a parallelogram.

In their worksheets, nineteen students had written the same proof as Emi's, and four students had instead showed the congruence of triangles ADE and CBF to deduce AE = CF; four students wrote out both proofs [5]. Another three wrote different valid proofs, and the remaining students could not reach full proofs or wrote incorrect proofs.

Non-examples

The teacher started the second lesson by questioning whether quadrilateral AECF was still a parallelogram, even if the shapes of parallelogram ABCD were different from the attached diagram (Figure 6). Many students answered yes; only Naoki responded that it might be not be in some cases. The students then started to draw various shapes of parallelogram ABCD to investigate the teacher's question. After that, the teacher asked whether anyone had found an impossible case. Many students gave a square and rhombus, as they could not produce quadrilateral AECF in these figures because points E and F coincided on the intersection point of the diagonals of parallelogram ABCD (Figure 7, top and middle, overleaf) [6]. These students seemed to consider these figures as special cases of a parallelogram, based on their understanding of the hierarchical classification of quadrilaterals.

Next, Rie drew the parallelogram ABCD, shown in Figure 7 (bottom), on the blackboard and stated that:

> This is a very thin parallelogram. I drew diagonal BD, and wanted to draw a perpendicular line from here [point A]. But, if I draw the perpendicular line [the dotted line in Figure 7, bottom], it does not go into the parallelogram, and point E appears here [on the extended line of diagonal BD]. If I draw from here [point C], point F appears here [on the extended line of diagonal BD]. So, I do not think the parallelogram is made.

As mentioned in the example about isosceles triangle in the previous section, the three cases in Figure 7 are not counterexamples but non-examples, since they do not satisfy the assumption of the primitive problem; this problem implicitly assumes that it is possible to construct quadrilateral AECF, and under this assumption, it claims that quadrilateral AECF is a parallelogram.

The classroom discussion then focused on the case in Figure 7 (bottom). The following are some representative comments by students.

> *Ken:* In Rie's diagram, it is first of all impossible to draw [perpendicular lines] to diagonal BD. It is impossible to make the quadrilateral [AECF] itself. If BD is not a diagonal but a line, it would extend without limit, so I think it is possible to make the quadrilateral, it is possible to make the parallelogram. But, because of diagonal BD, I think it is indeed impossible.

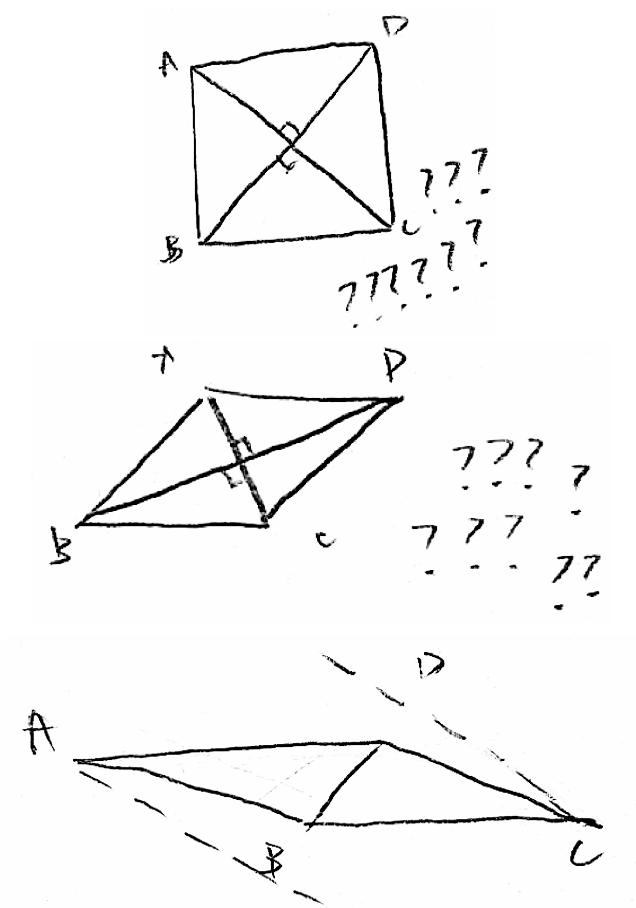

Figure 7. *Students' non-examples.*

Makoto: Maybe Ken meant that a diagonal is a segment. We can't extend a diagonal. I think it becomes possible if we extend [diagonal BD], but [I am puzzled] whether we can extend it.

Asuka: I think similar things to them [Ken and Makoto]. I think that if we extend diagonal BD, we can make perpendicular lines. Then, the parallelogram [AECF] is made, which is bigger than the parallelogram [ABCD].

Like Rie, Ken stated that it was impossible to construct quadrilateral AECF in the case of Figure 7 (bottom). He mentioned the possibility of extending diagonal BD, but finally came to have the same idea as Rie. Makoto also mentioned this possibility, but was unsure whether extending the diagonal was permitted. In contrast, Asuka argued strongly for extending diagonal BD (Figure 8).

Extending the statement

The third lesson dealt with a class of non-examples in which the perpendicular lines from points A and C did not intersect with diagonal BD. In the beginning, the students independently drew various diagrams of the non-examples in their worksheets (Figure 9) and, therefore, considered not one diagram, but a general class of objects in which the perpendicular lines intersected with line BD but not diagonal BD.

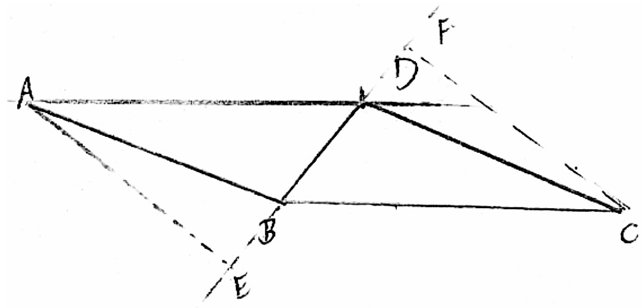

Figure 8. *Extension of diagonal BD.*

The subsequent classroom discussion was as follows:

Teacher: If we draw perpendicular lines to the extended line of BD, we can make quadrilateral AECF. In this case, what is quadrilateral AECF?

Students: Parallelogram.

Teacher: Really? I can't see this as parallelogram.

Student: (*Uncertain*) Yes [parallelogram].

Teacher: For example, does it result in a quadrilateral whose condition is stronger?

Student: (*Uncertain*) Rhombus.

Teacher: Yeah, is there any case in which this becomes rhombus?

Student: (*Uncertain*) Square.

Teacher: Someone said square. Is there any case which gives square? If we think so, what this [quadrilateral AECF] becomes is still unknown. [7]

In this exchange, the students initially conjectured that quadrilateral AECF would become a parallelogram for the case of Figure 9. The teacher then said that the quadrilateral might be a more special parallelogram, and the students answered that it might become a rhombus or square. In response to their answers, the teacher stated that what type of quadrilateral AECF could be was still unknown.

The students then investigated this teacher's question. During this investigation, the teacher mentioned Emi's proof from the first lesson, and told the students that they could utilize this proof or construct a new proof from the beginning. After about fifteen minutes, the teacher had Satoshi write his idea on the blackboard.

Satoshi's proof (summary):

In ΔAEB and ΔCFD,

$\angle AEB = \angle CFD = 90^\circ$ … (1)

AE // CF … (2)

AB = CD … (3)

$\angle ABD = \angle CDB$ … (4)

$180 - \angle ABD = 180 - \angle CDB$ … (5)

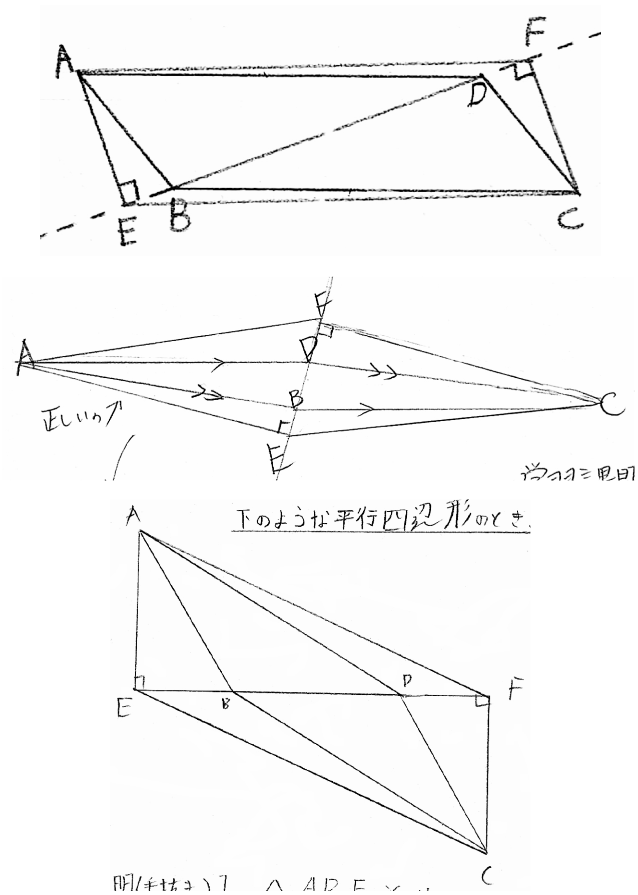

Figure 9. *Quadrilateral AECF constructed by the extension of diagonal BD.*

$\angle ABE = \angle CDF$ … (6)

From (1), (3), and (6), $\Delta AEB \equiv \Delta CFD$, and therefore $AE = CF$ … (7)

From (2) and (7), quadrilateral AECF is a parallelogram.

Satoshi said that he constructed his proof from the beginning without utilizing Emi's proof. Then, the teacher asked whether any student used a different method to Satoshi, and Maika answered:

> I made [a proof for the case in Figure 9] by utilizing the proof [by Emi] for the previous diagram [Figure 6], where E and F were inside the parallelogram [ABCD]. Only the positions of alternate angles were different, and the others were all the same. It does not matter whether E and F are inside or outside, and I think [quadrilateral AECF] still becomes a parallelogram.

Thus, Maika intended to utilize Emi's proof by changing only the part that was not applicable to the case in Figure 9 and by applying the other parts directly to this case. When deducing the congruence of triangles AEB and CFD, it is necessary to change the part that shows the congruence of angles ABE and CDF. However, Maika wrote on her worksheet "from alternate angles, $\angle FDC = \angle EBA$". That is, although she noticed the necessity of changing this part, she was not able to modify this appropriately.

At the end of the third lesson, the teacher said, "This is the problem in the first lesson, but you previously said that you could not produce this figure [quadrilateral AECF] in this case [Figure 7, bottom]. Now, how should we change this problem so that it can include this case?" Emi and Daisuke answered that it was sufficient to change diagonal BD in the first lesson's problem to line BD. Thus, the students were able to extend the primitive problem and to invent a more general statement such that in parallelogram ABCD, if quadrilateral AECF is constructed by drawing perpendicular lines AE and CF to *line* BD from points A and C, respectively, this quadrilateral is a parallelogram. As mentioned previously, they considered a general class of objects rather than one diagram, and therefore found that this statement held for cases in which quadrilateral ABCD was a parallelogram in general, with the exception of a square and a rhombus.

Concluding remarks

Lakatos wrote his book *Proofs and Refutations* to show that mathematics progressed gradually with problems, conjectures, proofs, and refutations, and he regarded increasing content by deductive guessing as an important mathematical action for this progress. In this article, we have shown how proof problems with diagrams can function as an opportunity for students to experience a process of proofs and refutations, and so contribute to the introduction of authentic mathematical activity in regular classrooms based on standard tasks in school geometry. Of course, we do not intend to claim that all proof problems with diagrams guarantee the achievement of proofs and refutations. It is necessary to utilize suitable problems in which counterexamples or non-examples can be found by changing the location or shape of the attached diagram.

Among the characteristics of increasing content by deductive guessing, we have focused on the extension of a conjecture without emphasizing how this extension is performed. The task in our episode had the possibility of an opportunity in which students could deductively extend their previous conjecture. For example, at the end of the third lesson, Daisuke wrote on his worksheet, "I could determine the uncertain shape *through proving*" (our emphasis). This comment showed that Daisuke engaged in increasing content by deductive guessing. Nevertheless, many students did not rely on deductive methods, such as utilizing their previous proofs, when they first made the extended conjecture. Hence, it may be that our task actualizes quite an ordinary process of conjecture-by-staring-at-the-diagram to proof. However, a feature of our task is its potential to enrich students' activities *after* proving a primitive statement, and a proof problem with a diagram can facilitate students' attempts to refute the primitive statement by discovering its counterexamples and non-examples and to overcome this refutation by inventing a more general statement.

Finally, we mention two implications for research. First, the episode we have described illustrates a difficulty that students may face during the extension of a conjecture. In the third lesson, if the students had utilized Emi's proof constructed in the first lesson, it would have been easier to prove that quadrilateral AECF in Figure 9 was a parallelogram, because it was sufficient to change only the part that showed the congruence of angles ABE and CDF [8]. However, few students reflected on Emi's proof, and even Maika, who mentioned her attempts to use this proof, was unable to modify it appropriately. This episode implies a necessity to develop an instructional strategy that can lead students to prove an extended conjecture more efficiently.

Second, although the students in our episode extended the primitive problem by non-deductive methods, it is still unclear how and why they did not use a deductive method. We have some hypotheses about the students' behavior. They might think that if they could construct quadrilateral AECF as shown in Figure 9, the quadrilateral would automatically become a parallelogram. In other words, they might not be able to differentiate between whether they could make quadrilateral AECF by extending diagonal BD and whether that quadrilateral became a parallelogram. The students also might make their conjecture analogically, according to their feeling of some similarity and continuity between Figures 6 and 9, or empirically, from the appearance of Figure 9.

Mathematics education researchers have deepened their understanding of students' behaviors about restricting a conjecture to exclude its counterexamples by, for example, describing the students' behaviors or constructing frameworks that enable this description (*e.g.,* Balacheff, 1991; Larsen & Zandieh, 2008). In the future, it will be necessary to construct a framework to describe and analyze how students invent a conjecture that holds for previous counterexamples and non-examples in order to design an instructional approach based on students' existing strategies.

Notes

[1] Some researchers have dealt with generalizing a statement by its proof though they do not refer to increasing content by deductive guessing. See, for example, De Villiers (1990), Hanna and Jahnke (1996) or Miyazaki (2000).

[2] Though Lakatos introduced the concept of normal polyhedra when describing this process, we omit it for simplicity.

[3] It is possible to find this conjecture by changing the reason for the congruence of angles BAD and CAE from the common angles to vertical angles.

[4] Though Emi wrote all the reasons in her proof, we omit them here for simplicity. We do the same for Satoshi's proof, shown later.

[5] In the lesson, Kaori also wrote her proof on the blackboard that deduced the congruence of triangles ADE and CBF. Because we gathered the students' worksheets after the first lesson, some students might have copied the proofs by Emi or Kaori.

[6] All diagrams in Figures 7–9 are the students' actual drawings.

[7] Though the teacher's comments in this excerpt make sense in a natural language sense, they do not make sense in a mathematical sense, because despite the hierarchical classification of quadrilaterals, the teacher denied the possibility of a parallelogram and accepted the possibility of a rhombus or square.

[8] Kaori's proof could be applied directly to Figure 9 without any modification.

Acknowledgements

We are grateful to the anonymous reviewers for their valuable suggestions on earlier drafts of this article. This research is supported by Grants-in-Aid for Scientific Research (No. 23330255, 24243077, 24730730, and 24830017).

References

Balacheff, N. (1991) Treatment of refutations: aspects of the complexity of a constructivist approach to mathematics learning. In von Glasersfeld, E. (Ed.) *Radical Constructivism in Mathematics Education,* pp. 89-110. Dordrecht, The Netherlands: Kluwer Academic Publishers.

De Villiers, M. (1990) The role and function of proof in mathematics. *Pythagoras* **24**, 17-24.

Hanna, G. & Jahnke, H. N. (1996) Proof and proving. In Bishop, A.Y., Clements, M. A., Keitel, C., Kilpatrick, J. & Laborde, C. (Eds.) *International Handbook of Mathematics Education,* pp. 877-908. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Herbst, P. (2002) Engaging students in proving: a double bind on the teacher. *Journal for Research in Mathematics Education* **33**(3), 176-203.

Herbst, P. & Brach, C. (2006) Proving and doing proofs in high school geometry classes: what is it that is going on for students? *Cognition and Instruction* **24**(1), 73-122.

Jahnke, H. N. (2007) Proofs and hypotheses. *ZDM* **39**(1-2), 79-86.

Komatsu, K. & Tsujiyama, Y. (2013) Principles of task design to foster proofs and refutations in mathematical learning: proof problem with diagram. In Margolinas, C. (Ed.) *Task Design in Mathematics Education: Proceedings of ICMI Study 22,* pp. 471-479. Oxford, UK: ICMI.

Lakatos, I. (1976) *Proofs and Refutations: The Logic of Mathematical Discovery.* Cambridge, UK: Cambridge University Press.

Lampert, M. (1990) When the problem is not the question and the solution is not the answer: mathematical knowing and teaching. *American Educational Research Journal* **27**(1), 29-63.

Larsen, S. & Zandieh, M. (2008) Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics* **67**(3), 205-216.

Miyazaki, M. (2000) What are essential to apply the "discovery" function of proof in lower secondary school mathematics? In Nakahara, T. & Koyama, M. (Eds.) *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education,* vol. 4, pp. 1-8. Hiroshima, Japan: Hiroshima University.

Motterlini, M. (2002) Professor Lakatos between the Hegelian devil and the Popperian deep blue sea. In Kampis, G., Kvasz, L. & Stöltzner, M. (Eds.) *Appraising Lakatos: Mathematics, Methodology and the Man,* pp. 23-52. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Reid, D. A. (2002) Conjectures and refutations in Grade 5 mathematics. *Journal for Research in Mathematics Education* **33**(1), 5-29.

Stöltzner, M. (2002) What Lakatos could teach the mathematical physicist. In Kampis, G., Kvasz, L. & Stöltzner, M. (Eds.) *Appraising Lakatos: Mathematics, Methodology and the Man,* pp. 157-188. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Yim, J., Song, S. & Kim, J. (2008) The mathematically gifted elementary students' revisiting of Euler's polyhedron theorem. *The Montana Mathematics Enthusiast* **5**(1), 125-142.