MOVING IN EARLY GEOMETRY EDUCATION

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> Caminante, son tus huellas el camino, y nada mas;
> caminante, no hay camino, se hace camino al andar
> [Wanderer, your footsteps are the road, and nothing more; wanderer, there is no road, the road is made by walking.] (Machado, 1978, 82f).

Imagine you are in a kindergarten. A 4-year-old pupil, Vincenzo, stands in front of a large sheet of paper placed on a painting easel and holds a brush with his right hand. While he is observing the movements of one of his classmates' hands along a strip of adhesive tape, he draws a line by sliding the brush on the sheet; then, he turns to the teacher and says, "Teacher, don't worry, the dots are all hidden in the strip." [1] Witnessing this episode would likely leave you amazed, as you would rightfully wonder how Vincenzo, despite his young age, conceives the line which he has created with a continuous hand gesture, as constituted by a series of points, recalling the sophisticated idea of locus of a point of the (Western) modern mathematical culture?

In this article, we introduce and discuss theoretically a design study of an educational path for a kindergarten class aimed at allowing children to explore some features of straight, jagged, and curved lines as geometric entities. During the design, our focus was on body movements and on their interactions with the environment, which we promoted and oriented to the perception of some geometric properties of lines. This work is framed in the enactive view of cognition as embodied action (Varela, Thompson & Rosch, 1991), and within it we adopt the theoretical perspective of sensuous cognition (Radford, 2013).

Sense of movement and foundations of geometry

Our approach to the idea of movement derives from the suggestions of the French neurophysiologist, psychologist, and engineer, Alain Berthoz (2000). For Berthoz, perception is a multi-sensorial construction and movement is to be considered as a *sixth sense* that collaborates with the other ones in cognitive processes. Berthoz elaborated a biological theory of the 'sense of movement' and highlighted its importance, observing that the survival of all animal species depends on it. Nevertheless, it is the least evident to human consciousness in comparison to the other senses as its receptors are hidden; indeed, the motion receptors are inside the muscles, the joints, and the vestibular system of the ear. Berthoz, who is an avid advocate for the necessity of a multidisciplinary approach to the study of perception, also reflected on the link between human movements and the origins of geometry. In particular, he supported the vision of Poincaré (1905), according to which the notion of geometric space itself is not given *a priori,* but it is built on the human possibilities of moving. Poincaré stated:

> To localize an object simply means to represent to oneself the movements that would be necessary to reach it. I will explain myself. It is not a question of representing the movements themselves in space, but solely of representing to oneself the muscular sensations which accompany these movements and which do not presuppose the preexistence of the notion of space. (as quoted in Berthoz, 2000, p. 37)

In the same spirit, we concur with the position put forth by Longo (2016), which traces the idea of line back to the experience of gesture, of movement. Longo opined that the Euclidean definition of line, as length without breadth, is a 'philosophical decision' that led mathematical line to be far from sensorial experience. Nevertheless, its meaning arises from a sensorial experience itself:

> Such a notion of line can only be said and written, in the language, and is the result of a philosophy of ideas, but its meaning, its continuity, is in a *practice* of gesture, of drawing: to understand it, one must reactivate the sense, go back to the line drawn on the blackboard by the first teacher, to his gesture-trajectory in space. (Longo, 2016, p. 18, our translation)

More generally, Husserl (1939) posited that geometry, which he considered as a paradigmatic example of science, is 'transcendental' in the sense that it has been developed through idealizations produced and reproduced, starting from and through the practical actions of sensible human beings. From an educational point of view, the transcendental and material nature of the discipline makes "the emergence of a microculture of geometry" possible in school classrooms (Roth & Thom, 2009, p. 92), through an embodied and collective work that takes place in a pre-geometric context.

An enactivist view of mathematical learning

The idea of an experiential foundation of geometry fits perfectly with the enactivist view of cognition as "a history of structural coupling that brings forth a world" (Varela, Thompson & Rosch, 1991, p. 206), *i.e.,* as a co-evolution process which involves perceivers and their environment, triggered by their recurrent interactions (Reid & Mgombello, 2015). As noted by Radford, Arzarello, Edwards and Sabena (2017), enactivism proposes a middle view between rationalist and subjectivist epistemological positions. On the one hand, enactivists disagree with rationalist positions, according to which the world has pregiven properties which are independent of human cognition, and instead claim that "cognition does not represent a world, it creates one" (Reid & Mgombello, 2015, p. 176). On the other hand, against the subjectivist view, in which a cognitive system projects its own world, the enactive program states that human cognitive categories depend both on the biological nature and on the cultural world of the perceiver. Enactivism connotes cognition as experiential and conceptualizes it as *embodied action* (Varela, Thompson & Rosch, 1991), whereby knowing is based on sensory and motor experiences which are both conceived as active processes occurring through the interaction between the perceivers and the environment. Briefly, in Varela's (1999) words, "The world is not something that is given to us but something we engage in by moving, touching, breathing, and eating. This is what I call cognition as enaction since enaction connotes this bringing forth by concrete handling" (p. 8).

Starting from a general perspective on cognition that arises from enactivism, and drawing upon the work of Vygotsky and his cultural-historical psychology, Radford (2013) conceptualized cognition in mathematics education as simultaneously conceptual, embodied, and material. In this theoretical approach, named *sensuous cognition,*

> [human] responsive sensation evolves—both at the phylogenetic and ontogenetic levels—intertwined with the material culture in which individuals live and grow. As a result, cognition can only be understood as a culturally and historically constituted sentient form of creatively responding, acting, feeling, imagining, transforming, and making sense of the world. (Radford, 2013, p. 144)

Thus, in this perspective, learning takes place when, in their interaction with the material world, students start adopting a cultural and historical mathematical way of sensibly perceiving, acting, and thinking.

This is possible owing to the *plasticity* of human sensations. As Radford (2013) observed, unlike many animal species, the sensorial functions with which humans are born are highly unspecialized, but develop into very complex forms over the course of life. Recall that, in an enactivist view of cognition, human senses are conceived as co-evolving with culture. Thus, they are not to be considered solely as a part of our biological apparatus as they undergo a *cultural shaping.* The extent of the cultural shaping depends on the immersion in a particular society, and ultimately results in a culturally and historically determined way of perceiving. Moreover, akin to Berthoz, Radford underlined the *multimodality* of human sensations, asserting that their functioning and development are interrelated, due to which "the human mind and cognition are not merely sensuous but also multimodal" (p. 146).

Concerning the interaction between the learner and the *material world,* sensuous cognition is inspired by the work by Luria and Vygotsky who opined that artifacts play a key role in learning and are considered as bearers of sedimented human intelligence and cultural production. However, according to the sensuous cognition, artifacts are more than mediators of human thinking and experience. In fact, sensuous cognition dwells upon dialectical materialism (Leont'ev, 1978) which "offers a conception of knowledge and the knowing subject as cultural, historical entities entangled in, and emerging from, material human activity" (Radford *et al.,* 2017, p. 711).

Semiotic node and semiotic contraction

Within the sensuous cognition approach, two tools are used to analyze mathematical activity and to capture the eventual emergence of new mathematical awareness—the correlated concepts of *semiotic node* and *semiotic contraction,* respectively. A semiotic node is a segment of classroom activity (or rather, its description and interpretation provided by one or more researchers) which offers a unitary picture of the inseparable conceptual, embodied, and material constitutive aspects of a mathematical form of thinking. In sensuous cognition, thinking consists of perceptions, outer and inner speech, gestures, other signs, sensorial modalities of imagination, and material activities involving cultural artifacts, all of which are captured by a semiotic node. In this work, we refer to the definition of 'semiotic node' given by Radford *et al.* (2017), in which multimodality is highlighted both in the coordination of semiotic registers and in the functioning of senses:

> A semiotic node is not a set of signs. It is a segment of joint activity that usually includes a complex coordination of various sensorial and semiotic registers that the students and teachers mobilize in order to notice something (*e.g.,* a mathematical structure or a mathematical concept at work). (p. 714)

Instead, the expression 'semiotic contraction' (Radford *et al.,* 2017) is used to indicate the evolution of a semiotic node. This phenomenon occurs when students and teachers no longer act, or make shorter and more direct, gestures, movements and signs, because they no longer consider the old ones necessary inside their collective activity. This type of contraction is interpreted in sensuous cognition as a manifestation of new mathematical learning.

In approaching our design work, we were inspired by the general theoretical framework of enactivism. In particular, we agreed with the idea of cognition as embodied action, which bases knowing on sensory and motor experiences. Enactivism also suggested us to consider each perceiver in our kindergarten classroom as a system that co-evolves with its environment. From this perspective, in which human cognitive categories are both biologically and culturally determined, we embrace the sensuous cognition framework. Its conception of learning as cultural shaping of senses, emerging from material activity, guided the design and implementation of our didactical path, which paid attention to the perceptual and motor experiences and assumed the multimodality of senses' functioning and development. Moreover, sensuous cognition provided us with the methodological tools of semiotic node and semiotic contraction to give sense to the episode described at the beginning of the paper, and to look at it as a particularly meaningful manifestation of new mathematical learning.

Educational path design

In this section, we describe the activities that constitute the educational path, justifying the particular design choices we made. For clarity, we also include some images from its realization, allowing the reader to 'see' the sequence of activities as they were implemented in practice.

The teaching-learning experience was designed for an Italian kindergarten class of 28 four-year-old preschoolers. It lasted one month with two activities per week. The research team comprised two mathematics education researchers (Gemma Carotenuto and Maria Mellone) and the class's research-teacher (Marina Spadea).

Methodological preamble

The didactical path comprises of three phases of activities, involving straight, jagged, and curved lines, respectively. In accordance with the enactivist perspective on cognition, we designed an educational path with the aim of allowing pupils to *experience* different types of lines that arise from their own *movements*, through particular walking patterns, or painting and drawing actions, which are seen as movements that leave traces.

In the design of the activity sequences, within each phase, the distinction between full body movements and manipulative movements, acting on material objects, plays a crucial role. Pupils performing manipulative activities involve in a significant and conscious way only certain parts of the body. For example, in a painting task, pupils may not even be aware of their postural changes, but would clearly perceive the intentional movements of their arms and hands, which are more extended in space. More and more, if we look at the muscles, the joints and the vestibular system of the ear as receptors of the sense of movement, as Berthoz suggested, we have to recognize the great cognitive potential, *e.g.,* of a walk through the classroom space. We claim that, for the purposes of mathematical learning, at the kindergarten level and beyond, full body movements are at least as important as interacting with cultural artifacts, although these are often underestimated in practice. This is the reason why, for each of the three types of line to be experienced, we designed a sequence of activities that initially triggered full body movements and subsequently transitioned into associated manipulative tasks. Specifically, in our educational path full body movements consisted of particular walks through the classroom space, which were supposed to nurture different perceptions of types of lines, through activation of the vestibular organs' sensing of linear acceleration, as well as through motion of muscles and joints.

Our path design was guided by the integrative nature of the human sensorial development and functioning, as posited in the sensuous cognition approach. We paid attention not only to the 'sense' (Berthoz, 2000) of movement. As we show below, pupils' sensorial experiences also included *touch* (*e.g.,* realized by putting one foot in contact with the other), *sight* (which is wide awake in every task), and *hearing* (*e.g.,* engaged by listening the sounds produced during a playful context of a car race).

Phase 1: straight lines

The initial perceptual experiences in the educational path triggered full body movements: they were walks on straight lines which crossed the classroom space. At first, every pupil walked along a quite wide balance beam (Figure 1a), while the teacher guided the pupil in paying attention to the sensorial perceptiveness of *going straight* without falling off the beam, also engaging the vestibular system. Indeed, the principal goal of this task was to prompt the children to be conscious of the uniformity of such a walk. *Uniformity* is an essential quality of straight lines which can already be found in the Euclidean definition as 'line which lies evenly with the points on itself'. The subsequent activities were oriented towards the objective of tactilely and motorally perceiving the topological properties of *connectedness* which is common to all lines. Every pupil walked on a tape strip which traced a straight line on the floor, and was thinner than the balance beam. Consequently, movements had to be more precise than before. There was also a new constraint, as pupils were instructed to put one foot immediately after the other *consecutively* (Figure 1b). Next, every pupil was required to put one hand after the other on the same tape strip, with an analogous motor and tactile constraint of touching with previous hand the successive position of the other hand. At the same time, another pupil was involved in a manipulative task: that student reproduced the walks by creating a point for each classmate's step on a large sheet of paper, using brush and paint (Figure 1c–1d). We emphasize that the latter was not only a representative task, as we also had to consider the motor experience of drawing points on the paper. The episode involving Vincenzo, which we described at the beginning of the paper, took place while he was completing this task. In these and in the following activities, children who were not directly performing the task were invited to observe and give feedback to the classmates. Thus, they represented to themselves the muscular sensations which accompany other pupils' movements (Poincaré, 1905).

Phase 2: jagged lines

The second set of activities was designed to address the geometric entity of jagged line. First, pupils experienced jagged lines with full body movements by walking on several pieces of tape purposely prepared on the floor in order to draw a jagged line (Figure 2a). From the motor-experiential viewpoint, these 'zig-zag' walks were very different from the previous ones. Indeed, while the previous walks were characterized by uniformity of movement, now children's whole bodies had to *rotate while passing through the extreme points* of the different segments which made up the jagged line. This experience aimed at motivating the pupils to distinguish a jagged line from a straight line, with the sense of movement supporting cultural evolution of the sight. In the next activity, children were divided into small groups, each of which took part in the manipulative task of constructing a race track for toy cars by using a sheet of paper and some tape strips. Starting from motivations belonging to the playful context of a car race, pupils were asked to go through some specific points previously indicated by the teacher with dots on the sheet (Figure 2c). This

Figure 1. (a) A pupil walks on a balance beam; (b) A pupil walks along a straight tape strip; (c) A pupil puts one hand after the other on the tape strip and a classmate reproduces the movement on a piece of paper; and (d) Detail of the drawing task.

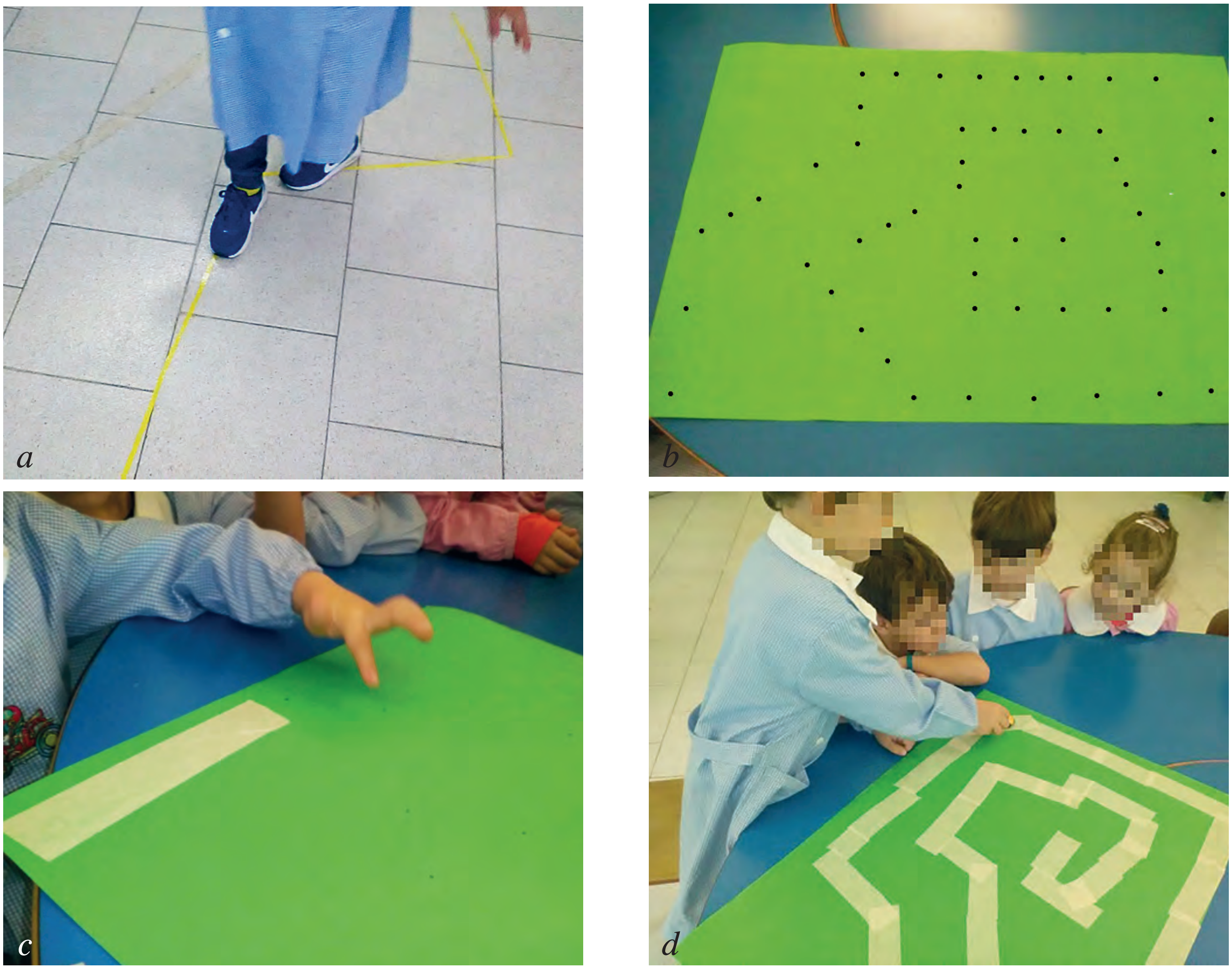

Figure 2. *(a) A pupil walks on a 'zig-zag' line on the floor; (b) A dotted large sheet of paper (the points have been highlighted in the image); (c) A group of pupils constructs a race track for toy cars; and (d) A pupil plays with his group's race track.*

Figure 3. *(a) A pupil walks on a rope that forms curves on the floor; (b) Pupils individually build a city model; and (c) A pupil paints a street in a large city model.*

particular constraint was combined with the use of tape strips whose materiality allowed and suggested the creation of segments, as the resulting race track shapes were reminiscent of jagged lines in geometry (Figure 2b, 2c). Finally, children were left to play freely with the toy cars on their tracks and to perceive their shape through sight, but also through their movements as they played (Figure 2d).

Phase 3: curved lines

The last group of activities involved a more general type of line—curved lines. Again, the class exploration began with full body movements through individual walking tasks. Now pupils walked along a rope that formed smooth curves on the floor and perceived a *fluid rotation of their body* as they advanced (Figure 3a). Later, the children themselves created curved lines in individual and group manipulative tasks immersed in the playful context of building city models. As the first step, the pupils individually built their own city models by positioning Lego blocks to represent buildings on a small sheet and by using a marker to freely trace the streets on the working table (Figure 3b). Then, the children played an analogous game in small groups, but now on large sheets and by representing buildings with bigger Lego constructions. The pupils were still quite free in the creation of the streets, but were now using a brush and some paint with the sole constraint imposed by the presence of Lego blocks. Notice that this particular design choice prompted children to create smooth curved lines. In fact, if we compare these individual and group city creation tasks with that involving race track construction, we can note two substantial differences: (1) the absence of specific crossing points and (2) the use of markers and brushes instead of a tape strip. These material aspects, which correspond to two different playful contexts, encouraged children to create curved lines in one case and jagged lines in the other.

Vincenzo's sophisticated form of seeing

The collected data consist of photos and videos taken during the educational activities, records of the classroom discussions, and the teacher's notes about her direct observations. Our approach to data analysis perfectly fits with the enactivist position on methodology in mathematics education offered by Maheux and Proulx (2015). Indeed, we are aware of the fact that what observers describe is not what happened, but their interpretations of the phenomenon in which they were involved. Thus, what researchers can produce are personal—and hopefully productive—interpretations of a mathematical activity in which they participated. Therefore, we are driven by the aim to "revisit, nuance, and expand how we (and our readers) may conceive students' or teachers' actions" (Maheux & Proulx, 2015, p. 213).

Giving sense to Vincenzo's movements and words

We can now return to the episode described at the beginning of the paper, which involved the 4-year-old pupil Vincenzo inside his classroom. It took place during the second day of the didactical path when the pupils' hands replaced their feet in the straight walks in the classroom, and a pupil created a point on a large sheet of paper for each step by using brush and paint (Figure 1c−1d). The whole class and the teacher were involved, albeit in different ways, in the walk with the hands on the tape strip and in the associated drawing activity. In fact, the two tasks were performed by only two children, but everybody in the classroom participated by observing and giving feedback. We describe what attracted *our* attention about the moment immediately before the episode by capturing it as a semiotic node comprising of:

a. The coordinated perception (motor, tactile, and visual) of the pupil walking with her hands
b. The coordinated perception (motor, tactile, and visual) of the pupil drawing on the paper
c. The teacher's gestures and words who rhythmically repeated "step and point" and those of the observing pupils who commented and gave feedback
d. The iconic representation of the steps made by hands on the paper
e. The visual perception of the pupils and the teacher observing the realization of the two tasks

At some point, Vincenzo was called to draw. He did not use the brush in order to create points, like other children had already done, but instead he allowed the brush to flow across the paper. Then, pointing to the continuous line he had traced, he said to the teacher, "Teacher do not worry, the dots are all hidden in the strip." Vincenzo demonstrated awareness of his motor-representative choice and its associated meanings; he preferred to trace a line, but he was able to see the line as a configuration made of points. Thus, the sophisticated idea of geometric locus *emerged* (Roth & Thom, 2009).

Notice that the walking−drawing activity implemented in this case corresponds to a discretization of the line, as pupils experienced and represented paths generated by different but contiguous steps. At the same time, the need for greater material comfort, in the movement of the arm and the hand that guides the brush, prompted Vincenzo to draw an almost smooth line. As Vincenzo, some other children who drew before and after him left traces of ever-smoother lines on the paper (Figure 4).

We look at this phenomenon as a semiotic contraction (Radford, 2017), *i.e.,* a transformation of the semiotic node described above, and in particular of its (b) and (d) components. It seems that Vincenzo deemed the continuity of the

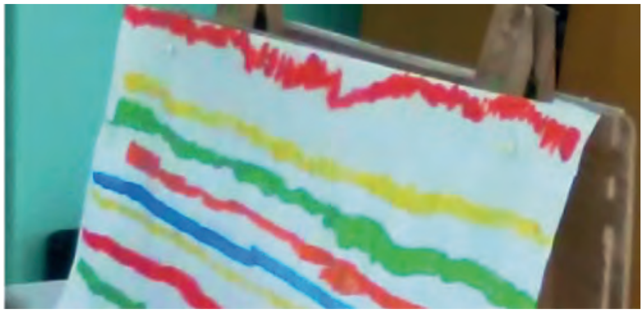

Figure 4. The initial sequence of the drawn lines (Vincenzo created the blue line).

drawn figure relevant while perceiving its explicit point composition as irrelevant. We claim that this plausible awareness, which is suggested by Vincenzo's reference to the dots being "hidden in the strip," could be a result of the specific didactic design which is also oriented to a cultural mathematical development of the sense of sight, supported by the senses of movement and touch.

Moreover, we claim that Vincenzo's and other pupils' motor task evolution toward the naturalness of a fluid movement could have a double material nature, connected with both the chosen drawing tools and the configuration of the human body itself. Indeed, in this experience, the drawing tools—the brush, the paint, and the large sheet of paper—"are not conceptually neutral" (Radford *et al.,* 2017, p. 711). If we had given the children ink stamps, for example, instead of brush and paint, their representative and motor experience would have been very different. They would have been more comfortable in creating a single sign for each step, the sequential signs produced on the sheet would have been more homogeneous, and perhaps points would be aligned more precisely. On the other hand, children would in all probability not have opted for a continuous gesture of the arm. Indeed, the corporeal-kinetic configuration formed by the human arm with the brush, which we see as a historical and cultural human production, is the system that led children to live the embodied experience of *tracing a line.*

Comments and didactical implications

In the field of mathematics education, body movement has started to capture considerable attention during the last twenty years, prompted by the diffusion of the work on the embodied mind and its relations with mathematics by Lakoff and Núñez (2000). Fortunately, this research interest is consistent with the educational didactic tradition which, at least in Italy and in many Western countries, recognizes the importance of placing children's bodies and material manipulations at the core of the educational proposals for the kindergarten curriculum, albeit rarely with specific mathematical objectives.

Guided by an enactivist spirit, the original purpose of our work was in line with the more general aim of *imagining possibilities* for mathematics education (Maheux & Proulx, 2015), in the particular field of the geometric learning in kindergarten, and through the body movements and its interactions with the environment. Adopting the perspective of learning as cultural shaping of human senses, which underlines the multimodality of senses' functioning and development (Radford, 2013), during our design work, we considered body movements as interrelated with pupils' other sensorial experiences, through touch, sight, and hearing. In our work, we wanted to emphasize the importance of a careful educational design, that incorporates embodied and material aspects and links these to specific mathematical concepts. For example, we documented the choice of planning different walks to experience different lines, and of using different materials to trigger different movements. In order to reflect on the possible advantages of this type of design, we analyzed an episode involving a very young pupil who appeared capable of seeing the points 'hidden' in the line he traced with a continuous hand motion. Due to purposely designed mathematical activities, Vincenzo seems to have developed a very special historical and cultural mathematical form of seeing.

In many current school practices, full body movements are unfortunately often considered almost as an obstacle to the learning process; for example, young children who 'move too much' are seen as not completely suitable for the school environment. For this reason, as researchers and teacher educators, we felt the need to show not only that full body movements are not enemies of learning, but also that they can be powerful allies if seriously considered in the educational design. As emphasized by Varela, Thompson and Rosch (1991), "the dissociation of mind from body, or awareness from experience is the result of habit, and these habits can be broken" (p. 25).

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Note

[1] Translated by the authors, as the dialogue in the class was held in Italian.

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