Learning Mathematics through Conversation: Is It as Good as They Say? [1]

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A. Why do we believe in learning mathematics through conversation? An introduction

Anna Sfard

There seems to be a strong consensus among educators about the need to foster students' ability to 'talk mathematics'. *The Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) includes 'learning to communicate mathematically' among five 'new goals for the student':

> The development of a student's power to use mathematics involves learning the signs, symbols, and terms of mathematics This is best accomplished in problem situations in which students have an opportunity to read, write, and discuss ideas in which the use of the language of mathematics becomes natural. (p. 6)

All this sounds quite reasonable. In a world overpowered by cellular telephones, fax machines and computer networks, where 'communication' has become the name of the game and where people are consumed with the desire to talk on everything to everybody, the request to turn mathematics classrooms into another site for interpersonal exchange seems only natural. No wonder then that we may be ready to join the 'communication camp' without questioning. [2]

Most of us would accept the claims about the need to foster students' ability to speak mathematically as a basic, almost self-evident truth. It is the goal of the present discussion to turn the seemingly obvious into an object of critical inspection. In what follows, the various authors try to unpack their respective faiths in the importance of mathematical conversation in an effort either to reinforce it or to refute it through a disciplined analysis At the outset, I scrutinise three basic arguments which can often be heard from advocates of mathematical communication, coming from well-developed, overarching conceptual frameworks which attract many followers. In spite of such impressive foundations, on closer examination each one of the three lines of argument reveals some weaknesses

1. A cognitivist argument

As students communicate their ideas, explain the *Standards'* authors, "they learn to clarify, refine, and consolidate their thinking" (p. 6). This clearly implies that the ability to participate in mathematical conversation is to be promoted not only for its own sake, but also for its expected effects on the process of learning and on the quality of the resulting knowledge Journals in mathematics education abound in article titles which bring a similar message: for example, 'On the learning of mathematics through conversation' (Haroutunian-Gordon and Tartakoff, 1996), 'Reflection, communication, and learning mathematics' (Wistedt, 1994) and 'Journal writing and learning mathematics' (Waywood, 1992).

In short, mathematical conversation is believed to be good for your mathematical thinking The questions which should be answered by those concerned with cognition are of an intricate nature and require a deep insight into the workings of human mind

- What is it in the mechanism of mathematical thinking which makes verbalisation of mathematical ideas beneficial to the whole process?
- What are the relations between such different cognitive procedures as languaging and symbolising on the one hand, and conceptualising and reasoning on the other?
- What does our ability to solve mathematical problems have to do with our capacity for communicating mathematics to others?

In identifying all these questions, I am not implying that they cannot be answered On the contrary: this is the kind of problem which seem to be a perfect theme for study by a researcher belonging to the cognitivist framework All I am trying to say is that, in spite of the long-standing interest in the complex relationship between thought and language (and this refers us, of course, to Piaget and Vygotsky, among numerous others), we still do not know enough to turn a *belief* in the power of mathematical talk into a theoretically sound *assertion*

2. An interactionist argument

For a steadily growing number of researchers, the idea of learning through conversation is a natural by-product of the conception of learning as an initiation into a 'community of practice' (Lampert, 1990; Lave and Wenger, 1991). The essence of the argument can be presented as follows. Mathematics students are beginning practitioners or, as Streefland (unpublished) calls them, 'junior researchers'. If we want to have them act accordingly, it is our job as teachers to turn the classroom into a 'community of inquiry' (Schoenfeld, 1996), one which would be as close as possible in its norms and practices to those of 'expert practitioners' So far, so good. The argument certainly sounds convincing in the eyes of those who adopt the metaphor of learning as 'legitimate peripheral participation' in such a community. The only remaining question is what should count as mathematical practice

As it turns out, if talking and communicating are the hallmarks of the educators' vision of this practice, then those 'expert practitioners' are likely to take exception to the whole project. Indeed, we require our 'junior researchers' to communicate and verify ideas among themselves, first in small collaborative groups, and then again, in what has come to be known as 'whole-class discussion'.

Communities of mathematicians, however, do not necessarily work this way. On the contrary, the loneliness of the professional mathematician is notorious. The motif of isolation and lack of meaningful communication with others returns time and again in mathematicians' accounts of their own practice. Here is a representative example, in the form of advice to a beginner, coming from a well-known 'expert practitioner', Camille Jordan.

> you must learn to enjoy [mathematics] alone [...] You will be a subject of astonishment to those close to you. You will not be much better understood by the scholarly world [...] even [mathematicians] do not always read each other (from Schmalz, 1993, p. 54)

Of course, this does not have to be true for all mathematicians, many of whom do collaborate and communicate with others. The point is, however, that the frequently reported reticence of the mathematician seems to have good reasons behind it, and these reasons may well also be in force when it comes to 'junior mathematicians'. Being extremely demanding in terms of concentration and intellectual effort, mathematical problem solving can perhaps best be practiced in seclusion, where all effort may be focused on the problem at hand, and on this problem alone. Communication with others, itself another strenuous activity, may distract a researcher and make her problem-solving attempts less effective. This argument should certainly be given much thought by the advocates of mathematical conversation

3. A neo-pragmatist argument

The third argument for the centrality of conversation derives from another new metaphor for knowledge and knowing, the same one which gave rise to the now popular discursive approach to research on thinking and mind (Harré and Gillet, 1995). Rather than merely treat conversation as a secure route to knowing, more and more thinkers *equate* knowledge with conversation (Rorty, 1979; Foucault, 1972; for a survey of relevant literature, see Ernest, 1993). The gist of the idea is epitomised by Rorty in the following statement:

> If we see knowing not as having an essence, to be described by scientists or philosophers, but rather as a right, by current standards, to believe, then we are well on the way to seeing conversation as the ultimate context within which knowledge is to be understood. (p. 390)

Those, however, who use the authority of post-modern philosophers to account for their fascination with co-operative problem solving and classroom discussion may be taking the metaphor of conversation a bit too literally. While talking about knowledge as a 'conversation of mankind', Rorty does not necessarily refer to the most immediate, literal meaning of the component terms. For him, as well as for his many colleagues on both sides of the Atlantic Ocean, conversation is a broad metaphorical idea which includes all kinds of human communication - from an interactive, real-time oral exchange, to written correspondence, and to articles and books.

Mathematics is certainly a conversation in this last sense, but is it also one in the first sense? Again, mathematicians may not be too eager to put an equality sign between their professional practices and the activity of 'interactive oral exchange'. This, of course, does not mean that language and communication do not have any role in learning mathematics. It only implies that those who try to substantiate their belief in the power of talking with the metaphorical equation 'knowledge = conversation' may not be using the right type of argument.

Finally, here is the question that was posed to the panel.

> Do you believe that mathematics can and should be learned through conversation (in the sense of an interactive oral exchange)?

B. Talking mathematically versus talking about mathematics
Pearla Nesher

In response to this question, I would like to make a distinction between *talking mathematically* and *talking about mathematics*. I will start with Anna's cognitivist argument, which is based on the connection between language and thought: language is the major route to the articulation of ideas.

The basic question which is raised within this argument is: what 'language' are we talking about? Do we talk about the natural language that serves us for natural communication in our everyday life, in classroom conversations, in describing situations; or about the formal language of mathematics? These two are not the same. I am worried that speaking about 'language' and 'communication', in general terms, contributes to the fuzziness and ambiguity of the discussion. The NCTM *Standards* speak clearly about mathematics as the language of signs, symbols, etc. Symbols and signs do not form a language unless they stand for something. They, of course, express thought, and in our case these are thoughts about the mathematical concepts, objects, relations, etc.

When problem situations are discussed in mathematics classrooms, several levels of conversation are going on at the same time. There is the use of the *symbolic mathematical language* to describe mathematically the phenomena (usually) expressed *in natural language*. We employ both languages, the one that describes our world in natural language, and the one that attends to it in a formal language. Yet, it is important to keep in mind that these are two distinct languages.

Natural language is limited in its ability to describe mathematical notions. Take the following trivial example: write down the expression "a fourth of a number decreased by five". It looks to be part of natural language; at least it is not written in a symbolic manner. This is, of course, an ambiguous expression. It can also be written, either as $\frac{1}{4}(n - 5)$ or as $\frac{1}{4}n - 5$.

From a mathematical point of view, these expressions have different meanings. However, it is *only* with formal notation that we can differentiate between them. To talk mathematically means in part to be able to formulate expressions in a way that will distinguish between the two meanings, and this cannot be accomplished without formal notation. Mathematicians, by agreeing about the role of parentheses in mathematics' formal notation, are able to make such distinctions, whereas the users of natural language must have recourse to intonation, which is missing in the written form of natural language

There are many terms in mathematics and science which do not exist or have a different meaning in natural language Here are a few examples:

function, root, integral, height, point

Some suggest that discussion in natural language about these notions will clarify their meanings in mathematics. We have, however, a dilemma which is not easy to resolve How can natural language clarify notions which are not known within natural language? In the process of learning, we try by conversations in natural language to tie new notions to old ones which are known to the child. Yet, we should admit that this is almost a 'mission impossible'. This kind of conversation is doomed to become, in many cases, blurred and confusing rather than clarifying. Nothing in the square root of a number can be understood by the aid of natural language. It is the understanding of the notion of 'power' (which, in its turn, is based on the notion of 'multiplication' and of 'inverse' operation) that can help in understanding the notion of square root in its mathematical meaning. Thus, meanings of mathematical concepts need to be acquired in the context of mathematical notions

We use natural language to speak about mathematical notions. We listen to students' talk in order to evaluate what they have understood. Learning about the square root as a mathematical notion, however, will be evaluated by the learners' ability to apply this notion in various contexts that call for the use of this formal concept. In other words, students will be evaluated according to their ability to *talk mathematically* in using this term, and not by the way they *talk about it in natural language*.

While acknowledging that it is important to talk with children about their formal mathematical activities and that explaining their own actions may, indeed, support their learning of mathematics, we should keep in mind the main issue: that talking mathematically, means, first of all, acquiring the ability to describe the world's situations with the formal models of mathematics.

This brings me to the neo-pragmatist argument What does it mean that knowledge is socially derived? A mathematician's creation has a social context The mathematician does not work in a vacuum, even when he works alone. The acceptance of his creations as a part of 'standard mathematics' has, of course, a social aspect to it. Yet, what we teach at school are the standard notions and methods of doing mathematics. Therefore, we must accept the universality of mathematics as a language in which to talk about events detached from contexts. We usually ground the mathematical knowledge in various contexts and we apply it in real-life situations which we first describe in everyday language. We talk also about mathematical activities in natural language, which would then serve as a meta-language Yet, in essence, mathematics is a formal language which is helpful because of its universality and due to its special syntax and semantics This is the understanding of mathematics which guides our teaching.

Thus, although both teaching and creating mathematics take place in social contexts (in different ways), the most essential characteristic of what mathematicians are creating (and what we are demanding our students learn) is its universality and independence of context. To these two features mathematics owes its strength as a tool of communication.

I see many cases in which conversations can help: in realising the difficulties children encounter, in learning the variety of applications, in trying to clarify to oneself the thinking process, etc., and this brings me to the interactionist argument. I would like to differentiate among:

1. *Conversations among mathematicians* in the process of creating mathematics This kind of conversation usually takes place in professional journals, at conferences, etc. This is a special type of discourse accepted by the mathematical community. The rules of argumentation obeyed by mathematicians are agreed and are unique to the formal language of mathematics

2. *Conversations between a teacher and student* that serve many purposes. From the constructivist point of view, it is the major means by which the teacher has the opportunity to learn about the student's thinking and have a real dialogue

3. *Conversations among children* in which the students try to explain their methods. In a way, this is parallel to the mathematicians' conversation. In the process of education, we could benefit from this kind of conversation, *if* (and this is a big IF) the rules of the game were clear The teachers, as a part of their role, could help the students learn what is a convincing argument in mathematics and how it is similar to, or different from, an ethical or artistic argument

Having this kind of conversation, where children argue with each other about valid arguments in mathematics, could be the core of our mathematics teaching.

C. The conversation of mathematicians vs. students' mathematical discussions
Leen Streefland

In order to illustrate my position with respect to the focal question, I have selected three examples. Two of them come from the history of mathematics and one is taken from classroom observation Several basic conditions need to be fulfilled if mathematical communication is to take place There is also the question of basic assumptions. One such may be as follows:

> Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only a difference of degree, a difference of level, both works being of similar nature. (Hadamard, 1945, p. 104)

First historical example:
Johann Bernoulli challenges his colleagues

In the *Acta Eruditorum* of June 1696 (ill p 269), Johann Bernoulli issued "an Invitation to all mathematicians to solve a new problem" (one which he called the brachystochrone problem - in Greek, *brachus* means "short" and *chronos* "time").

If in a vertical plane two points, A and B, are given; one is asked to determine the trajectory (or curve) AMB for the moving point M, along which M will move by its own weight, starting from A and arriving at B as fast as possible.

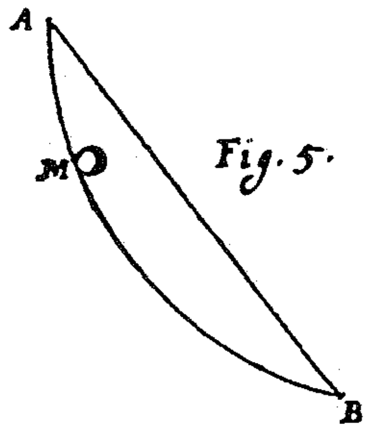

Figure 1

Bernoulli assured his audience that applications of the solution of this problem would go well beyond mechanics. At the end, he warned his colleagues not to judge too fast. By saying this he excluded as a possible solution the straight line from A to B, which, in spite of its being the shortest path, would not require the shortest time. He also promised that by the end of that year he would mention the curve AMB in his solution and added that this curve was well known to geometrists (van Maanen, 1995). On June 9, 1696, Bernoulli sent the problem to Leibniz, who answered already on June 16. He suggested 'stachystoptote' - the 'line of fastest descent' - for the name of the required curve. He also said the problem was wonderful and added: "Although I resisted, it attracted me like the apple attracted Eve"

Bernoulli published his solution in the *Acta Eruditorum* of May 1697. I will only mention one of his ideas - the one which was deemed as brilliant by other mathematicians. He imagined that the problem was not about a ball or a point mass rolling down, but about a ray of light shining through a material with variable density. According to Fermat's principle, as referred to by Bernoulli in his solution, light always takes the path which ensures the shortest transition time. Therefore, the ray would follow the brachystochrone if it passed through a medium in which at height x it would have exactly the same transit velocity as the known rate of fall of the little ball at height x.

Bernoulli sent his solution to Leibniz already on July 21, 1696 and Leibniz reacted to it on July 31. He wrote, among other things:

> I was glad to see the correspondence between our solutions of the problem you posed, because we gave constructions of the same line, although the constructions themselves were different. [...] I like the term 'brachystochrone', because of its general meaning; the term 'stachystoptote' could be acceptable for a particular case, because it is about the fall or descent through weight. I find your variant with the path of rays of light through a continuous medium very beautiful.

Second historical example: the case of Andrew Wiles

Andrew Wiles worked on the proof of Fermat's last theorem for seven years. As Wiles himself told us in the television documentary on his achievement, he had had a passion for Fermat's problem since his adolescence For him, it was "a beautiful problem, a great challenge". "Mathematicians just love challenge. It was a my private battle", he said

In this project, Wiles went through several stages. Working alone in complete isolation was the first phase (even admitting to working on Fermat's last theorem would certainly have raised some eyebrows!) Wiles decided to transfer elliptical curves into Galois representations in order to make them countable. This step took three years. Later, Wiles hoped to make his counting strategy complete by applying the Iwasawa theory. [3] Then he skipped from here to the class number theory (formulas of Flach and Kolyvagin). All this took seven years. It was only then that he eventually presented his findings at a conference, after talking to one or two colleagues,

Before publication, his colleague Nick Katz went through the manuscript and emailed Wiles once or twice daily with questions such as "What do you mean by this?" or "What does that mean?" It turned out that Wiles' line of thought contained a mistake, which was discovered by Katz Several mathematicians, such Richard Taylor, a former student of Wiles, came to his assistance in an attempt to overcome the difficulty. The proof was eventually completed with the help of Taniyama-Shimura conjecture, which had already played a role at an earlier stage of Wiles' work.

John Mason has made the following observation of what mathematicians are usually doing.

> They do work alone, then with a small group, then with the wider community They work away on a problem, discuss things with colleagues sometimes, then offer a colleague their tentative theorems and proofs and use this to modify and clarify. (from an email message sent to his co-panel-members)

This description fits perfectly Wiles' ways of working, as it equally fits the working habits of many other mathematicians Still, Andrew Wiles' story is special. He solved what seemed to be unsolvable. He proved what appeared to be unprovable in spite of Fermat's famous note made on the margin of a book more than three centuries ago. He did what no one else was able to do. But, in so doing, he paved the way for some others who, although unable to reach so far on their own, could now discover a mistake and thus contribute to the final proof, after all

To sum up, mathematical productivity, communication and meta-cognitive reflection go hand-in-hand and seem to constitute an all-important triad. As far as I am concerned, mathematical communication owes its significance to the co-existence of mathematical production and reflection. This has already been illustrated by the example of communication between Johann Bernoulli and Leibniz on the brachystochrone problem. Notice that proving something does not mean making sure that one's thinking is true beyond doubt; rather, it shows that proving means, first and foremost, increasing coherence by connecting and integrating different mathematical ideas and theories.

Third example: impression from a mathematics classroom – measuring the height of a tower

Some sixth graders estimated the height of a tower by comparing it with objects of reference like a door, the height of a van, the estimated length of a segment of the tower, and so on. They did so with the help of a photograph of the tower. The two teachers simultaneously present in the classroom invited the children both to think about the problem and to compare their methods. One teacher hinted that on a sunny day there would be a special method to think up.

The children worked in small groups first, and then there was a classroom discussion. It can be ascertained from the protocols that there were only two small groups of children who took the sun-and-shadow approach seriously, with boys only in both groups. When during the whole-class discussion the teachers invited the children to propose their solutions, Micha was the first one to come up with the idea using a sun-and-shadow approach. His reasoning boiled down to the following comment:

> You use the shadow by measuring it and then you calculate the height of the tower.

Several interventions followed. The children, as well as one of the teachers, made remarks about the length of the shadow and the height of the tower. Some of them mentioned the incompleteness of the suggested solution, and there were a number of conflicting opinions. All this did not take long, however. Soon, the teacher managed to reproduce with the whole class the thought experiment on the varying shadows of a vertical object, which had been performed earlier by one of the observed groups. A striking feature of this joint thinking process was that among its many active participants there were students who had not dealt with the sun-and-shadow questions in their groups, with girls taking part in the dispute along with the boys.

There was a repeated intervention by Bart, who seemed to have reached a global perspective on the sun-and-shadow question. He claimed that a vertical object and its shadow must be equal twice during one day; however, the boy did not succeed in making his point in a comprehensible manner, and, as a result, his input initially did not influence the discussion.

Then, there was a contribution from another boy, Peter, who during the group work period acted like a little Andrew Wiles in that at a certain stage he stopped listening to what the different members of his group were saying and began to sail his own course. Now, he may have saved the situation by supporting Micha's suggestion.

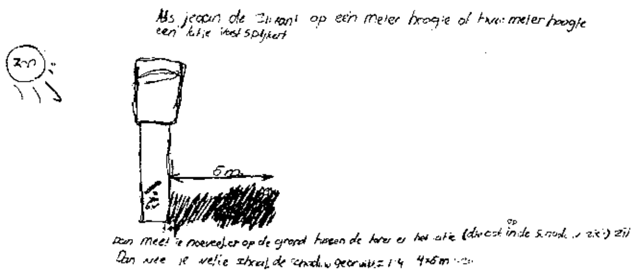

Figure 2

> Nail a little lath to the tower at a height of two meters, for instance. Next, measure the distance from the foot of the tower to its shadow; then I can calculate the scale of the shadow. [He literally said: "Then I know what scale the shadow uses."] Since the shadow of the whole tower can be measured at the same time, the height of the tower can be calculated from it by means of the scale of the shadow.

After some elucidations and additional remarks from others, Micha voiced his objections saying that the tower is a monument, and therefore "You are not supposed to nail a lath to it". At the same time, however, he had to admit that Peter's idea worked nicely. Saskia proposed a compromise: "Why don't you use your own shadow?" Bart intervened again and suggests that Peter's lath could also be used for his own idea, namely to produce a shadow at a distance of two metres from the foot of the tower. Finally, Micha blew the whole theory by asking:

> What if we had a sloping tower like the one in Pisa? Then you still do not know anything.

Comments

This brief classroom observation, which has been presented here only in general terms, reveals that:

- being constructive and productive is a necessary condition for effective classroom interaction;
- this classroom interaction brings about different levels of insight, ranging from ones having to do with some particular, local phenomenon of the sun-and-shadow question right up to those which deal with a global, overall perspective;
- creating a climate which encourages broad participation also can provoke reflection in students who did not deal with the given question before (compare this with the finishing touches to Wiles' proof!);
- some pupils behave like little mathematicians: like Wiles, they would initially work in a solitary manner, and later they would make creative contributions which would allow others to join in whenever the breakthrough ideas resonated with their own experiences. [4]

Conclusion

To sum up, classroom discussion provokes a lot of reflection and gives an opportunity to compare, criticise, refute, complete, reject, and so on. Meta-cognitive shifts are commonplace in such process (compare the historical examples in Streefland, 1996). There is a triad of principles which must be observed if the discourse is to be effective.

The participants have to

(a) be constructive and creative;
(b) communicate mathematically in a productive manner;
(c) allow for meta-cognitive shifts.

The possibility of a productive contribution to mathematical communication by its participants is the first thing to consider, in my view. This brings me back to the starting question:

> Can and should mathematics be learned through conversation?

To this, my answer is as follows. Learning by mathematical communication? Yes, but only if this communication is nourished by the constructive input of all of its participants, and if it promotes the mathematical progress of the classroom community as a whole These examples both from history and the classroom reflect this with clarity

D. Theorizing about mathematical conversation and learning from practice

Paul Cobb [5]

My immediate response to Sfard's question is to ask what the alternative might be, given that conversations occur in all classrooms. For example, Japanese elementary mathematics teaching is often held up as an exemplary form of instructional practice, in which students are encouraged to engage in discussions by explaining and justifying their reasoning (Stigler, Fernandez and Yoshida, 1996) However, when the number of words spoken per minute is taken as a crude measure of the amount of talk occurring in classrooms, the results of the recent TIMSS video-study indicate that there is actually more discourse in traditional American elementary classrooms than in Japanese classrooms (Stigler, personal communication). This finding conflicts with expectations in that traditional American classrooms are often portrayed as being devoid of discourse.

The emphasis placed on discourse in recent reform recommendations (such as the NCTM *Standards*) in fact represents a reaction to this commonly-accepted view of traditional American instructional practices On my reading, the major point of the TIMSS finding is that although there is more discourse in American classrooms, less of mathematical significance is being said I therefore suggest that the question is not whether students should engage in conversations, but instead concerns the nature of those conversations that constitute productive situations for mathematical learning.

In each of the three arguments that Sfard rightly challenges, proponents of conversation attempt to derive instructional implications directly from general, orienting, background theories. In other words, the implicit strategy in each argument is to translate basic theoretical tenets into instructional prescriptions In my view, this rhetorical move involves a basic category error, in that the three theoretical perspectives are descriptive rather than prescriptive. Their basic tenets are concerned with how to interpret human activity and, as a consequence, they simply do not give rise to instructional implications.

The classic example of this type of category error arises when the theoretical commitment to analyze mathematical learning as a constructive process is translated into the instructional prescription that teachers should enable students to construct mathematical understandings for themselves by not telling them anything. Constructivism does not, however, have a monopoly on this type of error. It is also prevalent in discussions of distributed theories of intelligence (as intelligence is distributed, computers and other tools should always be available for students to distribute intelligence over them) and situated learning theories (as mathematical learning is situated, instructional situations should always involve authentic, real-world problems).

I might also note that attempts to derive recommendations for teaching directly from any philosophy of mathematics or science is similarly flawed. In my view, many of the questionable pronouncements that pass for instructional theory reflect this error, thereby subjugating the wisdom of practice to the fervor of ideological commitment.

If we do not rely on general background theories to tell us how to teach, how then might we approach the question posed? I am enough of an empiricist to suggest that we experiment in classrooms, that we look at the conversations that do arise, that we focus in particular on the nature of the students' participation in those conversations, and that we investigate what students might actually be learning in the course of that participation

I would readily acknowledge that this way of framing the question reflects my basic theoretical commitments. My focus on both participation and students' learning indicates that I subscribe to a social constructivist perspective. However, it is one thing for general theoretical commitments to orient how one casts questions, and another for these commitments to serve as the primary touchstone for instructional decisions.

I would also note that the way I have framed the question is unusual (at least in the American context), in that it attends to what students might actually be learning. Frequently, a particular vision of classroom discourse is held up as the ideal and instructional interventions are judged to be successful if actual classroom discourse approaches this ideal. In these interventions, shaping classroom discourse has become an end in itself, and the issue of whether students might be learning any mathematics that is worth knowing is not a focus of investigation. In my view, the justification for any intervention (including those that involve discourse) has to attend to the changing nature of students' participation in the practices established by the classroom community and thus to their mathematical learning

I will illustrate the general approach that I have outlined by describing two aspects of classroom conversations that I and my colleagues have found to be potentially productive for students' mathematical learning I should stress that these features of conversations were not derived *a priori* from theoretical principles, but have instead emerged in the course of a series of classroom teaching experiments conducted in collaboration with teachers over a ten-year period.

The first aspect draws on the work of Thompson and Thompson (1996) and concerns a distinction that they make between calculational and conceptual orientations in teaching We have found it useful to extend this distinction by talking of calculational and conceptual *discourse*. It is important to clarify that calculational discourse does not refer to conversations that focus on the procedural manipulation of conventional symbols that do not necessary signify anything for students. Instead, calculational discourse refers to discussions in which the primary topic of conversation is any type of calculational process This can be contrasted with conceptual discourse in which the reasons for calculating in particular ways can also become explicit topics of conversation. In this latter case, conversations encompass both students' calculational processes and the task interpretations that underlie those ways of calculating.

I can give a clarifying example by referring to a recently-completed seventh-grade teaching experiment that focused on statistics. One of our instructional goals was that the students would come to view data as a single entity rather than as a plurality of individual data values. To this end, we developed a sequence of instructional activities that involved analyzing one or more univariate data sets in order to make a decision or a judgment. In addition, we designed two computer-based analysis tools that students used when conducting their analyses. In one of these tools, individual data points were shown as dots located on an axis of values (i.e., an axis plot). This tool provided students with a variety of different options for acting on data sets. The least sophisticated of these options simply involved dragging a bar to a chosen location on the axis, thereby partitioning the data set into two groups of points. The number of points in each group was shown on the screen and adjusted automatically as the bar was dragged along the axis.

In one instructional task, the students analyzed the T-cell counts of two groups of AIDS patients who had enrolled in different experimental protocols. There were approximately 40 patients in one group and 150 in the other. In the setting of such a task, a calculational explanation involved describing the specific steps taken when conducting the analysis. For a relatively unsophisticated analysis, in which the bar is used to partition each data set, the students might simply explain that they placed the bar at a particular value and then report the number of points above and below this value in both data sets. As it so happened, three of fourteen groups of students conducted analyses of this type. In each case, they placed the bar so that what they called the "hill" of one data sets was mostly below it and the "hill" of the other was mostly above it.

A conceptual explanation of these solutions would involve describing not merely the steps of the two analyses but the reasons for carrying them out with respect to the issue at hand, that of judging the effectiveness of the two experimental protocols. In giving explanations of this type, students in the teaching experiment classroom clarified that they placed the bar at a particular location in order to develop a mathematical description of a perceived qualitative difference between the two data sets. Our classroom observations indicate that the teacher's expectation that the students would give conceptual explanations was crucial to the emergence of the critical notion of distribution, wherein data was viewed as an entity with properties and characteristics. This was because, in these explanations, students' use of the tools and thus their mathematical descriptions of the data were tied to qualitative judgments about the shape of the data sets viewed as single entities rather than as pluralities of data values.

Our experiences in this and other teaching experiments lead us to conclude that discussions in which the teacher judiciously supports students' attempts to articulate their task interpretations can be extremely productive settings for mathematical learning. As these articulations focus on the reasoning that lies behind solution procedures, students' participation in such discussions increases the likelihood that they might come to understand each others' thinking. Had the discussion in the teaching experiment classroom remained calculational, students could only have understood other students' explanations by creating a task interpretation that lay behind their use of the tools entirely on their own. In contrast, the students' participation in conceptual discourse provided them with resources that might have enabled them to reorganize their initial interpretations of tasks.

As the example makes clear, these resources are not limited to what is said, but also include inscriptions and notations that are pointed to and spoken about (cf. Thompson, Phillip, Thompson and Boyd, 1994). In general, we have found that the development of classroom discourse and the development of ways of symbolizing and notating go hand in hand and are almost inseparable. I would also note that in helping students explain their task interpretations, the teacher is simultaneously initiating and guiding the renegotiation of the socio-mathematical norm of what counts as an acceptable explanation. We have found that, within a few weeks, most students routinely give conceptual explanations as the need arises and that they ask others clarifying questions that bear directly on their underlying task interpretations.

The second aspect of discourse that we have found to be productive for students' mathematical learning builds on the first and involves what we call *reflective shifts in discourse*. A shift of this type occurred in the sample episode when the teacher and students compared and contrasted task interpretations. Initially, the task was simply to assess the two experimental treatment protocols and the students explained how they had analyzed that data to arrive at their conclusions. Later, both these analyses and their underlying interpretations became topics of conversation, in the course of which the teacher and students differentiated between additive and multiplicative reasoning about data.

In the case of additive reasoning, students who partitioned the data sets by using the bar compared the number of data points either above or below the bar (i.e., part–whole reasoning about data), whereas, in the latter case, they compared the proportion of the data points in the corresponding parts of the two distributions. What had previously been said and done in action was now a topic of conversation in that the theme of the discourse was the structural characteristics of the students' analyses. I and my colleagues have argued elsewhere that in making shifts of this type, the classroom community engages in a collective act of reflection (Cobb, Boufi, McClain and Whitenack, 1997).

In explaining why reflective shifts in discourse can be productive for students' mathematical learning, I draw heavily on Sfard's (1991) theory of reification. Briefly, she accounts for mathematical development in psychological terms by contending that students' initial, action-based operational conceptions evolve into object-like structural conceptions via a process of reification. When we analyze instances of reflective shifts in discourse, we see this same process occurring in the collective activity of the classroom community. Consequently, we hypothesize that opportunities for students to reflect on, objectify and reorganize prior mathematical activity arise as they participate in reflective shifts in discourse.

In making this conjecture, we note that students do not just all spontaneously happen to begin reflecting at the same moment. Instead, their reflection is enabled by their participation in the discourse, and by reflecting they contribute to the shift in discourse. Further, when viewed in broader terms, conceptual discourse that involves reflective shifts is consistent with Streefland's observation in this panel discussion that discourse needs to be both mathematically constructive and productive. In the sample episode, for example, both the notion of data as constituting a distribution and the contrast between additive and multiplicative reasoning came to the fore. These are significant mathematical ideas, the discussion of which served to advance our pedagogical agenda.

In concluding this brief discussion of classroom discourse, I should clarify that when I and my colleagues first began working intensively in classrooms ten years ago, we required students always to work in small groups and then to participate in whole-class discussions. Thus, if students in traditional classrooms were not allowed to explain how they actually interpreted and solved tasks, the students in these classrooms were not allowed to shut up and engage in mathematical activity on their own.

Over the years, we have modified the classroom activity structure in several ways. For example, the students now often work individually, but on the understanding that they can talk with peers of their choosing as the need arises. One of the teacher's responsibilities is then to help the students learn how to use their peers effectively as resources for their learning. In addition, we do not organize a whole-class discussion of students' individual or small-group work, unless we anticipate that mathematically significant issues that will advance our pedagogical agenda might emerge as explicit topics of conversation.

Thus, whereas we were previously satisfied if students engaged in a discussion that appeared to have a mathematical theme, we now consciously plan discussions by first monitoring the various ways in which students are interpreting and solving tasks. We then often select particular students to call on because we conjecture that a potentially significant issue might emerge with the teacher's guidance, either when discussing a particular solution or when comparing two or more solutions.

In taking this approach, we no longer value discourse for its own sake. Instead, each classroom discussion, viewed as social event in which students participate, has to be justified in terms of whether the issues that emerge contribute to the achievement of our potentially-revisable goals for students' mathematical learning.

Ironically, the discussions conducted in the classrooms in which we worked ten years ago often appear to the untrained eye to be superior in that they are characterized by a hubbub of activity. However, when we focus on the students' participation and what they are learning, the discussions conducted in recent classroom teaching experiments are strikingly superior to those in the first classrooms in which we worked. As was the case with the comparison of the Japanese and American classrooms, more that is mathematically significant is being said, even though there is less talk.

E. Talking while learning mathematics versus learning the art of mathematical conversation
John Mason

> Do you believe that mathematics can and should be learned through conversation (in the sense of an interactive oral exchange)?

No, but ...

> I know that mathematics is learned through, among other things, conversation (with oneself, with an imagined other, with colleagial others, with sceptical others, and with past and present authors).

How do I know?

> I know this from interrogation and analysis of my experience, validated through resonance from others.

Conversation and discussion

The term 'conversation' is sometimes taken to cover any verbal interchange, especially social interchange, between people. 'Conversation' has also been taken to refer to the entire interaction between the individual and the social, whether in the form of other people or with the culture (through acting and interacting) or more particularly, the historical detritus of previous 'conversations' in the form of genres of text and patterns of discourse. Thus, Maturana (1988) sees conversation as *the* way of being-in-the-world.

'Discussion' is usually taken to mean a more focused conversation. For example, in mathematics, Pirie and Schwarzenberger (1988) have suggested that it requires four elements: to be purposeful, on a mathematical topic, with genuine pupil involvement, and interactive. They were trying to circumvent the classic teacher report 'we discussed', simply meaning that the teacher expounded. At issue for most practitioners is achieving productive mathematical discussion: one common teacher fear is that students will chat about other things, and in some classrooms, as well as mathematical discussion, there is a great deal of unfocused or off-task interaction which may or may not be seen as productive and conducive to learning mathematics. Note, though, that adults in the work-place also talk about a variety of topics, because personal communication is vital to effective task communication in a non-authoritarian environment – but like anything, it can be overdone.

Fundamental to productive mathematical conversation and conversation about mathematics is the development of a conjecturing atmosphere (Mason *et al.*, 1984; see also Baird and Mitchell, 1986, and Baird and Northfield, 1992). In a conjecturing atmosphere, everything that is said is said as a conjecture, uttered with the intention of externalising thoughts so as to be able to examine them critically, and to modify them, often as a result of other people's comments. This is by way of contrast with an atmosphere in which utterances are expected to be pre-formulated, correct, and justifiable. In a conjecturing atmosphere, there is a shared struggle to find ways to express and convince which can be understood and appreciated by others, as well as challenged, exemplified, amplified, varied, generalised, etc. by them.

Some people seem to describe conversation as though it were the principal or even the sole means whereby students (re-)construct ideas; that collectively students can (re-)discover the essential features of mathematics. But this makes no sense in any reasonable perspective, as without encounters with the ideas of others, whether from the historical past or the relatively-expert present, most students are unlikely to reconstruct the important ideas of mathematics. They need to be in the presence of more highly structured awarenesses, in the form of carefully constructed tasks, exposition, and people. Or as Vygotsky (1981) put it, in the presence of 'higher psychological processes', which they then reconstruct and adapt.

Thus, a critical component of effective discussion is the presence (possibly virtual) of a relative expert. This presence is not mere physical presence but rather the presence of awareness of mathematical thinking processes as well as content. For example, a group of dedicated and disciplined students can have fruitful discussions based on reading exposition by an author; the effectiveness of their discussions will be determined by the way of working which the group develops, including mutual support, individual and group intention, as well as group discipline.

The social constructivist-based argument that since mathematics is a discourse, students will somehow learn mathematics through developing a discourse amongst themselves, is as specious as the behaviourist-based argument that students will learn mathematics by being conditioned or trained to employ techniques and algorithms correctly on typical test items.

Conversation as a mode of interaction

Conversation is a form of interaction. In Mason (1979), based on ideas of Bennett (1966), I elaborated how action is essentially triadic in quality, involving affirming, mediating, and responsive impulses. I label the six actions which arise when the tutor (standing for a relative expert), student, and content take up those three roles in all possible ways. The result is what I call the six Ex's, with names chosen because they come close to capturing the essential quality of the respective modes - but note that these words are being used in technical ways. Each triangle can be 'read' using the formula 'the *affirming* acts upon or contacts the *response*, facilitated or enabled by the *mediator*'. In the diagram below, the affirming is at the top, mediating in the middle, and responding at the bottom.

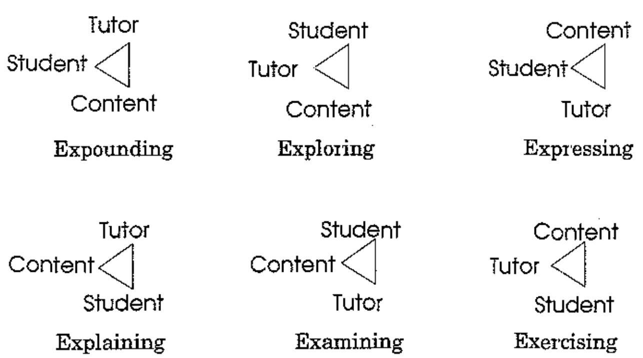

Figure 3

It is easy to generate the six diagrams mathematically, but more difficult to make sense of them. In particular, some traditional meanings of the labels have to be amplified or modified in order to capture the quality of the corresponding mode. Briefly summarised, the six modes are as follows.

Expounding

Expounding is what produces exposition, the form which is most familiar in texts. True exposition occurs when the presence (actual or virtual) of the audience enables the expounder to make contact with the content in fresh and vital ways. The experience of preparing a session, with a flood of rich connections amongst which choices have to made, is typical of the energy of this action. The audience is attracted into the world of experience of the expounder, whose contact is heightened by the presence of the audience. The expounder could be said to be in conversation with the content, enabled by the students' presence.

Explaining

True explaining comes about when the tutor or relative expert tries to enter the world of the student, by means of, through, and concerning, the content. As soon as 'the difficulty' becomes clear, the temptation is very strong to slip into expository mode. To an observer, tutor and student may be in 'mere' conversation, but the energy and attention is highly focused on the student's experience.

Exploring

The student takes the initiative and makes contact with the content through the presence and guidance of the tutor. The student might be reading a text and actively making sense (reconstructing, re-specialising, re-generalising, conjecturing, re-justifying) guided by the text, or by materials designed by a relative expert. In a sense, the student is in conversation with the content, mediated by a virtual or actual relative expert manifested in person or as structured materials, text, etc.

Examining

The students submit to the relative expert in order to validate their own judgement that they understand and have re-constructed appropriately. This is different in energy from the degenerate form of assessment in which students are tested because someone else has decided the time and place independently of whether any particular student is ready or not. It is vital that students validate their own criteria against those of the expert, rather than simply submitting to being tested like a quality-control check on a production line. In some forms of examining, such as oral testing, a conversation could be said to be going on, but the nature and focus of that conversation is assessment orientated.

Expressing

In expressing, the content bursts forth spontaneously, often unstoppably, using the presence of an audience to release energy of connection-making and understanding, however conjectural. This is characteristic of brainstorming, and of the triggering of idiosyncratic connections which marks social conversation, but which also plays an important part in working on mathematics. It is almost as if the content is using the student in order to participate in conversation!

Exercising

Here the content draws the student into rehearsal to mastery, as a child will repeat certain actions over and over until they are automatic. Note the contrast from the degenerate form in which the tutor urges exercises upon the student who then completes them with a minimum of energy and attention. This action could be seen as a conversation between the student's awareness and some of their functioning selves as they integrate skills and techniques into their soma.

Problematicity

For me, the various modes of interaction are mutually supportive, so that personal and collective work prepares the ground for hearing what an expert has to say, and hearing what an expert has to say prepares the ground for personal and collective work. Engaging students in mathematical conversation and in conversation about mathematics is not the issue, but there are some significant issues which do arise.

Intention

Education is not something that is done to people; it is an action leading to activity in which people participate in different ways and with different intensities. This view is consistent with that of Leont'ev (1981), who distinguished three levels of activity: energised or motivated activity, action defined by a goal, and operational activity. Operational activity is unaware of goals or subgoals, and energised or motivated activity may not have conscious goals

Thus, conversation-discussion is not in itself valuable Rather, its value at any time depends on the role played in the situation, and the commitment of individuals and of the group collectively

Problematicity of changing modes

Getting meaningful discussion started is not necessarily difficult in itself. The technique of talking in pairs is an excellent way to start, having students rephrase something for themselves to each other before inviting them to contribute to a more plenary discussion. The opportunity to try something out in relative safety, discovering that what you think is indeed worth saying and not ridiculous, supports more sophisticated contributions to discussions Once established, a conjecturing atmosphere as described earlier supports people in exposing their uncertainties as conjectures which can then be worked on, modified, and developed without fear of ridicule or unpleasant exposure.

Moving into individual work is just as important as collective work, in order to allow individuals to reconstruct ideas, situations, and techniques for themselves Merely being present when someone else does something can be sufficient for some people in some circumstances, but usually individuals need to spend time re-visiting and re-constructing for themselves, so that they can hope to reconstruct again in the future when it is required.

Most difficult is moving from individual work to collective work: listening, adapting to and building on others' thinking, learning to suppress one's own approach in order to appreciate someone else's, learning to express one's own approach in ways which others can enter and appreciate

Acquainting students with the different modes, their requirements and their effects helps educate their awareness and so help them to learn from experience.

F. Believing in learning through conversation is not enough: a conclusion

Anna Sfard

So, should we teach mathematics through conversation or not? All the authors may not be entirely univocal on the issue and, in the present context, the notion of conversation may not mean exactly the same thing to each of them However, in the wide range of differing opinions and arguments, there is a common hard core. Albeit with varying emphases, all the authors agree that mathematical conversation does seem to have great potential as a mode of learning; yet, on the other hand, only certain types of conversation are likely to bring this potential to fruition.

In spite of Cobb's and Mason's sober reminders that only experience can bring an ultimate verification of the claims for the pedagogical advantages of conversation, I feel a summary urge to make a theoretical comment. I wish to admit now my own faith in the power of conversation and to stress that this faith is not an isolated opinion, but rather a matter of a world-view according to which *all* our thinking, with mathematical thinking being no exception, is essentially discursive. [6]

Like interactionists and neo-pragmatists, and probably like most of the panelists in this group, I view our conceptual systems, and thus both our human selves and the world each one of us lives in, as created through and within the activity of speaking, whether public or private Being discursive creatures, we cannot simply step out of the discourse Discourse is where all our cognitive acivities start, exist, and come to closure.

As such, all these activities are essentially social, and even their occasional appearance as leading to universal and mind-independent results is but a discursive by-product. If so, the more aware we are of the discursive processes that constitute our mathematical activity, the better chance we have of attaining appropriate control of these processes; the better our control, the more effective our students' learning In short, the question is not *whether* to teach through conversation, but rather *how* Since learning mathematics may be equated to the process of entering into a certain well-defined type of discourse, we should give much thought to the ways students' participation in this special type of conversation might be enhanced.

And when it comes to directing and orchestrating a helpful exchange, all the panel participants seem to agree on the decisive role of the teacher (or any other 'relative expert'). To a great extent, it is up to him or her whether a given mathematical conversation, designed for the purpose of learning, will be a success or a failure. There are many ways to turn classroom discussion or group work into a great supplier of learning opportunities; there are even more ways to turn them into a waste if time, or worse than that - into a barrier to learning.

As it happens, futile, useless, and even potentially harmful types of discursive activities can be observed only too often in mathematics classrooms all over the world. One such case, now being described by Carolyn Kieran and myself (Sfard and Kieran, in preparation), comes from our own teaching experiment carried a few years ago in Montreal. Two reasons may be responsible for this state of affairs First, to charge the teachers with the responsibility for the effectiveness of the conversation is easy, but to support them in attaining this goal with helpful advice is difficult. Orchestrating a productive mathematical discussion or initiating a genuine exchange between children working in groups turns out to be an extremely demanding and intricate task

Second, our Montreal experience has shown with particular clarity what psychologists have long known: communication skills cannot be taken for granted. Children are not necessarily born with a natural talent for transferring their intentions to others or with knowledge of how to interact with others to enhance everybody's understanding. If conversation is to be effective and conducive to learning, the art of communicating has to be taught. How this is to be done, and what exactly should be learned by the students remains a question to which the mathematics education community has yet to give much thought.

Perhaps all one can say right now is that for a conversation to be productive, it has to have the characteristics of a true dialogue. For a top-level description of the task, it may be useful to consult Gadamer (1975), according to whom a true dialogue:

> is a process of two people understanding each other. Thus it is characteristic of every true conversation that each opens himself to the other person, truly accepts his point of view as worthy of consideration and gets inside the other to such extent that he understands not a particular individual, but what he says. (p. 347)

I hope that the present exchange among we five authors may serve as an illustration of the kind of conversation Gadamer had in mind

Notes

[1] This debate originally took place orally in June 1997 in Calgary, Canada, at the Third International Conference of History, Philosophy and Science Teaching The panel was chaired by Anna Sfard, who also edited this written version

[2] The shift of the academic focus toward the social and away from the pyschological has played an important contributory role.

[3] During this period, Andrew Wiles walked a lot along a lake, where it was quiet and peaceful. According to his own account, it was good to relax there, to concentrate and to let the 'subconscious mind' work

[4] The loneliness of a top mathematician is present in the position of Bart, whose initial insights are beyond those of the rest of the group

[5] The analysis reported here was supported by the U S Office of Educational Research and Improvement (OERI) under grant number R305A60007 The opinions expressed do not necessarily reflect the views of OERI.

[6] See Edwards and Potter, 1992; Harré and Gillet, 1995

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