Using Code-Switching as a Tool for Learning Mathematical Language

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Mathematical language and its uses by learners and teachers has attracted the interest of many mathematics educators. However, despite significant attention to specific terminology used or misused in particular mathematical topics, expressions pertaining to elementary number theory have not been mentioned among the numerous examples reported. Recent research on learning number theory concepts (Zazkis and Campbell, 1996a, 1996b; Zazkis, 1998a) has provided an opportunity to attend to the language used in these topics.

This article is a reflection on four years of teaching a course 'Principles of mathematics for teachers' for pre-service elementary school teachers. In what follows, I identify and describe the discord between the formal mathematical language and their informal language used in the context of elementary number theory. I share my unsuccessful attempts to address the problem. Further, I describe encouraging results of a 'code-switching experiment', that was inspired by several articles in the February 1998 special issue **18**(1) of *For the Learning of Mathematics*.

Code switching and pedagogical dilemmas

Code switching can be described as alternation in use of more than one language in a single speech act (Adler, 1998). It may occur when a multilingual person is addressing another multilingual person. It may consist of a single word or phrase or it may involve several sentences. Though code-switching can be criticized as an inappropriate mixture of languages, it has important social aspects.

Code-switching has received attention from investigating communication in classrooms where the official language of mathematical instruction, frequently English, is different from the native language of the students and (perhaps) their teacher. In these settings, teachers were observed to switch codes in order to translate or clarify instructions but also to reformulate and model appropriate mathematical language use. Students switched codes to seek clarification and to express their ideas or arguments. Code-switching was noted as a valuable educational resource, and as means to foster mathematical understanding of the students (Adler, 1998; Setati, 1998).

Though code-switching is usually mentioned in a multilingual context, it can also occur between dialects and registers.

> Mathematics can be singled out, among other forms of human imagination and ingenuity, by the very specific linguistic register in which its ideas are formulated. (Winsløw, 1998, p. 19)

Extending the idea of code-switching between languages to code-switching between the mathematical and everyday registers of English invited an exploration of code-switching in a monolingual mathematics classroom.

Adler (1998) has elaborated several dilemmas of mathematics teachers working in multilingual mathematics classrooms:

> - the *dilemma of code-switching* (of developing spoken mathematical English vs. ensuring mathematical meaning);
> - various *dilemmas of modelling mathematical English* (of whether such modelling is 'talking too much');
> - the *dilemmas of mediation* (of validating pupil meanings vs. developing mathematical communicative competence, [...]);
> - the *dilemma of transparency* (of the visibility vs invisibility of language as a resource for learning) (p. 32)

The last dilemma needs clarification. Transparency of resources, according to Lave and Wenger (1991), involves their dual visibility and invisibility. A tool must be visible to be noticed and used and yet simultaneously be invisible in order not to become the sole object of attention. Language, as a communication tool in a mathematics classroom, must be visible (so it is clearly identified) and simultaneously invisible (so it can be utilized when discussing mathematical meaning).

My dilemmas in English-only mathematics classrooms were very similar to those described by Adler. I wished to model appropriate mathematical usage of concepts without 'talking too much' (modelling). I wished students to be encouraged to express their ideas but also to develop appropriate communication skills for those ideas in the mathematical register (mediation). I searched for a balance between emphasizing appropriate language *per se* and utilizing mathematical language in mathematical activity or problem situations (transparency).

The dilemma of code-switching - to switch or not to switch? - is presented as a major challenge. Having made several unsuccessful attempts at helping students to communicate mathematically (as described in a subsequent section), I wished to explore the idea of code-switching, appreciating that students' language and the mathematical

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register presented quite different codes. In the context of code-switching between registers, there is a widespread value judgement at work: mathematical formal code is preferable since students' non-mathematical informal code is frequently vague and inadequate for describing mathematical relationships.

Nevertheless, the non-mathematical code, at least initially for students, has a greater power to communicate their intended meaning. On the one hand, I wished to acknowledge the explanatory power of non-mathematical code; on the other, I was not prepared to switch to this code myself as it contradicted my belief in proper modelling. As a teacher, I could not allow myself imprecise reference to mathematical situations. A possible resolution of this code-switching dilemma follows the elaborated summary of the different codes used in describing elementary number theory concepts and relationships.

Setting

For four years, I have taught the course 'Principles of mathematics for teachers', which is a core mathematics content course in a teacher education program for certification at the elementary level. There were between sixty and eighty students enrolled in each offering of the course, and approximately twenty students from each cohort volunteered to participate in a clinical interview.

The interview questionnaires were designed as a part of an on-going research project investigating the learning of number theory by pre-service elementary school teachers. The interview questions investigated participants' learning and understanding of concepts of elementary number theory, such as divisibility, prime decomposition, factors, divisors and multiples. Students' use of language was not a focus of investigation when the questions were designed and the interviews were conducted. Analyzing these interviews strengthened my interest in mathematical language in general and language related to introductory number theory in particular.

Formal mathematical vocabulary

The main mathematical focus of this article pertains to the idea of divisibility. Divisibility is one of the fundamental concepts in elementary number theory, one that can be captured with a variety of lexical descriptions. In a formal mathematical definition, we say that a natural number A is divisible by a natural number B if and only if there exists a natural number C such that $A = B \times C$. This definition is interpreted in some texts for elementary school students and teachers in terms of division: if the result of division of A by B is a natural number, then we say that A is divisible by B. There are five equivalent statements to express the idea of divisibility:

(1) A is divisible by B;

(2) B divides A;

(3) B is a factor of A;

(4) B is a divisor of A;

(5) A is a multiple of B.

I will refer to the use of these expressions as utilizing formal or mathematical vocabulary. These terms and expressions were familiar to students in the sense that they were defined in the textbook and used in class work for approximately four weeks while attending to the topic on number theory in the course. All the interviews took place upon completion of the unit on elementary number theory.

Informal vocabulary

From the first set of interviews, it was evident that the language used by the interviewers differed from the language used by the interviewees. In the first cohort of students, only three out of a group of twenty-one used mathematical terminology consistently. Other participants used mathematical terminology and informal non-mathematical descriptions interchangeably, with a preference for the latter.

Below are several examples of forms of expression (italicized by me – Int = interviewer) used by students:

Marty: Prime number. A prime number is a number that *can only be divided* by one and itself, there is *no other factors that go into that number*

Jill: Because once again I know what M is, it's 1,575, and I know that 1,500 *can be divided um by 15 evenly and I know that 75 also can be divided evenly by 15*.

Barb: I looked at that, but I couldn't find a number that would *easily be divided into 391*, and so I just assumed that it was prime, and, and, I mean I could be wrong, I didn't, I didn't take 8 or something like that, but I just checked for the first five numbers

Int: Well, what you could do is, you can tell me how you would go about determining the number of *factors* that 90 has.

Barb: Okay. Well, um, *1 goes into it, then 2 goes into it, 3 goes into it, um, 4 doesn't go into it, 5 goes into, 6 goes into it*, 7 doesn't, 8 doesn't, 9 does, 10 does

Int: Is M *divisible* by 7?

Jack: By 7 *you should be able to divide it*, quite possibly have the potential of *complete divisibility*

Int: I'm going to ask you, is 391 *divisible* by 23?

Lisa: Hmm, [pause] um, well, [pause] um well, it is, but I'm not sure if it's *divisible evenly* or not.

'Goes into' was the most common informal wording used by students to communicate divisibility. Variations included expressions such as 'goes into completely', 'goes into easily', 'goes into evenly', 'can be put into', 'can fit into', etc. Another popular expression used by students in the discussion of divisibility was 'can be divided', as exemplified above by the excepts from the interviews with Marty and Jill. Variations added to this expressions descriptors such as 'evenly', 'easily' or 'absolutely'.

Many student references to divisibility were accompanied by overemphasizing, such as 'complete divisibility' or 'divisible evenly', as shown above in the excerpts from the interviews with Jack and Lisa. Some students described divisibility as being 'divisible without decimals' or 'exactly divisible without remainders'. In what follows, I will refer to these expressions as utilizing informal vocabulary

Though some of these expressions, such as 'divisible without remainder' can be seen as harmless redundancy, others, such as 'divisible evenly' are potentially misleading. Several students inquired for example, whether the number 14 could be considered as 'evenly divisible' by 2, because the division resulted in an odd number.

Modelling mathematical language

As a teacher, I had a strong desire to have my students use appropriate mathematical terminology. Listening to the class discussions, I was aware of students' difficulties in applying the vocabulary of the mathematical register and their preference for informal language. However, I realized the degree of seriousness of the problem only by reading through the transcribed interviews, which happened after the completion of the course

During the second offering of 'Principles of mathematics for teachers', I decided to emphasize modelling correct mathematical language. I praised every student who used a formal expression. Frequently, I asked them to rephrase informal expressions. I included exercises in which students were asked to apply formal terminology or rephrase statements using either better or simply different mathematical terms. The results were disappointing. The same 'goes into' persisted in students' talk

Before the third offering of the course, I conjectured that students were unable to adopt mathematical terminology, because too many equivalent expressions were presented to them simultaneously. Among the five equivalent expressions listed above, I decided to avoid (2) and (4). Expression (2) appears too close to the familiar 'A divided by B'. This could have been one of the reasons for confusion and misinterpretation. I believed that the intended meaning of words that were less familiar would be better accommodated. Expression (4) appeared problematic among the second cohort because of the double meaning of the word 'divisor'. 'Divisor' in the context on arithmetic operations is interpreted as a number we divide by: 'divisor', in the context of elementary number theory, describes a relation between numbers (Zazkis, 1998b).

It was my hope that using three expressions instead of five would help students accommodate to the terminology. Unfortunately, this was not the case. By analyzing their interviews, it became apparent that the use of language among the students in the third cohort was not different from the language of students in the first and second cohort. Obviously, intensive modelling by the instructor on its own was not beneficial in helping students to acquire communication skills within the mathematical register. The idea of code-switching described in the February 1998 issue of *For the Learning of Mathematics* inspired the experiment described below.

A 'code-switching' experiment

This experiment emerged spontaneously in an early classroom discussion. I was reluctant to repeat students' informal language: however, addressing it without repeating it created ambiguity of reference. My attempted solution was to put the words in someone else's mouth. I told my students a fictional story about a student named Simon, who was talented mathematically, but was unable to learn mathematical language. Simon expressed his ideas in his own everyday words. During the course, students' were invited to formulate Simon's ideas using mathematical vocabulary. At times, they were also asked to interpret mathematical statements using 'Simon's words'.

Below is an example of a typical classroom conversation.

Teacher: Yesterday we mentioned an important property of triangles. Can someone recall what it was?

Student 1: In a triangle there are 180 degrees.

Teacher: Can you be more precise?

Student 1: The angles, the angles of a triangle make 180 degrees.

Teacher: OK. Good idea. Simon says: "The angles of a triangle make 180 degrees". How do they make it? Is there a better way to say this in our mathematics class?

Student 2: Adding up the angles of a triangle gives 180 degrees

Teacher: This is indeed the idea. Can someone suggest another way to express it?

Student 3: The sum of the angles of a triangle equals 180.

Teacher: Good. Can we be more precise?

Student 1: The sum of the measures of the interior angles in any triangle equals 180 degrees.

Simon became a part of our classroom. Students spontaneously referred to Simon's code while correcting each other and clarifying the intended meaning. Furthermore, there were 'Simon questions' practiced on classroom assignments. For instance, a typical daily assignment was:

> Write Simon's ideas using appropriate mathematical terminology in two different ways.
>
> Simon says:
>
> (a) 3 goes into 6, therefore 3 goes into 42 because 6 goes into 42

(b) 16 is the smallest number that has the following property: *A* divided by 16 results in a whole number and *B* divided by 16 results in a whole number.

(c) 11 and 49 have nothing in common.

In addition to single statements, students were given lengthier explanations attributed to Simon, usually derived from the interviews with previous cohorts, and asked to rephrase them using the mathematical register. Students were successful with such exercises, indicating that they were able to apply appropriate vocabulary when explicitly invited to do so.

Some students reported that a translation from Simon's language to mathematical language was easier for them compared with the translation in the opposite direction. Noting students' successful application of a visible language, the question remained whether mathematical vocabulary would be applied by students spontaneously in a problem-solving situation, when language is invisible, when it is to be used a tool rather than being explicitly in focus.

The results were encouraging. Among nineteen volunteers interviewed from the fourth cohort, eleven consistently applied appropriate vocabulary. The remaining eight used both appropriate and inappropriate expressions. However, inappropriate expressions were used in most cases in conjunction with appropriate terminology. The following excerpts exemplify such uses.

Nick: A number is prime when it *can only be divided* by 1 and itself, that is, it has no other *factors* other than one and itself.

Int: How could you find numbers so that 117 is their multiple?

Mark: I would just start plugging in numbers, like I'd, I'd say, 3 would probably be the first thing that I'd do, and I'd go, in my calculator, I'd go 117 divided by 3, and then that would give me an answer, *if it goes into it evenly, as Simon says, [smiles] then I know it is a multiple of it*, and then I would try 6 or I would try 9, *multiples of 3* is what I would try.

Int: Can you find a prime number between 120 and 150?

Karen: I have to try. Do you want me to do this?

Int: Yes, please. You may start by describing how you would try.

Karen: Obviously, it's not 120 or 150 and not 130 or 140. I wouldn't try 121 because this is 11 squared. I wouldn't try 122, I wouldn't try even numbers, as they're *divisible by 2*, so I wouldn't try 123 because it has *a factor of 3* because 1+2+3 gives me 6, so this one isn't prime for sure and I will cross out all the numbers *divisible by 3*. So 124, no, 125, no, and will cross out all those that end with 5. So 127, *will 7 go into it*? I don't know. But this would be something to try. [Pause] No, *7 is not a factor of 127*, and no 11 is not ...

Nick, in the above excerpt, started with informal description, but then he code-switched to the mathematical register. Mark used informal vocabulary to explain how he understood the meaning of a mathematical word, 'multiple'. Karen appeared to use formal mathematical terminology when she was confident in her claims; however, she switched back to informal vocabulary when she was uncertain of the answer. It appears that for students who have not fully interiorized mathematical vocabulary and occasionally code-switch to informal vocabulary, these 'switches' or 'slips' are most likely to occur at the moments of uncertainty.

Convincing a colleague

Some colleagues of mine have suggested that as long as students understand the concepts, the way they talk about these concepts is of secondary importance: some readers may have a similar response. I would like to offer three comments with respect to this assertion.

One, I believe that learning mathematical language is a part of mathematical education, and students' familiarity with and ability to utilize mathematical language is no less important than their ability to apply certain algorithms, formulae or problem-solving strategies. This is consistent with Lave and Wenger (1991), who argue that learning in a community of practice involves learning the language of this community.

In particular, learning mathematics includes "appropriating ways of speaking mathematically" (Adler, 1998, p. 30), that is, learning the language of mathematicians. Mathematical language is characterized by lack of redundancy, mathematical nomenclature is minimal where every word has a meaning. In mathematics, we talk about 'whole' not 'entirely whole' numbers; 'proper' not 'absolutely proper' fractions. Furthermore, whenever an adjective or adverb accompanies a mathematical term, specific meaning is attached to it: consider, for example, *proper* subsets, *absolutely* converging series or a *uniformly* continuous function. The words describing subsets, convergence or continuity communicate mathematical properties, they modify the meaning. Therefore, in an effort to speak mathematically, a learner's attention should be directed to the conventions of the community of practice and the reasons for them.

Two, it may indeed be possible for a child to understand the idea of divisibility without the ability to express this idea in an appropriately mathematical fashion. However, my students were pre-service elementary school teachers, not elementary school students. I believe that at this level it is fitting to require and expect appropriate communication skills about mathematical ideas, especially given that these participants are about to teach mathematics, among other subjects, to elementary school students.

Of course, code-switching to informal language could be advantageous at times for the sake of communication with learners. However, use of informal code by a teacher should be a pedagogical choice, rather than a symptom of a lack of proficiency with the mathematical code itself.

Three, informal forms of language, at least in my experience, go hand in hand with incomplete understanding of the concepts involved. Though there might be exceptions to this rule, 'good language' is usually an indication of good understanding. That is to say that while mathematical ideas and understanding can be expressed informally, I have not found indications of inadequate understanding expressed in a correct formal mathematical language.

Language plays a critical role in concept development (Esty and Teppo, 1994) and may in fact assist the development of concepts. According to Vygotsky (1962), low-order concepts can be formed without the aid of language, while the emergence of higher-order concepts would seem to be inextricably linked with language. Austin and Howson (1979) cite Vygotsky [1]:

> The concept does not attain to individual and independent life until it has found a distinctive linguistic embodiment. (p. 167)

Mason (1982) offers three stages of convincing: convince yourself, convince your friend and convince your enemy. I am sufficiently convinced by the first argument. I believe that my friends would be convinced by the subsequent two. However, further research on the interrelationship of students' understanding and the language they choose to describe their mathematical ideas may be needed to convince the opposition.

Students' reactions

Students' reactions to the experiment varied. Although in a credit course students 'played by the rules', some did so reluctantly. The following outburst was not uncommon: "You like me to say 100 is divisible by 25 or 25 is a factor of it. OK, I can say it. But what it *really* means is that 25 goes into this number evenly. That's how we understand it."

Some students initially resisted the focus on mathematical terminology. In reaction to my comment that 'goes into evenly' is not an appropriate mathematical expression, I got the following response: "That's what I've said all my life and that's what I was taught, and that's what I heard all my life. Are you sure? Because that's how we say it *here*."

The emphasis on 'here' was most likely a student reaction to my obviously foreign accent. This comment shows a vicious circle of obscure language usage: we learned it from our teachers and will teach it to our students, who will do the same when become teachers. Can this circle be broken? Should it be?

I believe that for the majority of students initial resistance weakened and for some even turned to appreciation. For example, Judy claimed early in the course: "I'm not big on names, I, I like to know concepts. But the name is just kind of an extra something for me to remember. It's not, I don't consider it very important."

Towards the end of the course she seemed to change her perspective: "Simon has made me see that I have to be much more careful when, like if I teach math, how I teach it and vocabulary I use, and exactly what's going on, that I might not realize what's going on."

Reflecting on pedagogical dilemmas

Giving prospective teachers opportunities to discuss difficulties of their prospective students with subject matter is a powerful teaching strategy, because it creates an opportunity to address personal difficulties without explicitly acknowledging them. This belief is shared by many colleagues working in teacher education. Simon's existence assists along the same lines. Attributing certain expressions to imaginary Simon gives a chance to address students' non-mathematical, at times imprecise or confused language without focusing on any particular student.

A reference to Simon helped me implement code-switching, while code-switching assisted in addressing, though not reconciling, certain dilemmas of mediation and transparency. Code-switching was used as a continuing translation from students' (Simon's) informal language to the mathematical register. As such, it helped students elaborate on the meaning of the concepts in the mathematical register. It also assisted with the dilemma of modelling, because the responsibility for accurate and explicit modelling of mathematical language was shared between the instructor and the students. Therefore, not only were my pedagogical dilemmas similar to those faced by teachers working in multilingual classrooms, but also the consequences of code-switching.

Adler (1998) reported that an explicit focus on mathematical language *per se* (making it visible) may obscure rather than support mathematical practice. In the 'code-switching' experiment in my classroom, the obstacle presented by making language visible was temporary. In addition to introducing and modelling formal terminology within the mathematical register, which is an essential practice in any mathematics classroom, code-switching assisted in bridging the gap between the terminology of the mathematical register and the popular informal terminology students were accustomed to. I believe that with pre-service elementary school teachers a possible way to achieve effective invisible language is first to make it visible.

Conclusion

According to Winsløw (1998), it takes time, will and energy to gain access to the mathematics register. How is it possible to assist students in learning to communicate within the mathematical register? Winsløw believes that, similar to learning the first foreign language, communicating mathematically would become more natural if trained at an early age. However, having missed the opportunity to start training at a early age, is it possible to assist pre-service elementary school teachers in learning to communicate within the mathematical register?

There is no doubt that the best way to learn a foreign language is immersion. However, the conditions for immersion are not always available and there are thousands of students who attempt to learn a foreign language by constant translation via their native language. For those learners, translation back and forth between a foreign and their native language is a necessary exercise before one can communicate and eventually come to 'think' in a foreign language.

Translation from a natural language to the language of mathematics is often considered as a conversion from word problems to equations or from more complex, 'real-world' situations to mathematical models I took a different perspective on such translation, considering it as moving back and forth between the mathematical register and register of everyday English, that is, code-switching

> Troubles that learners experience with the language of mathematics, can be explained at least partly by the lack of insight of textbook authors and teachers into the complexity of the matter As those who are responsible for the instruction of this matter are not conscious enough of these details, they lack the insight into the possible sources of mistakes. (Freudenthal, 1983, p 472)

English teachers do not let their students get away with wrong grammar, wrong sentence structure or wrong syntax, even when great creative ideas are in place. I believe that mathematics teachers, while appreciating good ways of thinking and correct answers, should also enssure that these thinking strategies are properly expressed. It is important that incorrect versions are not allowed to become a habit (Oldfield, 1996, p 22).

For mathematics educators, teachers of teachers, the task is different This task is not only to introduce and enforce mathematical language, but also to supersede the robust, informal language that has been interiorized in school. Researchers reported that teachers were:

> having more success in teaching unfamiliar content than in reteaching (or replacing) familiar content (Markovits and Sowder, 1994, p 24)

Similarly, replacing previously acquired vocabulary could be more challenging than introducing a new vocabulary for new concepts: challenging, but not impossible

Freudenthal (1983) suggested that learning the linguistic element of structuring devices in mathematical language "requires a more conscious didactics" (p 473). Conscious and deliberate code-switching is one didactical idea that may assist pre-service teachers in acquiring the language of mathematics

Note

[1] No page reference is given for this citation in Austin and Howson Despite it being consistent with much of what is there, I have been unable to locate this purported citation in *Thought and Language*.

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