Kevin: a Visualiser Pupil

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1. Introduction

The role that mental images play in human thought processes and in the construction of mathematical knowledge has been the object of study, throughout time, by different people: philosophers, poets, speakers, religious teachers and psychologists (Kosslyn, 1983; Sommer, 1978). Over the last twenty years or so a number of researchers in mathematical education have dedicated some of their efforts to investigating where mental images and visualization have made an outstanding contribution to learning and teaching mathematics. [1] Some of them suggest that visual thought can provide an alternative and powerful resource in learning mathematics. Visualization opens a door for a wide range of ways of thinking, different from the traditional form where formalism and symbolism dominate teaching

Our research intends to make a modest contribution to the topics of images and visualization, its principal area being to analyze the use that teachers and students may or may not make of mental images, when they either teach or learn mathematics A careful study of some of our empirical data will be carried out, suggesting possible reasons that could help understand our results.

Although our work is fundamentally related to visual images, that is mental images with a strong visual component, the observation and analysis of the students' and teacher's work (that is, the external representations they make - the pupil, when s/he solves a problem or the teacher, when s/he is explaining), has allowed us to interpret the thought processes and the ways in which 'our' students and 'their' teacher made sense of the concepts and mathematical ideas.

There are some terms in studies related to mental images that are contentious, since they are defined in different ways by different authors. In our work, we call a *mental image* a cognitive construction that the mind creates through one or more senses; we shall use the verb *visualize* to refer to the process whereby mental images are related and the mind is actively changing (for example rotating, moving or otherwise transforming) the image The term *visualization* refers to the fact of being able to visualize.

Following Presmeg's (1986) investigations, a *visual method* of solution will be referred to as one which involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are employed A person's mathematical *orientation to the visual* is the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and non-visual methods.

'Good' visualisers are those individuals who, besides preferring the use of visual methods for mathematical problems, have the ability to turn, manipulate, slide and mentally transform an object being constructed, as well as relating those images to ones previously constructed.

In the research she carried out with secondary students and teachers, Presmeg (1985) identified five kinds of different imagery: *concrete imagery, pattern imagery, memory images of formulae, kinaesthetic imagery* and *dynamic imagery*. We make use of this classification to differentiate the forms of imagery used by the subjects in our study

Bishop (1983) draws attention to the fact that if we want to examine or to study visualization in mathematics, tests that include as many figurative elements as not should be used. Taking the above into account, we think that it is important in experimental studies such as ours, because visualization can not only exist in geometry, but also in algebra or arithmetic.

From a cognitive point of view, Hershkowitz *et al.* (1996) emphasize several aspects of the development of visual thought in the school years:

- visualization is an essential part of human intelligence;
- visual development does not occur through a linear approach;
- a phenomenological approach to learning mathematics can give the student a better understanding of space and shape (starting in a problem situation, searching for patterns, rich contexts instead of poor ones, the role of re-invention)

It is possible to develop and to improve the visualization capacity of pupils, permitting them to approach mathematical problems of a certain difficulty with a greater guarantee of success, and therefore visual thought can be an alternative and powerful resource to learning mathematics (Guzmán, 1996)

2. Experience report

Our research with students has had as its main purpose obtaining information on their mathematical comprehension level and on the use they may or may not make of visualization processes in solving mathematical problems, as well as to find out their beliefs about teaching and learning processes in general, and about mathematics in particular.

The data used in support of our ideas and conclusions has been obtained from different sources:

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- clinical interviews (both video and audio-taped);
- formal interviews with the teacher;
- students' classwork;
- exams;
- daily diary of the researcher

The criterion used for the selection of the students was their score on the WSAT (Wheatley Spatial Ability Test - Wheatley, 1978), which measures the students' spatial ability to rotate two-dimensions figures This test has been used in studies carried out at Florida State University (USA) by various researchers [2], who found that pupils who scored highly on the test were more 'competent' in mathematics when compared on non-routine problems than those who obtained low scores.

2.1 The case study: Kevin

Our objective is to describe and to analyze an uncommon situation: it deals with a student, Kevin, who judged against other studies (Presmeg, 1985; Zazkis, Dubinsky and Dautermann, 1996) and our own research can be considered as a good visualiser.

Kevin was selected for this research because of his high score on the WSAT. He scored 90.5 Out of 100 questions in the test, he answered 95 in the time allowed, correctly solving 92 of them In relation to the WSAT, he is a student with a high score. At the time, he was 14 years old

According to his teacher, he is a 'special' pupil, given the fact that he displays attitudes different from most students Our impression, based on the data and the time we have worked with him, is that he is very extrovert, with a great verbal facility and ideas of his own about school, teaching -learning processes and mathematics

In order to collect information about Kevin, we interviewed his teacher. She considered that he could be a good student:

> He is a problematic kid, but at a behavioral and personality level. At the knowledge level he isn't problematic [...] he's a pupil who could be brilliant if he were constant, if he turned up; if he didn't have those big problems in his family, of personality, of self exigency, he could be a very brilliant pupil

Due to his social environment, which is low, Kevin could be considered to be pupil 'with problems' (according to his teacher). Our opinion as investigators is that Kevin was a pupil who collaborated enthusiastically in the interviews, attending every one. There were nine in total, one every fifteen days for five months. He was always waiting with great enthusiasm on the day of the interview And after the interview he always helped us to take the equipment to the car.

During the interview, he endeavoured to express his thoughts and reasoning in the clearest possible way, in order to make the investigation work easier. He never got discouraged and his efforts and concentration were always focused while he was solving a problem.

We would like to remark on the different attitudes of the pupil in his school work and in his collaboration with this research. We explain this change in the interest as it raised his motivation because it valued his ability to solve problems and was not merely to achieve a mark.

2.2 Pedagogic beliefs

We interviewed Kevin in order to find out information about his beliefs and pedagogic conceptions: namely, where we could, to discover his ideas about mathematics, teaching and learning, how he perceived the profession of teaching, and how he felt in school What follows is part of this interview, which we consider relevant for this article.

Kevin perceives solving problems in the classroom as systematic and routine, something that does not happen to him with the interviews. He expresses this in the following manner:

> those of class are to learn to add, subtract, multiply, divide, square root, all that we do And what we do with you is to know, to learn to reason

Furthermore, he considers that the problems in class are "taught and explained", in contrast to the non-routine problems proposed in this investigation. He comments:

> those in class are taught and explained to us, but these are not, you have to build a kind of way to escape from there.

Clearly, Kevin exposes to us his idea of how a learning situation can be constructed.

Another idea that is underlying in the interview is that in the school, knowledge is uniform and all the student body learns the same content in the same way:

> In the school, if one is going through a route the others go along the same route, but here no, if one is going to one side, the others are going on other ones, and each one solves it according to his method.

Practice, if it is adequately formulated, carries us, in an inductive way, to build theoretical plans that help to structure scientific thought. Prior to this, Kevin alludes to his preference for practical teaching, in contrast to the theoretical, in the following way:

> If you are studying chemistry, for example, what do you like best? To think at best, that calcium has valency 2, or to put together calcium and magnesium and to mix it? Practice, not theory I don't like theory, I like practice.

We can elaborate an image of Kevin through the previous paragraphs which makes us perceive him as a pupil, as we have already said, with ideas of his own about school.

2.3. The resolution of mathematical problems

In this section, we will analyze Kevin's mathematical actions in two situations: the first one in relation to an interview and the second a written examination that he did as part of the mathematics assessment. Interpretation of the problems are based on the use Kevin makes of mental images and visualization.

2.3.1 Reproducing a geometrical model

The first task was part of an interview, where it was intended to reveal the quality of the student's images. It is a task that has already been used by other researchers (Brown and Wheatley, 1990; Brown, 1993) with good results and we have adapted it for our investigation.

The student was given the seven pieces of the tangram puzzle, material not familiar in the classroom. The investigator showed him, briefly, the models that appear in Figure 1, presented below, and he was asked to reproduce them by using the images he built in his memory.

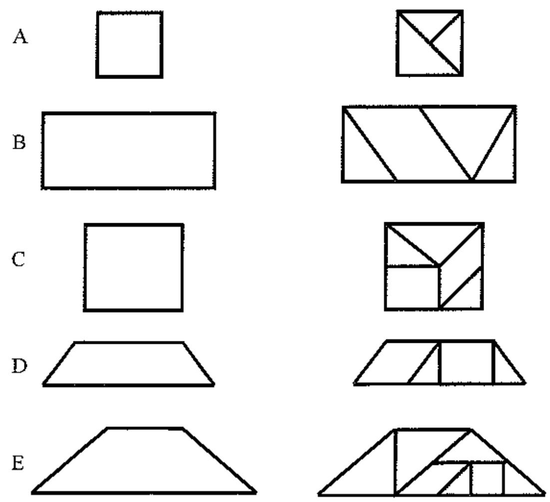

Figure 1

Kevin did not have any problem in solving this task.

We reproduce below part of the interview which includes model C (the large square):

Once Kevin had seen the model for some seconds, he began to have doubts and the teacher showed it to him again. Then he chose the required pieces and correctly built the model in a few seconds, but with the left and right parts inverted, as can be seen in Figure 2.

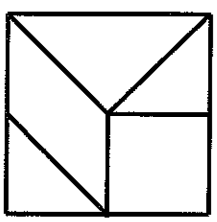

Figure 2

Interviewer: Is that the image you've got in your head?

Kevin: Yes.

I: Why do you think it is the same? Do you think it is the same or have you got any doubts?

K: I've got doubts

I: Why do you think you've got doubts? Do you believe that there is any difference between what you made and what I showed you?

K: Yes.

I: What do you think is different?

K: I know that there are some differences, but I don't know what the difference is. Show it to me again.

The interviewer shows it to him for a second.

I: Could you lay it out for me as I showed you?

Kevin puts a paper over the construction so that the pieces do not get displaced and makes a turn of 180 degrees in the space below the other of 180 degrees in the plane (as shown in Figure 3):

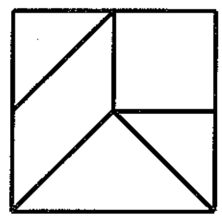

Figure 3

I: What image have you got in your head? Is it this or what I gave you?

K: At the moment of making it, I did it the easiest way for me.

I: Can you lay it as I showed you?

The pupil notes that somehow he has it upside down, but he argues that the images are the same; he takes the sheet and turns it 180 degrees in the plane obtaining the correct position (Figure 4).

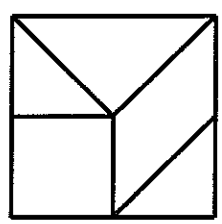

Figure 4

In doing this task, we observe Kevin's facility in building and transforming mentally his images – processes implicating the use of images Compared to the "poorest results" we have contemplated in other students interviewed, we now appreciate his strong visualization capacity of the relationship between mental images

Watching the videotape, it can be verified that the mental images Kevin uses are *dynamics* and *kinaesthetics*, shown by the continuous displacement we presume is being produced in the mind of the pupil and the actual movement of his hands, while he tries to explain the situation Images play an outstanding role in this task, where Kevin must anticipate the rotation of the pieces before placing them to form the proportionate model (Wheatley and Bebout, 1990)

2.3 Resolving three problems

This section starts with a general description of a test jointly proposed by the researchers and the teacher, consisting of two parts In the first part, the pupils must solve three equation systems by the methods of substitution, equalization and reduction, and in the second part they must establish and solve three verbal problems using systems of equations.

This last problem has a double objective: on the one hand, it serves to evaluate the algebraic knowledge after instruction, and on the other it provides opportunities for the students to solve a problem by visual methods This allows them to build mathematical ideas with comprehension and with meaning, where reaching the solution is not limited to the routine application of an algorithm.

The difference between these problems and that of the tangram is that the pupil may either 'invent' or 'use' an image or not, as a possible strategy to solve the proposed problem. This helps us to know the preferences of using images, while in the tangram problem, an image must necessarily be constructed as part of the task To interpret the results of the these three problems, we make use of the knowledge we have gained about Kevin He participated in seventeen problems outlined in the interviews.

His ignorance of any algebraic tools taught at school must be emphasised, as it is corroborated by the fact that he left blank the first part of the exam, that is the proposed equation systems. In addition, the teacher, during the above mentioned interview, talked about Kevin.

> He hasn't got the necessary tools to formulate them, he solves all the problems by trial and error, but when he finds a problem he can't solve by trial and error, he hasn't got the tools, he doesn't know about properties, he hasn't got methods, he hasn't got anything, because he hasn't acquired them here.

Next, we present the verbal problem, the solution and our analysis of the three problems done by Kevin.

Problem of the scale

> If you place a cheese on a pan of a scale and three-quarters of a cheese [cheese is *queso* in Spanish] and an x kg weight on the other, the pans balance How much does a cheese weigh? (Presmeg, 1985)

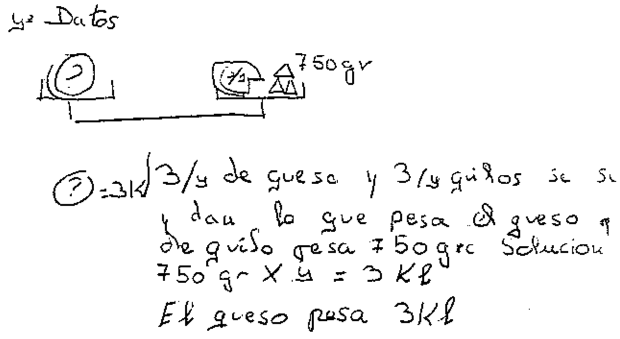

Figure 5

We can say Kevin uses logical reasoning in this problem Since he does not make use of algebraic tools, he has difficulty communicating his strategies. The way he expresses his images and mental representations is by making a drawing (Figure 5), but this drawing does not convey to us his actual thought processes. However, Kevin expresses all the concepts and ideas and the solution in the drawing. It is the teacher who does not value the interpretation of his representation, and therefore she did not accept his drawing or evaluate it

We think that Kevin uses a visual method; he relies on the drawing the verbal problem suggests to him; this figurative image corresponds to *concrete* images.

We note that Kevin uses the equivalence of 3/4 kg = 750 gr, a fact that he possibly has not learnt in school, but he has acquired it, probably from his experience in daily life. This self-gained knowledge is manifested further, by the fact that he writes in a natural way 3 kilos instead of 3 kg He also writes correctly 750 gr, something which does not give cause for mistakes, because he maintains a common unit vocabulary

While he makes the drawing, we think that Kevin has already solved the problem: he has got a visual image of the solution. His strategy to solve it agrees with the visual method. The explanation he uses under the drawing is only an attempt to justify the already-visualized result

Problem of the mother and the daughter

> A mother is seven times as old as her daughter If the difference between their ages is 24 years, what is the age of each one? (Presmeg, 1985)

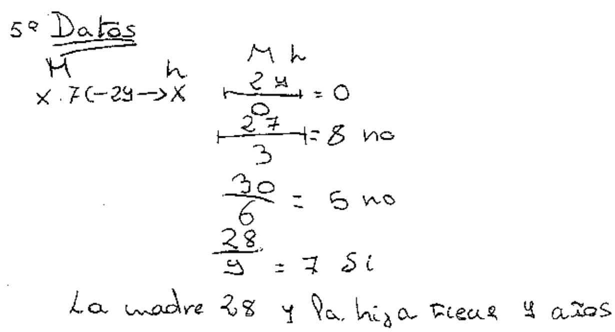

Figure 6

At first, Kevin tries to formulate the problem algebraically and he does it using only the unknown x that he associates with the age of the daughter, writing:

M h

$x.7 - 24 = x$

(M = *madre* (mother), h = *hija* (danghter)

Note that he does not write on the paper a second unknown quantity $y = x.7$ (the age of the mother), but he maintains that relationship in his mind. This allows us, in some way, to assume that he resorts to a mental image. On the other hand, it is important to note that he does not write the difference of the ages, which would be easy, and is the strategy used by most students ($y - x = 24$).

On the contrary he writes $x7 - 24 = x$, which indicates that he understands and gives meaning to the verbal problem. It is to be emphasized that he does not solve the problem by the simple equation that he has exposed, but he resorts to the trial and error method through a "table of double entry", where the variables are the mother and daughter's ages (numerator and denominator) and the objective is to find a seven, which he discovers at the fourth try, using logic.

Ducks and rabbits problem

> In a field there are ducks and rabbits comprising a total of 39 heads and 126 legs. How many animals are there of each type?

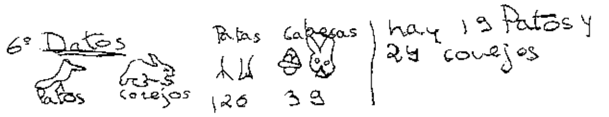

Figure 7

As it can be observed in Figure 7, Kevin notes correctly the result, 15 *patos* (ducks) and 24 *conejos* (rabbits), without showing any strategy of how he arrived at a solution. In fact, is there any teacher who may not have overlooked a result, leaving it unassessed, as no written method for the solution could be found?

We believe Kevin solved the problem by making use of a strategy: he uses mental images and he does not need concrete support, which might help him in his search for the solution. Our statement is based on the fact that Kevin solved two very similar problems in the clinical interview to those we analyze here using this method. One of them is formulated as follows:

> In a field there were rabbits and hens. When Jonás looked through the fence he could see 7 heads and 20 feet. How many rabbits were there? How many hens? (Adapted from Brown, 1993)

Part of the interview, where Kevin solved the previous problem, is transcribed below:

K: Seven and twenty feet, but seven heads ..... ?

I: In total.

K: In total. I'm going to solve first how many rabbits there were. If there were seven heads and twenty feet. We are going to solve the heads first

I: To solve the heads? How do you mean?

K: How many rabbit heads there were.

I: And how are you going to do it?

K: We are going to put three of rabbit and then three of hen, [...] the rabbits have got four feet, multiplied by four makes twelve. Twelve, four multiplied by two, it is already solved!

Kevin tackled the problem by trial and error. For his first trial, he took the number three, which he does not choose at random (as he explained to us in the subsequent part of the interview); on the contrary, he remembered at the same time the solution of a problem solved previously:

K: [...] three, because is like the half of seven.

In the problem that currently concerns us, and acting in a similar way, he took a number near half of the total number of heads, 20 in this case, and supposed that there are 20 rabbits, proceeding, according to our opinion, like this:

20c x 4p = 80p (80 feet);

126p - 80p = 46p (46 feet);

46p ÷ 2 = 23p (23 ducks);

20 rabbits + 23 ducks = 43 heads;

(c = head, p = feet)

As he does not obtain 39 heads, which form part of the data of the problem, he tries again to obtain the exact solution: 24 rabbits and 15 ducks is correct as can be observed in Figure 7, the only thing Kevin writes on the paper.

Briefly, Kevin established a relationship by mentally connecting the solution of this problem with other solutions, which is denoted as a cognitive act. The image of a previously solved problem helps him to solve an analogous situation (Presmeg, 1997).

In spite of the fact that Kevin solved the three formulated problems in a correct and creative way, using a visual method each time, he failed the exam. The explanation is that the teacher did not value these strategies, because she wanted to evaluate the pupils' knowledge of algebra; to assess if they could solve problems using systems of equations by the standard methods. This was something that Kevin did not demonstrate.

3. Conclusion

Through Kevin's mathematical actions, which were made clear in the previous problems, we can consider him a good visualiser, something which was manifest in his solution of the four problems.

- in the tangram problem, he makes use of dynamics and kinaesthetics images;

- in the scale problems, he makes use of the representation of a concrete image;

- in the problem of the ages, he makes use of the image of a relationship;

- in the problem about the ducks and rabbits, he makes use of the representation of a concrete image.

Kevin was a pupil who failed mathematics In addition to not passing the final exam, he also had many absences from class. Therefore the teacher, at the interview told us:

> I can't pass him even though he had made a brilliant test.

Presuming that assessment must be part of the teaching process, a reasonable doubt appears before us about the teacher's behavior in relation to the marks of Kevin

Assessment is very important for the pupil. A teacher can't just put a 'cross', or a 'no', or 'this is wrong', or ignore an issue, but must reflect a little more about what the pupil has written and what this means. We are conscious of the difficulties that individualized treatment implies in practice, due to several reasons.

We understand in this case the teacher's behavior, which coincides with that of many professionals, who consider it to be more important to fulfill the institutional rules than to take into account the specific situation and particular realities

However, in this specific case, visual thought was not valued and, quite probably, many other forms of it are not valued either, due to the ignorance of the existence of visualization in mathematics. That is why we consider, from our research, it is important to work with teachers, analyzing their educational practices in order to improve them. It would be desirable to have an investigation which would help the teachers to reflect on and give accounts of their thinking It might also be useful to monitor the effects this might have on future teaching methods and subsequently on student learning

The situation exposed in the previous paragraphs and the existence of pupils similar to Kevin (visualiser, observant, imaginative, creative, self-taught), who are not academically valued, is not new and happens quite often in the classrooms of many countries, where:

- visual education is neglected;

- the teaching system in general and the evaluation methods do not permit detection of these pupils;

- the educational system considers they are intelligent pupils (they are able to reason and act differently with new situations, etc.), but it fails them because they do not use standard tools;

- teachers tend not to take into account nor consider valuable knowledge acquired outside the classroom.

Nowadays, there is a consensus on the fact that in the search for standards and mathematical relationships, visualization shall be considered to be equally important, as calculation and symbolization (e.g. NCTM, 1989; Senechal, 1990). However, visual education is often a forgotten area in educational practice, in relation to the importance that the numerical and algebraic content have.

The US Curricular Standards (NCTM, 2000) proposes visualization as a fundamental part of the understanding of relationship in two- and three-dimensional geometry, especially for secondary students.

Since visual thought provides students with new ways to think and do mathematics, it would be desirable as far as mathematical education is concerned that in the gradual change between the activities, if systematic attention is paid to visualization and new technologies, then something which would actively help the students themselves in the learning situation were taken into account.

To conclude, we enumerate some didactic considerations that could attenuate some of the problems reported in this article:

- to stimulate visualization as a tool for the teaching and learning processes in other mathematical fields: algebra, analysis, statistics;

- to develop visual education in the classroom through adequate tasks; to seek mechanisms to detect visualisers and creative pupils (alternative exams, tests);

- to stimulate and to encourage these pupils with a particular follow-up, where their capacities could be developed; to value and not to discriminate against this kind of pupil, taking advantage of and supporting their innate capacities

Notes

[1] See Bishop (1983, 1989), Clements (1981, 1982), Presmeg (1985, 1997), Zimmerman and Cunningham (1991), Wheatley (1997), Sutherland and Mason (1995), Plasencia, Espinel and Dorta (1998)

[2] See Brown and Wheatley (1989, 1990), Brown and Presmeg (1993), Brown (1993), Wheatley, Brown and Solano (1994)

References

Bishop, A J (1983) 'Space and Geometry', in Lesh, R and Landau, M (eds), *Acquisition of Mathematics Concepts and Processes*, New York, NY, Academic Press, pp 175-203

Bishop, A J (1989) 'Review of research on visualization in mathematics education', *Focus on Learning Problems in Mathematics* **11**(1) 7-16.

Brown, D L (1993) *An Investigation of Imagery and Mathematical Understanding in Elementary School Children,* Unpublished Master's thesis, Florida State University, Tallahassee, FL

Brown, D L and Presmeg, N (1993) Types of imagery used by elementary and secondary school students in mathematical reasoning. *Proceedings of the 17th Psychology of Mathematics Education Conference*, Tsukuba. Japan, vol. 1, pp 137-144

Brown, D L and Wheatley, G (1989) 'Relationship between spatial ability and mathematics knowledge', *Proceedings of the 11th Psychology of Mathematics Education North American Conference*, New Brunswick, NJ, pp 143-148.

Brown, D L and Wheatley, G (1990) 'The role of imagery in mathematical reasoning', in *Proceedings of the 14th Psychology of Mathematics Education Conference*, Oaxtapec, México, vol. 1, pp 217

Clements, M A (1981) 'Visual imagery and school mathematics: part 1', *For the Learning of Mathematics* **2**(2), 2-9
Clements, M A (1982) 'Visual imagery and school mathematics: part 2', *For the Learning of Mathematics* **2**(3), 33-38.
Guzmán, M de (1996) *El Rincón de la Pizarra*, Madrid, Pirámide.
Hershkowitz, R., Parzsyz, B and van Dormolen, J. (1996) 'Space and shape', in Bishop, A J. *et al.* (eds), *International Handbook of Mathematics Education*, Dordrecht, Kluwer, pp 161-204.
Kosslyn, S M (1983) *Ghosts in the Mind's Machine*, New York, NY, W W Norton and Co.
NCTM (2000) *Principles and Standards for School Mathematics* Reston, VA, National Council of Teachers of Mathematics.
Plasencia, I, Espinel, C and Dorta, J A (1998) 'Visualización y creatividad', *Educación Matemática* **10**(2), 102-120
Presmeg, N. C. (1985) *The Role of Visually Mediated Processes in High School Mathematics: a Classroom Investigation*, Unpublished Ph D thesis, Cambridge, University of Cambridge.
Presmeg, N C. (1986) 'Visualization in high school mathematics', *For the Learning of Mathematics* **6**(3), 42-46
Presmeg, N C (1997) 'Generalization using imagery in mathematics, in English, L D. (ed.), *Mathematical Reasoning*, London, Lawrence Erlbaum, pp. 299-312
Senechal, M. (1990) 'Shape', in Steen, L A. (ed.), *On the Shoulders of Giants: New Approaches to Numeracy*, Washington, DC, National Academic Press, pp 139-181.
Sommer, R (1978) *The Mind Eye*. New York, NY, Delacorte Press
Sutherland, R and Mason, J (eds) (1995) *Exploiting Mental Imagery with Computers in Mathematics Education*, Oxford, Springer.
Wheatley, G (1978) *The Wheatley Test of Spatial Ability*, West Lafayette, IN, Purdue University
Wheatley, G. (1997) 'Reasoning with images in mathematical activity', in English, L D. (ed.), *Mathematical Reasoning*, London, Lawrence Erlbaum, pp 281-297
Wheatley, G and Bebout, H (1990) 'Mathematical knowledge of young learners', in Steffe, L P and Wood, T (eds), *Transforming Children's Mathematics Education*, Hillsdale, NJ, Lawrence Erlbaum, pp 107-111.
Wheatley, G, Brown, D and Solano, A (1994) 'Long-term relationship between spatial ability an mathematical knowledge', *Communication presented in the North American Chapter of the Psychology of Mathematics Education*, Baton Rouge, LA.
Zazkis, R, Dubinsky, E and Dautermann, J (1996) 'Coordinating visual and analytic strategies: a study of students' understanding of the group D4', *Journal for Research in Mathematics Education* **27**(4), 435-457
Zimmermann, W and Cunningham, S (eds) (1991) *Visualization in Teaching and Learning Mathematics*, Washington, DC, Mathematical Association of America