Improving Diagrams Gradually: One Approach to Using Diagrams in Problem Solving

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It is widely claimed that drawing diagrams or pictures is helpful for solving problems in general [e.g. Gick, 1989], for solving mathematical problems in particular [e.g. Hembree, 1992; Kersh & McDonald, 1991; Lakin & Simon, 1987], and that this kind of activity is considered to be one of the problem-solving strategies [e.g. Charles *et al*, 1985; Eicholz *et al.*, 1985]. Its usefulness lies, we think, in the fact that it can show relationships among elements in the problem clearly. For example, Nickerson, Perkins & Smith [1985] state:

> The intent of this heuristic is to concretize the problem. Part of such concretization has to do with visual thinking: once a graph or diagram is drawn, the problem solver can bring perceptual processes to bear on it. Also, a visual representation of a problem can make apparent certain relations among parts that might otherwise go unnoticed. [pp 75-76]

It is usually believed that relations among the parts of the problem comprise an inherent structure of that problem or a part of that structure. If diagrams show appropriate relations among the parts of the problem, then diagrams must reflect the inherent structure, especially in successful problem solving. The survey implemented by Doishita *et al* [1986], however, queries the optimism about the function which diagrams are expected to have in problem solving processes. They had fourth graders solve word problems and draw diagrams as part of their solutions. Many of the successful students wrote number sentences first, and then drew diagrams, while many unsuccessful students drew diagrams first, and then wrote number sentences

Furthermore, the diagrams drawn by unsuccessful students were often incorrect. Doishita *et al.* [1986] said: "More successful students can draw correct diagrams because they can solve the problem. Less successful students cannot solve the problem because they cannot draw correct diagrams." [p. 77] This finding suggests that the function of diagrams to show appropriate relations cannot directly help unsuccessful solvers. In order that we can treat the heuristic of drawing diagrams as a really helpful method for unsuccessful solvers, we need to explore diagrams that appear during the problem-solving process in detail.

1. Diagrams and solvers' structures of the problem situation

In the problem-solving process, the solver gives a certain structure to the problem situation in which the question is asked. This structure consists of those elements the solver recognizes in the situation, the relationships (s)he establishes among these elements, and the senses (s)he gives to the elements or the relationships. So this structure is a product of the interaction between the solver and the problem situation. Let us call it *the solver's structure of the problem situation* [Nunokawa, 1990]. It does not necessarily remain the same during the problem-solving process. Rather, it usually changes in the direction of the mathematical structure inherent in the problem [Nunokawa, 1992a].

Diagrams can be considered to be one kind of outer representation of the solver's structure of the problem situation. So diagrams must reflect the solver's structure. But unlike the inherent structure of the problem, the solver's structure often changes as the solving process proceeds. Therefore, in such cases, diagrams can also be expected to change according to the alteration in the solver's structure. In addition, the comments by Nickerson *et al.* [1985] that diagrams can show the solver a new relationship (s)he had not noticed before, can be interpreted as a suggestion that diagrams can trigger a change in the solver's structure of the problem situation. That is, diagrams and the solver's structure of the problem situation influence each other, and both of them are changing progressively during the problem-solving process.

Particularly when solving non-routine problems, the naive structure the solver constructs from the problem statement must differ from the mathematical structure inherent in the problem. (For otherwise, the solver could solve the problem at once and the problem cannot have been a non-routine one.) Therefore, it is natural to think that a change of the solver's structure must occur during the solving of non-routine problems.

From this point of view, diagrams drawn in the problem solving process are not necessarily complete, nor suggest the final answer from the outset. Diagrams are useful in so far as they can trigger a change in the solver's structure of the problem situation, even if such changes are themselves incomplete.

2. How the way of drawing changes during the solving process

Consider the following problem-solving session as an illustration. This session is one of nine implemented by the author. The subject was a graduate student whose major was in mathematics education. He was asked to solve the problem in think-aloud mode, and the whole session, including the interview, was both audio-and video-taped. (More detail can be found in Nunokawa, in press.) In the session we analyze here, the following problem was used:

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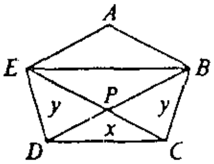
> A given convex pentagon ABCDE has the property that the area of each of the five triangles ABC, BCD, CDE, EDA, and EAB is unity. Show that all pentagons with the above property have the same area, and calculate that area. Show, furthermore, that there are infinitely many non-congruent pentagons having the above area property. [Klamkin, 1988, pp. 1-2]

It took about 90 minutes for him to reach his tentative solution, which was not complete. This indicates that this problem was challenging for him and required genuine problem solving.

According to Klamkin [1988], the area of the pentagon having the above property can be calculated in the following way:



Let [ABC] denote the area of triangle ABC, etc. Since [EDC] = [BCD] = 1 (see figure), both of these triangles have equal altitudes to side CD. Hence DC \\ EB. Similarly, the other diagonals are parallel to their respective opposite sides. Thus, ABPE is a parallelogram and [PEB] = 1. Letting x = [PDC] and y = [BPC] = [EDP], we havethat $x + y = 1$. Also

$$\frac{[\mathrm{EDP}]}{[\mathrm{EEPB}]} = \frac{\mathrm{DP}}{\mathrm{PB}} = \frac{[\mathrm{PDC}]}{[\mathrm{PCB}]}$$

or

$$\frac{y}{1} = \frac{x}{y}$$

Eliminating x, we get $y^2 + y - 1 = 0$, so that

$$y = \frac{\sqrt{5} - 1}{2} \text{ and } x = y^2 = \frac{3 - \sqrt{5}}{2}$$

Then area

$$[\mathrm{ABCDE}] = y + x + y + 2 = \frac{5 + \sqrt{5}}{2}$$

[p. 52]

In the solving process the subject drew 19 diagrams of the pentagon satisfying the given conditions. These diagrams can be categorized into three types. Figure 1, Figure 2, and Figure 3 show examples of these three types of diagrams. The differences among these diagrams, however, may not be clear only from looking at these particular diagrams. In fact, the differences lie in the ways of drawing each diagram.

When drawing diagrams like Figure 1, he drew the five sides of the pentagon first. Then he drew the line segment AC in order to construct ΔABC (Figure 4a), the line segment BD in order to construct ΔBCD (Figure 4b), and so on.

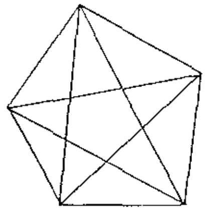


Figure 1

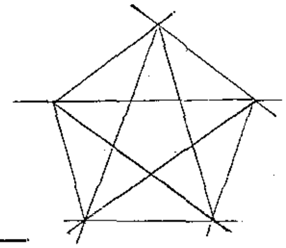


Figure 2

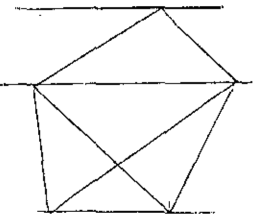


Figure 3

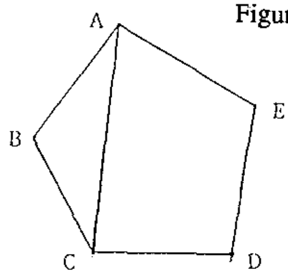


Figure 4a

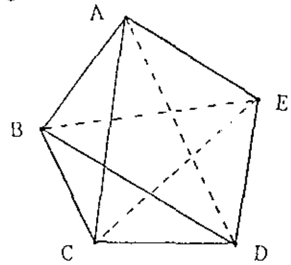


Figure 4b

When drawing the diagram represented by Figure 2, he drew two parallel lines which were horizontal. Then he drew two more parallel lines, shown as γ and δ in Figure 5a. In a similar way, he added two parallel lines three more times (Figure 5b, Figure 5c, and Figure 5d). As the result of this process, he obtained the pentagon shown in Figure 2.

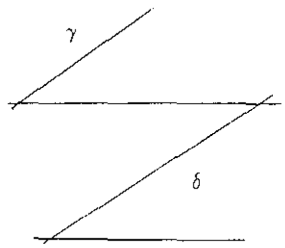


Figure 5a

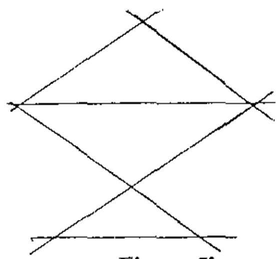


Figure 5b

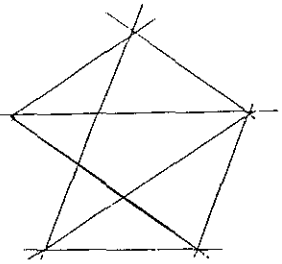


Figure 5c

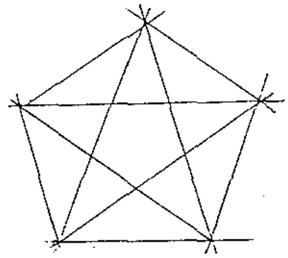


Figure 5d

When drawing diagrams of the type shown in Figure 3, he drew two parallel lines in a horizontal position just as he drew in the second type of diagram. Next however, he drew two additional lines to construct a triangle which lies between the parallel lines (Figure 6a). Then he drew two more lines to construct another triangle which lies between the parallel lines (Figure 6b). He then added a triangle on the above line (Figure 6c), and drew a line parallel to the parallel lines drawn before (Figure 6d), and lastly added another triangle between the new parallel lines.

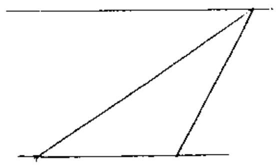


Figure 6a

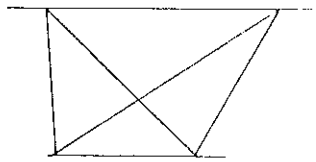


Figure 6b

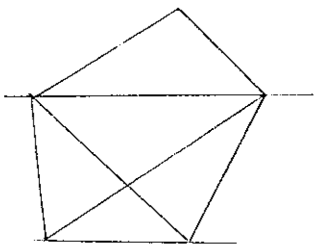


Figure 6c

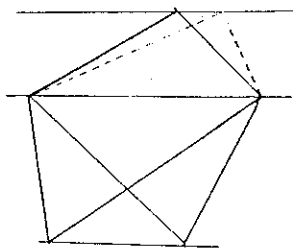


Figure 6d

Even though the ways of drawing are different, the results of the drawing, i.e. the diagrams as such, are almost identical. So it is possible that the difference in the ways of producing these drawings occurred out of caprice. Analysing the protocol of this session together with the diagrams, however, we find that each way of drawing the diagrams reflects the solver's structure of the problem situation at that time. In this problem the situation is the pentagon satisfying the given conditions, and this remained fixed during the problem solving process. But the structure the solver constructed in response to this situation, or the way he saw this situation, changed as the solving process proceeded.

When he drew one diagram in the way shown in Figure 4, he was reading the problem statement. In fact, the order of drawing the elements of the figure corresponds to the order of their appearance in the problem statement; from the five sides of the pentagon to the lines forming the sides of the small triangles. In other words, the first type of diagram reflects the naive structure constructed during reading the problem statement.

Just before drawing a diagram the second way for the first time, he had recognized that each diagonal was parallel to the opposite side (he said: "then each pair of lines makes parallels"). Using the second way of drawing, he drew five pairs of parallel lines. One of the two lines of each pair became a side of the pentagon and another became a diagonal. The second way of drawing reflected the relation the solver had established in the pentagon, i.e. the problem situation, at that time. To this solver, the pentagon was not only the figure satisfying the given conditions, but also the figure which had five diagonals parallel to the respective opposite sides, or even the figure constructed from five pairs of parallel lines.

Shortly, before drawing a diagram in the third way, the solver tried to show that there are infinitely many pentagons satisfying the given conditions. To show this, he used the idea that if a triangle was transformed between the parallel lines with the base fixed (Figure 7), the area of the triangle did not change (this explanation was made using a diagram drawn in the first way).

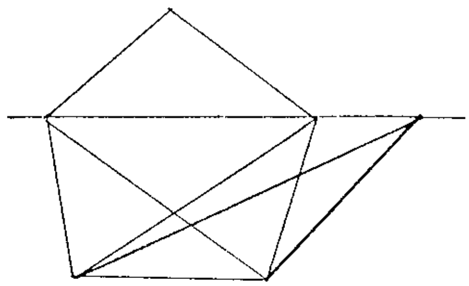


Figure 7

This idea might have called his attention to the pair of parallel lines and the triangles lying between these lines. Furthermore, he said that he could treat the triangle at the top of the pentagon separately, and he called it "a cap". So we find that he now considered the pentagon in the problem situation as a figure having two triangles between a pair of parallel lines and an additional "cap". It is clear that the third way of drawing a diagram reflected this version of the solver's structure of the problem situation.

From the above discussion, we can say that the way of drawing the diagram reflected the way in which the solver saw the pentagon, i.e. the solver's structure of the problem situation, at that time. These two things, the diagrams and the solver's structures, went hand in hand. We should notice, however, that we can reach this finding only when we take account of the way, or the order, of drawing a diagram as an important aspect of the drawing process. Attending to the way, or the order, of drawing a diagram can allow us to recognize this difference. And it also allows us to be more sensitive to the difference between the solver's structures of the problem situation reflected by these diagrams.

The diagrams shown in Figure 1 are very similar, which suggests that they remained almost unchanged through the solving process. However, taking account of the way of drawing, we can keep to our standpoint from which diagrams change as the solving process proceeds. Indeed, the solver improved his diagrams gradually.

3. The meaning of diagrams and their senses

In the last section, we discussed the possibility that, by taking account of the way of drawing, we can find the subtle differences among diagrams, differences which reflect variations in the solver's perceived structure of the problem situation. The content the way of drawing indicates can only be comprehended when we take account of the protocol and put the diagrams into the context of the solver's solving process.

Wertsch [1985] distinguishes two aspects of what is referred to by the words "meaning" and "sense", terms which are due to Vygotsky. Meaning (*znachenie*, Vygotsky, [1982]) is the decontextualized aspect of words, while sense (*smysl*) is the contextualized aspect of words. "Vygotsky considered the ways in which the structure and interpretation of utterances depend on their relationship with their extralinguistic and intralinguistic contexts." [Wertsch, 1985, p. 53] The solvers sometimes introduce extra information, which is not necessarily comprehended from the problem statement by the other people present. Such content of the problem statement is dependent on the specific solver's problem-solving activity; therefore, it is the sense given to the problem statement by the solver. The sense can play a positive role in the solving process. [Nunokawa, 1992b]

Here, let us consider diagrams or pictures from this point of view. If we read the statement of the problem mentioned above and then look at the diagrams shown in Figures 1 to 3, we can see in them the pentagon and several triangles inside this pentagon. These elements can be comprehended from these diagrams even if we do not stay with the solver and share the context with him. Since the information that there is a pentagon having some triangles in it, which is shown by the diagram, is independent of the context of this solver's activity, it becomes the *meaning* of this diagram.

The information, however that the pentagon is constructed of one pair of parallel lines, the triangles between these lines, and the "cap", cannot be comprehended only from the problem statement and the picture shown in Figure 3. As discussed in the last section, this information can only be comprehended by us when we consider the way of drawing the diagram as well as the protocol of the solver's problem-solving activity. In other words, trying to understand the context of this solving activity makes this information, which we can call the *sense* of this diagram, explicit.

The solution that the subject in the last section achieved at last involved adding the area of the trapezoid, which was made up from the pair of parallel lines, and the area of the "cap", in order to get the area of the pentagon. This slightly differs from the solution Klamkin gave. And it clearly reflects the solver's structuring of the problem situation corresponding to the third way of drawing a diagram: the pentagon is constructed from one pair of parallel lines and a "cap". This means that the sense of the diagram is closely related to his solution and plays an important role in his problem-solving process. That is, we can understand his problem-solving process only when we pay attention to the sense of the diagram

From this discussion in the last section, we can say that the way of drawing the diagrams helps us understand the sense of these diagrams. In other words, the way of drawing the diagrams reflects the sense of the diagrams and can complement the meaning which is displayed by the diagrams themselves, i.e. the final products of the solver's drawing.

4. Conclusion

As mentioned at the beginning of this paper, most educators and researchers agree that diagrams are useful for mathematical problem solving. Attention has, however, been paid mainly to the diagrams themselves, which are the products of drawing activities. For example, Larkin & Simon [1987] discussed the power of diagrams, and their attention was focused on what makes a "good" diagram even when they mentioned the knowledge of how to construct a diagram [p. 99].

Taking the view that the problem-solving process consists of changes to the solver's structure of the problem situation, the revision of diagrams usually occurs according to changes in the solver's structure. So our attention should be paid not only to a "good" diagram but also to the intermediate diagrams appearing during the revising process. In Larkin & Simon [1987], the diagram indeed changed the data structure, but the diagram remained the same throughout the solving process. In this paper, it is expected that diagrams interact with the solver's structure of the problem situation, and the diagrams themselves can also change.

But some changes in the diagrams cannot be noticed from their outer appearance. Recognizing these changes requires taking account of the way of drawing the diagrams. Taking the way of drawing as an important aspect of a diagram, one which is closely related to the sense of the diagram, we can analyze the detailed relationship between diagrams and the problem-solving process and understand the roles diagrams can play in that process. Such understanding will lead to the construction of appropriate instructional suggestions concerning diagrams

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Outside and inside form a dialectic of division, the obvious geometry of which blinds us as soon as we bring it into play in metaphorical domains. It has the sharpness of the dialectics of *yes* and *no*, which decides everything. Unless one is careful, it is made into a basis of images that govern all thoughts of positive and negative. Logicians draw circles that overlap or exclude each other, and all their rules immediately become clear. Philosophers, when confronted with outside and inside, think in terms of being and non-being. Thus profound metaphysics is rooted in an implicit geometry which—whether we will or no—confers spatiality on thought; if a metaphysician could not draw, what would he think? Open and closed, for him, are thoughts. They are metaphors he attaches to everything, even to his systems. [...] Hyppolite spoke of "a first myth of inside and outside." And he added "you feel the full significance of this myth of inside and outside in alienation, which is founded on these two terms. Beyond what is expressed in their formal opposition lie alienation and hostility between the two." And so, simple geometrical opposition becomes tinged with aggressivity. Formal opposition is incapable of remaining calm. It is obsessed by the myth. But this action of the myth throughout the immense domain of imagination and expression should not be studied by attributing to it the false light of geometrical intuitions

Gaston Bachelard