IV. Luis was sure that at least two opposite sides of the hexahedron were rectangles. What happens if we eliminate this condition in III and all the sides are simply parallelograms?

Twist a cube, keeping two opposite sides parallel.

How many sides does the new solid have?

Can we come back to the original cube by equating diagonals?

Can we deform a cube in order to get a non-convex dodecahedron?

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Three worlds and the imaginary sphere

MATTHEW INGLIS

Gray and Tall (2001) recently suggested that mathematics can be split up into 'three worlds': the embodied, the proceptual and the axiomatic. They claim that objects from each of these worlds are formed, and reasoned about, in significantly different ways. During the course of this article I consider the three world's theory and present the view that, as currently described, it is not sufficient to characterise all mathematical objects.

In doing this, I discuss the investigations of Johann Heinrich Lambert into non-Euclidean geometry, and claim that, *for Lambert*, the so-called 'imaginary sphere' does not fit comfortably into any of the three worlds. Through a discussion of an example from the classroom, I suggest that, looking from the point of view of certain learners, many objects may not satisfactorily fit into Gray and Tall's worlds.

Lambert and his imaginary sphere

Johann Heinrich Lambert, a Swiss mathematician who lived in the eighteenth century, was highly influential in his day. His scope of influence was wide, ranging from philosophical ponderings on rationalism to the proposal that the Milky Way was finite. In mathematics, he famously proved that π and e were irrational. I shall be discussing his work relating to non-Euclidean geometry (Gray, 1989, is the source for the historical detail).

Ever since Euclid published his *Elements*, in around 300BC, mathematicians had been attempting to prove his fifth postulate using only the other four. The so-called parallel postulate was noticeably more complicated (and less obviously true) than the others, leaving mathematicians down the ages dearly wishing to reclassify it from postulate to theorem. However, in the 2000 years that passed between the time of Euclid and the 1800s, nobody had managed to produce a correct proof.

Instead of trying to construct a direct proof, Lambert went about the task somewhat differently. Following the work of Saccheri, he showed that there were three possibilities.

1. If there is a triangle with angle sum < 180°, then every triangle has angle sum < 180°

2. If there is a triangle with angle sum = 180°, then every triangle has angle sum = 180° (this is standard Euclidean geometry)

3. If there is a triangle with angle sum > 180°, then every triangle has angle sum > 180°

Lambert attempted to show that cases one and three were contradictory, which would imply that Euclidean geometry - and therefore the parallel postulate - was 'true'. He managed to do this with the first case, but deriving a contradiction with the third proved more difficult. We now know that this task is impossible. Mathematicians had to wait until the work of Bolyai and Lobachevskii (in the 1820s) who paved the way for Beltrami (in 1868) to show that the third hypothesis was as consistent with Euclid's first four axioms as the second.

Although Lambert did not successfully find a contradiction, he did manage to derive some rather interesting results. In particular, he found that the area of a triangle (with angles α, β and γ) in the third case was proportional to $\pi - (\alpha + \beta + \gamma)$.

Recalling that the area of a triangle on a sphere of radius r is: $r^2(\alpha + \beta + \gamma - \pi)$, he noticed that letting $r = i$ gives: $i^2(\alpha + \beta + \gamma - \pi) = \pi - (\alpha + \beta + \gamma)$, which is the formula Lambert derived for the angle sum under the third hypothesis. He considered this for a while before making the following observation.

> I want to say: if of two triangles one has a greater area than the other then the angle sum of the first triangle is smaller than that of the other [...] I should almost therefore put forward the proposal that the third hypothesis holds on the surface of an imaginary sphere. (Fauvel and Gray, 1987, pp. 518-520)

Gray (1989) comments on the significance of this remark:

> it marks [Lambert] as a correct and inspired thinker [...] To enter the land which Lambert's vision was the first to descry was to take mathematics another hundred years. (p. 75)

It is important to emphasise here that Lambert is not defining his 'imaginary sphere' by the geometry that is true on its surface. He is pointing out that, given a sphere of radius i, the geometry on its surface would satisfy the third hypothesis. So what exactly is this imaginary sphere? Modern mathematicians would refer to such an object as a surface of constant negative curvature, but such notions were not available in Lambert's day. He lacked the mathematical tools to formalise his intriguing idea. If Lambert is seen as a learner striving to understand mathematics, how would Gray and Tall's theories apply?

The three worlds of mathematics

What exactly are mathematical objects? How do learners construct them? Mathematicians appear to have no problems referring to, and thinking about, highly abstract constructions as objects; but what exactly are these things? Tall *et al.* (1999) propose a simple and elegant solution:

> We therefore suggest that the total cognitive structure of the concept image of number, with its power to *manipulate* the symbols and to think of their properties, gives number its most powerful status as an object. What matters more is not what it *is* but what we can *do* with it. (p. 229)

In effect, what is being said here is that it does not really matter whether the number 3 exists or not. The point is, that when you are doing mathematics, you think about it, and you do things with it as if it were an object. This alone is enough to say that it actually *is* an object.

Dubinsky (1991) proposed that mathematical objects are formed during a four-part process known as APOS (Action, Process, Objects and Schema). An action becomes a process when the individual concerned can "describe or reflect upon all the stages in the transformation without necessarily performing them," and a process becomes an object when "the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations" (Cottrill *et al.*, 1996, p. 171).

The APOS theory was later modified to allow for the possibility that objects can also be formed through the encapsulation of schemas - coordinated collections of objects and processes. Sfard (1991) proposed a somewhat similar idea. She drew a distinction between 'structural' and 'operational' conceptions, describing how a process could be solidified into a static object through reification in a manner akin to Dubinsky's encapsulation.

Although these theories work well for large parts of mathematics, it has been pointed out that it is difficult to see how geometrical and axiomatic mathematics can be created through encapsulation. Where are the processes involved in understanding, say, Euclidean geometry (Tall, 1999)? To deal with this problem, Gray and Tall (2001) proposed the 'three worlds of mathematics':

> For several years now [...] we have been homing in on three [...] distinct types of concept in mathematics. One is the *embodied object* [...] Another is the *symbolic procept* [...] The third is an *axiomatic concept* in advanced mathematical thinking. (p. 70)

So, Gray and Tall would have us believe that there are three different types of mathematical object [1]:

> *Embodied objects* are those we can perceive and that are direct abstractions from our perceptions. School-level geometry lives in this world.
>
> *Proceptual objects*, or procepts, are represented by a symbol which can evoke flexibly either a process or an object. School-level algebra lives in the proceptual world, and many concepts from university analysis can also be found here e.g. What are the results of the sum $\sum_{n=1}^{\infty} \frac{1}{n^{a}}$ and the limit $\lim_{x\to} f(x)$? They are the real numbers $\sum_{n=1}^{\infty} \frac{1}{n^{a}}$ and $\lim_{x\to} f(x)$. The notation for the process is the same as the product of the process (Gray and Tall, 1994).
>
> *Axiomatic objects* are formed using axiomatic definitions and reasoned about using logical proof. An example would be a topology; if an object meets the axioms in the definition of a topology, then it *is* a topology and all the theorems about topologies apply to it as well. Most of advanced mathematics studied and researched at university lives in this world.

The 'three worlds' theory is a neat solution to the problem of applying APOS to geometry and advanced mathematics - they are different worlds! Tall (2002) goes on to explain how objects in each of these three worlds are handled in distinctly different ways, claiming that some individuals find moving between them extremely difficult.

Although neat, I believe that the three worlds theory is inadequate. It is not the case that every mathematical object fits into one of the three worlds.

What sort of object is Lambert's sphere?

Within Tall *et al.*'s (1999) 'definition', the imaginary sphere appears to have mathematical object status - Lambert was able to consider the sphere as a whole and rationally investigate its properties. Indeed, Lambert himself was able to derive results about the sphere's geometry despite hoping its existence would turn out to be contradictory. This closely mirrors the proposal put forward by Tall and colleagues: whether an object really 'exists' or not is irrelevant (and possibly meaningless), what is more important is whether you can study it.

So, if we accept that the imaginary sphere is an object, the next question is: how does it fit into Gray and Tall's scheme of 'three worlds'? It is not based upon real-world perception, it is not proceptual (it has neither symbol nor process) and, at least for Lambert, it is not axiomatic. Indeed, he believed it *could not* be axiomatic as that would be contrary to the 'true' postulates of Euclidean geometry. Of course, it is quite correct to say that the imaginary sphere can be made axiomatic, and indeed it was by Riemann in the 1850s. It is even true to say that the imaginary sphere can be embodied using Poincaré's disk model of hyperbolic space. The point is that *for Lambert* it was neither. In fact, most (if not all) objects in the proceptual and embodied worlds can be formalised. For example, real numbers can be defined as equivalence classes of Cauchy sequences of rational numbers, but this does not mean they are not embodied points on a line for most schoolchildren.

Lambert's conception of a sphere of radius i is an example of a mathematical object that does not easily fit into Gray and Tall's theory Invented by analogy, neither based on perception nor formal definition, it seems to lie somewhere between the embodied and the axiomatic worlds.

The importance of analogy – or 'structural metaphor' (Pimm, 1987) – in creating mathematics is well documented Davis (1984) noted:

> learning is primarily *metaphoric* – we build representations for new ideas by taking representations of familiar ideas and modifying them as necessary. (p. 313)

Changing one aspect of the familiar idea of a sphere (letting $r = i$ instead of $r = 1$) permitted Lambert to conduct an (admittedly limited) study of the properties of his new imaginary version.

I believe there are many examples of mathematical objects that can be studied in such a way. One example stands out. The novel 'Flatland' (Abbott, 1884), aimed at schoolchildren, accomplished the feat of combining a subtle sociological critique of Victorian Britain, with an introduction to higher dimensional geometry. Towards the end of the book, Abbott introduces his readers to the notion of the fourth dimension. He uses analogy. He notes that in one dimension a moving point produces a line (with two terminal points), that in two dimensions a moving line produces a square (with four terminal points) and that, in three dimensions, a moving square produces a cube (with eight terminal points) Pointing out that the sequence 2, 4, 8, ... is a geometrical progression, he draws the conclusion that in four dimensions a moving cube would result in a "divine Organisation" (a hypercube) with 16 terminal points

Similarly, Abbott points out that, since a line has two bounding points, a square has four bounding lines, and a cube has six bounding squares, one would expect the hypercube to have eight bounding cubes – in line with the arithmetical progression 2, 4, 6, 8, ... (Abbott, 1884, pp. 72-73).

After having read the book, a schoolchild is left in a position to wonder about (and provide answers to) such questions as: How would I measure such an object's '4D volume'? What would the 'surface volume' of hypercubes of various sizes be? Using analogies with lower dimensional objects it is possible to build up a fairly sophisticated, intuitive understanding of certain properties of hypercubes.

Similarly, as for Lambert and the imaginary sphere, *for the child* the hypercube is none of proceptual, embodied or axiomatic (indeed, they might not be aware of what an axiom is). Instead, it is created using an analogy with a more familiar situation.

Pimm (1987) describes what happens when an analogy is used to 'create' new mathematics:

> Immediately whole theories, comprising definitions, concepts and theorems line up for examination, 'translation' and exploration [...] We can translate and extrapolate knowledge and insight from a more familiar setting. (p. 102)

Just as Lambert translated knowledge about triangles on the sphere to the imaginary sphere, after reading 'Flatland' the child is able, albeit in a slightly less groundbreaking fashion, to translate knowledge about lengths, areas and volumes from three to four dimensions. All of these mathematical investigations take place outside of the three worlds.

Final thoughts

Lambert's imaginary sphere does not fit satisfactorily into any of Gray and Tall's worlds. It was proposed by a highly imaginative and insightful mathematician who managed to draw an analogy between the behaviour of a familiar sphere and the behaviour of some speculative algebraic equations. Although analogous in some respects to the traditional sphere, Lambert's imaginary version is not based upon a real-world perception. There are similarities between Lambert's situation and that of a child having read 'Flatland'. Tall *et al.* (2000) defined an object as, roughly, something you can manipulate and study. Using this 'definition', and looking from the point of view of certain learners, perhaps there are many objects that do not comfortably fit into Gray and Tall's worlds.

It is perhaps reasonable to argue that it should be the goal of the learner to try to put all objects into one of these three worlds. After all, it can be far easier to study an object based on firm axiomatic foundations than to study one using hazy analogies. It is, however, not correct to say that all learners do this; for whatever reason, many do not As things stand, I believe Gray and Tall's theory of three worlds is insufficient to encompass every learner's experience of mathematics.

Acknowledgements

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Notes

[1] The word Gray and Tall actually use is 'concept'. They have the frustrating habit of using the words 'concept' and 'object' interchangeably

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On priorities of research in mathematics education

ALEXANDER KHAIT

The purpose of this article is to provoke discussion on the priorities of research in mathematics education. Mathematics education is a vast field, and there is always a possibility that no one direction stands out in its importance. Each researcher has some list of priorities and 'pet topics'. I venture to propose a research program that could be the most fruitful research direction. Actually, candidates for the role of the most important research direction can be found in the literature. For example, the late Professor Amitsur suggested the goal of understanding the nature of mathematical thinking (see Sfard's talk with Amitsur in Sfard, 1998) and Schoenfeld (2000) stated at the beginning of his paper that, "without a deep understanding of thinking, teaching, and learning, no sustained progress [in improving mathematics instruction] is possible". I respectfully disagree. We really do not know how the mind works. Cognitive processes involved in mathematical learning are of astonishing complexity (Niss, 1999). A demand to understand the thinking processes, as a preliminary condition to working on improving mathematics instruction, is like saying that it is impossible to develop physics prior to understanding the nature of matter. This aim might be compared with a Holy Grail: the ultimate goal being to provide an everlasting inspiration to generations of researchers. This search can yield important results. However, one cannot bank on its ultimate success, and so it is not of the highest priority from the practical point of view.

Freudenthal (1981) formulated several problems in mathematics education. The first was, "Why can Jennifer not do arithmetic?" (p. 135). Later, Freudenthal's questions were repeated in a way similar to the famous Hilbert unsolved problems of mathematics (Adda, 1998). In his later book, *Revisiting mathematics education*, Freudenthal (1991) writes, "[it] is not a question anymore, because today Jennifer is eleven and excels in arithmetic" (p. 174). So, mechanisms of learning are certainly interesting from a theoretical point of view. Better understanding can facilitate learning of children in general, and especially of children with difficulties. But the fact that Jennifer (i.e. just a regular child for this matter) can do arithmetic without relying too much on research in mathematics education is crucial. It is not the most important problem, not Problem One. I once heard a remark attributed to Otto von Bismarck, that the Franco-German war of 1870 was won by a school teacher, meaning that the German soldiers had a higher level of literacy (basic skills in reading, writing and arithmetic). Since then, long experience of developed countries has shown that if the government supplies enough competent teachers, they succeed in implanting these skills in most children, no matter what the (reasonable) methodology. The basic method is sufficient practice in a supportive atmosphere. As far as arithmetic learning is concerned, wars will not be won by educational researchers.

Niss (1999) gave the following definition:

> The didactics of mathematics, alias the science of mathematics education, is the scientific and scholarly field of research and development which aims at *identifying*, *characterising*, and *understanding* phenomena and processes actually or potentially involved in the *teaching* and *learning of mathematics at any educational level* (p. 5)

This assumes that the subject of learning (i.e. mathematics) is taken for granted, and this is exactly what I want to question. The scope of mathematics is in perpetual change. For example, in the Pythagorean tradition, music is a part of mathematics while logic is not! In medieval groupings of subjects for educational purposes into the *trivium* and the *quadrivium*, geometry and arithmetic composed the *quadrivium* together with music (harmonics) and astronomy. The *trivium* was formed of logic (dialectic) together with grammar and rhetoric. Indeed, prior to the middle of the nineteenth century, logic was not recognized as a part of mathematics.

Ours is a dynamic society. For the educational context it creates new needs, unexplored by previous generations of educators. Formulation of these needs is not a simple task, and even more so supplying these needs. As an illustration let us consider the junction between secondary and tertiary education on the one, and between mathematics and computer science education on the other hand. A typical student I have in mind is an individual, not naturally inclined to study mathematics, learning it exclusively for practical purposes (e.g. to find a decent job). Most of these students will work in computer-related technologies, that is, their needs in mathematics are closely connected to various computer applications. From the point of view of a mathematics educator there *never* was so large a population that needs a solid mathematical background just to earn their living.

> *The sort of mathematics that arises in a computing context is not necessarily what most people would consider to be mathematics at all.* Its character may seem more like that of 'mere' organization, symbol management, or data manipulation. (Truss, 1999, p. v; *emphasis added*)

There is no tradition in teaching this kind of mathematics to such a population: many topics of mathematics needed for work with computers used to be purely mathematical research areas before the dawn of the computer age and were taught exclusively to the mathematically-inclined minority.