Current Trends in Mathematics and Future Trends in Mathematics Education*

PETER HILTON

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Introduction

My intention in this talk is to study, grosso modo, the dominant trends in present-day mathematics, and to draw from this study principles that should govern the choice of content and style in the teaching of mathematics at the secondary and elementary levels. Some of these principles will be time-independent, in the sense that they should always have been applied to the teaching of mathematics; others will be of special application to the needs of today's, and tomorrow's, students and will be, in that sense, new. The principles will be illustrated by examples in order to avoid the sort of frustrating vagueness which often accompanies even the most respectable recommendations (thus, "problem solving [should] be the focus of school mathematics in the 1980's" [1]).

However, before embarking on a talk intended as a contribution to the discussion of how to achieve a successful mathematical education, it would be as well to make plain what are our criteria of success. Indeed, it would be as well to be clear what we understand by successful education, since we would then be able to derive the indicated criteria by specialization.

Let us begin by agreeing that a successful education is one which conduces to a successful life. However, there is a popular, persistent and paltry view of the successful life which we must immediately repudiate. This is the view that success in life is measured by affluence and is manifested by power and influence over others. It is very relevant to my theme to recall that, when Queen Elizabeth was recently the guest of President and Mrs Reagan in California, the "successes" who were gathered together to greet her were not Nobel prize-winners, of which California may boast remarkably many, but stars of screen and television. As the London *Times* described the occasion, "Queen dines with celluloid royalty". It was apparently assumed that the company of Frank Sinatra, embodying the concept of success against which I am inveighing, would be obviously preferable to that of, say, Linus Pauling.

The Reaganist-Sinatrist view of success contributes a real threat to the integrity of education; for education should certainly never be expected to conduce to that kind of success. At worst, this view leads to a complete distortion of the educational process; at the very least, it allies education far too closely to specific career objectives, an alliance which unfortunately has the support of many parents naturally anxious for their children's success.

We would replace the view we are rejecting by one which emphasizes the kind of activity in which an individual indulges, and the motivation for so indulging, rather than his, or her, accomplishment in that activity. The realization of the individual's potential is surely a mark of success in life. Contrasting our view with that which we are attacking, we should seek power over ourselves, not over other people; we should seek the knowledge and understanding to give us power and control over things, not people. We should want to be rich but in spiritual rather than material resources. We should want to influence people, but by the persuasive force of our argument and example, and not by the pressure we can exert by our control of their lives and, even more sinisterly, of their thoughts.

It is absolutely obvious that education can, and should, lead to a successful life, so defined. Moreover, mathematical education is a particularly significant component of such an education. This is true for two reasons. On the one hand, I would state dogmatically that mathematics is one of the human activities, like art, literature, music, or the making of good shoes, which is intrinsically worthwhile. On the other hand, mathematics is a key element in science and technology and thus vital to the understanding, control and development of the resources of the world around us. These two aspects of mathematics, often referred to as pure mathematics and applied mathematics, should both be present in a well-balanced, successful mathematics education.

Let me end these introductory remarks by referring to a particular aspect of the understanding and control to which mathematics can contribute so much. Through our education we hope to gain knowledge. We can only be said to really know something if we know that we know it. A sound education should enable us to distinguish between what we know and what we do not know; and it is a deplorable fact that so many people today, including large numbers of pseudosuccesses but also, let us admit, many members of our own academic community, seem not to be able to make the distinction. It is of the essence of genuine mathematical education that it leads to understanding *and* skill; short cuts to the acquisition of skill, without understanding, are often favored by self-confident pundits of mathematical education, and the results of taking such short cuts are singularly unfortunate for the young traveller. The victims, even if "successful", are left precisely in the position of not knowing mathematics and not knowing they know no mathematics. For most, however, the skill evaporates or, if it does not, it becomes out-dated. No real ability to apply quantitative reasoning to a changing world has been learned, and the most frequent and natural result

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is the behaviour pattern known as "mathematics avoidance" Thus does it transpire that so many prominent citizens exhibit both mathematics avoidance and unawareness of ignorance.

This then is my case for the vital role of a sound mathematical education, and from these speculations I derive my criteria of success.

Trends in mathematics today

The three principal broad trends in mathematics today I would characterize as (i) variety of applications, (ii) a new unity in the mathematical sciences, and (iii) the ubiquitous presence of the computer. Of course, these are not independent phenomena, indeed they are strongly interrelated, but it is easiest to discuss them individually

The increased variety of application shows itself in two ways. On the one hand, areas of science, hitherto remote from or even immune to mathematics, have become "infected". This is conspicuously true of the social sciences, but is also a feature of present-day theoretical biology It is noteworthy that it is not only statistics and probability which are now applied to the social sciences and biology; we are seeing the application of fairly sophisticated areas of real analysis, linear algebra and combinatorics, to name but three parts of mathematics involved in this process.

But another contributing factor to the increased variety of applications is the conspicuous fact that areas of mathematics, hitherto regarded as impregnably pure, are now being applied Algebraic geometry is being applied to control theory and the study of large-scale systems; combinatorics and graph theory are applied to economics; the theory of fibre bundles is applied to physics; algebraic invariant theory is applied to the study of error-correcting codes Thus the distinction between pure and applied mathematics is seen now not to be based on content but on the attitude and motivation of the mathematician No longer can it be argued that certain mathematical topics can safely be neglected by the student contemplating a career applying mathematics I would go further and argue that there should not be a sharp distinction between the methods of pure and applied mathematics. Certainly such a distinction should not consist of a greater attention to rigour in the pure community, for the applied mathematician needs to understand very well the domain of validity of the methods being employed, and to be able to analyse how stable the results are and the extent to which the methods may be modified to suit new situations.

These last points gain further significance if one looks more carefully at what one means by "applying mathematics". Nobody would seriously suggest that a piece of mathematics be stigmatized as inapplicable just because it happens not yet to have been applied Thus a fairer distinction than that between "pure" and "applied" mathematics would seem to be one between "inapplicable" and "applicable" mathematics, and our earlier remarks suggest we should take the experimental view that the intersection of inapplicable mathematics and good mathematics is probably empty. However, this view comes close to being a subjective certainty if one understands that applying mathematics is very often not a single-stage process. We wish to study a "real world" problem; we form a scientific model of the problem and then construct a mathematical model to reason about the scientific or conceptual model (see [2]). However, to reason *within* the mathematical model, we may well feel compelled to construct a new mathematical model which embeds our original model in a more abstract conceptual context; for example, we may study a particular partial differential equation by bringing to bear a general theory of elliptic differential operators Now the process of modeling a mathematical situation is a "purely" mathematical process, but it is apparently not confined to pure mathematics! Indeed, it may well be empirically true that it is more often found in the study of applied problems than in research in pure mathematics. Thus we see, first, that the concept of applicable mathematics needs to be broad enough to include parts of mathematics applicable to some area of mathematics which has already been applied; and, second, that the methods of pure and applied mathematics have much more in common than would be supposed by anyone listening to some of their more vociferous advocates For our purposes now, the lessons for mathematics education to be drawn from looking at this trend in mathematics are twofold; first, *the distinction between pure and applied mathematics should not be emphasized in the teaching of mathematics*, and, second, *opportunities to present applications should be taken wherever appropriate within the mathematics curriculum.*

The second trend we have identified is that of a new unification of mathematics This is discussed at some length in [3], so we will not go into great detail here We would only wish to add to the discussion in [3] the remark that this new unification is clearly discernible within mathematical research itself. Up to ten years ago the most characteristic feature of this research was the "vertical" development of autonomous disciplines, some of which were of very recent origin. Thus the community of mathematicians was partitioned into subcommunities united by a common and rather exclusive interest in a fairly narrow area of mathematics (algebraic geometry, algebraic topology, homological algebra, category theory, commutative ring theory, real analysis, complex analysis, summability theory, set theory, etc., etc.). Indeed, some would argue that no real community of mathematicians existed, since specialists in distinct fields were barely able to communicate with each other. I do not impute any fault to the system which prevailed in this period of remarkably vigorous mathematical growth—indeed, I believe it was historically inevitable and thus "correct"—but it does appear that these autonomous disciplines are now being linked together in such a way that mathematics is being reunified We may think of this development as "horizontal", as opposed to "vertical" growth Examples are the use of commutative ring theory in combinatorics, the use of cohomology theory in abstract algebra, algebraic geometry, functional analysis and partial differential equations, and the use of Lie group theory in many mathematical disciplines, in relativity theory and in invariant gauge theory

I believe that the appropriate education of a contemporary mathematician must be broad as well as deep, and that the lesson to be drawn from the trend toward a new unification of mathematics must involve a similar principle. We may so formulate it: *we must break down artificial barriers between mathematical topics throughout the student's mathematical education.*

The third trend to which I have drawn attention is that of the general availability of the computer and its role in actually changing the face of mathematics. The computer may eventually take over our lives; this would be a disaster. Let us assume this disaster can be avoided; in fact, let us assume further, for the purposes of this discussion at any rate, that the computer plays an entirely constructive role in our lives and in the evolution of our mathematics. What will then be the effects?

The computer is changing mathematics by bringing certain topics into greater prominence—it is even causing mathematicians to create new areas of mathematics (the theory of computational complexity, the theory of automata, mathematical cryptology) At the same time it is relieving us of certain tedious aspects of traditional mathematical activity which it executes faster and more accurately than we can It makes it possible rapidly and painlessly to carry out numerical work, so that we may accompany our analysis of a given problem with the actual calculation of numerical examples However, when we use the computer, we must be aware of certain risks to the validity of the solution obtained due to such features as structural instability and round-off error. The computer is especially adept at solving problems involving iterated procedures, so that the method of successive approximations (iteration theory) takes on a new prominence. On the other hand, the computer renders obsolete certain mathematical techniques which have hitherto been prominent in the curriculum—a sufficient example is furnished by the study of techniques of integration

There is a great debate raging as to the impact which the computer should have on the curriculum (see, for example, [6]). Without taking sides in this debate, it is plain that there *should* be a noticeable impact, and that every topic must be examined to determine its likely usefulness in a computer age. It is also plain that no curriculum today can be regarded as complete unless it prepares the student to use the computer *and to understand its mode of operation.* We should include in this understanding a realization of its scope and its limitations; and we should abandon the fatuous idea, today so prevalent in educational theory and practice, that the principal purpose of mathematical education is to enable the child to become an effective computer even if deprived of all mechanical aids!

Let me elaborate this point with the following table of comparisons On the left I list human attributes and on the right I list the contrasting attributes of a computer when used as a *calculating engine.* I stress this point because I must emphasize that I am not here thinking of the computer as a research tool in the study of artificial intelligence I should also add that I am talking of contemporary human beings and contemporary computers. Computers evolve very much faster than human beings so that their characteristics may well undergo dramatic change in the span of a human lifetime. With these caveats, let me display the table.

HUMANS	COMPUTERS
Compute slowly and inaccurately	Compute fast and accurately.
Get distracted.	Are remorseless, relentless and dedicated
Are interested in many things at the same time	Always concentrate and cannot be diverted
Sometimes give up	Are incurably stubborn
Are often intelligent and understanding	Are usually pedantic and rather stupid
Have ideas and imagination, make inspired guesses, think	Can execute "IF ELSE" instructions

Human and computer attributes

It is an irony that we seem to teach mathematics as if our objective were to replace each human attribute in the child by the corresponding computer attribute—and this is a society nominally dedicated to the development of each human being's individual capacities. Let us agree to leave to the computer what the computer does best and to design the teaching of mathematics as a generally human activity. This apparently obvious principle has remarkably significant consequences for the design of the curriculum, the topic to which we now turn.

The secondary school curriculum

Let us organize this discussion around the "In and Out" principle. That is, we will list the topics which should be "In" or strongly emphasized, and the topics which should be "Out" or very much underplayed We will also be concerned to recommend or castigate, as the case may be, certain teaching strategies and styles We do not claim that all our recommendations are strictly contemporary, in the sense that they are responses to the current prevailing changes in mathematics and its uses; some, in particular those devoted to questions of teaching practice, are of a lasting nature and should, in my judgment, have been adopted long since

We will present a list of "In" and "Out" items, followed by commentary We begin with the "Out" category, since this is more likely to claim general attention; and within the "Out" category we first consider pedagogical techniques

OUT (Secondary Level)

1. TEACHING STRATEGIES
   - Authoritarianism
   - Orthodoxy.
   - Pointlessness.
   - Pie-in-the-sky motivation.
2. TOPICS
   - Tedious hand calculations
   - Complicated trigonometry.
   - Learning geometrical proofs.
   - Artificial "simplifications".
   - Logarithms as calculating devices

Commentary

There should be no need to say anything further about the evils of authoritarianism and pointlessness in presenting mathematics They disfigure so many teaching situations and are responsible for the common negative attitudes towards mathematics which regard it as unpleasant and useless By orthodoxy we intend the magisterial attitude which regards one "answer" as correct and all others as (equally) wrong. Such an attitude has been particularly harmful in the teaching of geometry. Instead of being a wonderful source of ideas and of questions, geometry must appear to the student required to set down a proof according to rigid and immutable rules as a strange sort of theology, with prescribed responses to virtually meaningless propositions

Pointlessness means unmotivated mathematical process. By "pie-in-the-sky" motivation we refer to a form of pseudomotivation in which the student is assured that, at some unspecified future date, it will become clear why the current piece of mathematics warrants learning. Thus we find much algebra done because it will be useful in the future in studying the differential and integral calculus—just as much strange arithmetic done at the elementary level can only be justified by the student's subsequent exposure to algebra. One might perhaps also include here the habit of presenting to the student applications of the mathematics being learnt which could only interest the student at a later level of maturity; obviously, if an application is to motivate a student's study of a mathematical topic, the application must be interesting.

With regard to the expendable topics, tedious hand calculations have obviously been rendered obsolete by the availability of hand-calculators and minicomputers To retain these appalling travesties of mathematics in the curriculum can be explained only by inertia or sadism on the part of the teacher and curriculum planner. It is important to retain the trigonometric functions (especially as functions of real variables) and their basic identities, but complicated identities should be eliminated and tedious calculations reduced to a minimum Understanding geometric proofs is very important; inventing one's own is a splendid experience for the student; but memorizing proofs is a suitable occupation only for one contemplating a monastic life of extreme asceticism. Much time is currently taken up with the student processing a mathematical expression which came from nowhere, involving a combination of parentheses, negatives, and fractions, and reducing the expression to one more socially acceptable This is absurd; but, of course, the student must learn how to substitute numerical values for the variables appearing in a natural mathematical expression

Let us now turn to the positive side. Since, as our first recommendation below indicates, we are proposing an integrated approach to the curriculum, the topics we list are rather of the form of modules than full-blown courses.

IN (Secondary Level)

1. TEACHING STRATEGIES
   - An integrated approach to the curriculum, stressing the interdependence of the various parts of mathematics.
   - Simple application.
   - Historical references.
   - Flexibility.
   - Exploitation of computing availability

2. TOPICS
   - Geometry and algebra (e.g., linear and quadratic functions, equations and inequalities).
   - Probability and statistics
   - Approximation and estimation, scientific notation.
   - Iterative procedures, successive approximation
   - Rational numbers, ratios and rates
   - Arithmetic mean and geometric mean (and harmonic mean)
   - Elementary number theory
   - Paradoxes

Commentary

With respect to teaching strategies, our most significant recommendation is the first. (I do not say it is the most important, but it is the most characteristic of the whole tenor of this article) Mathematics is a unity, albeit a remarkably subtle one, and we must teach mathematics to stress this. It is not true, as some claim, that all good mathematics—or even all applicable mathematics—has arisen in response to the stimulus of problems coming from *outside* mathematics; but it is true that all good mathematics has arisen from the then existing mathematics, frequently, of course, under the impulse of a "real world" problem Thus mathematics *is* an interrelated and highly articulated discipline, and we do violence to its true nature by separating it—for teaching or research purposes—into artificial watertight compartments. In particular, geometry plays a special role in the history of human thought It represents man's (and woman's!) primary attempt to reduce the complexity of our three-dimensional ambience to one-dimensional language. It thus reflects our natural interest in the world around us, and its very existence testifies to our curiosity and our search for patterns and order in apparent chaos We conclude that geometry is a natural conceptual framework for the formulation of questions and the presentation of results. It is not, however, in itself a method of answering questions and achieving results. This role is preeminently played by algebra If geometry is a source of questions and algebra a means of answering them, it is plainly ridiculous to separate them How many students have suffered through algebra courses, learning methods of solution of problems coming from nowhere? The result of such compartmentalized instruction is, frequently and reasonably, a sense of futility and of the pointlessness of mathematics itself.

The good sense of including applications and, where appropriate, references to the history of mathematics is surely self-evident. Both these recommendations could be included in a broader interpretation of the thrust toward an integrated curriculum The qualification that the applications should be simple is intended to convey both that the applications should not involve sophisticated scientific ideas not available to the students—this is a frequent defect of traditional "applied mathematics"—and that the applications should be of actual interest to the student, and not merely important. The notion of flexibility with regard to the curriculum is inherent in an integrated approach; it is obviously inherent in the concept of good teaching. Let us admit, however, that it can only be achieved if the teacher is confident in his, or her, mastery of the mathematical content. Finally, we stress as a teaching strategy the use of the hand-calculator, the minicomputer and, where appropriate, the computer, not only to avoid tedious calculations but also in very positive ways. Certainly we include the opportunity thus provided for doing actual numerical examples with real-life data, and the need to re-examine the emphasis we give to various topics in the light of computing availability. We mention here the matter of computer-aided instruction, but we believe that the advantages of this use of the computer depend very much on local circumstances, and are more likely to arise at the elementary level.

With regard to topics, we have already spoken about the link between geometry and algebra, a topic quite large enough to merit a separate article. The next two items must be in the curriculum simply because no member of a modern industrialized society can afford to be ignorant of these subjects, which constitute our principal day-to-day means of bringing quantitative reasoning to bear on the world around us We point out, in addition, that approximation and estimation techniques are essential for checking and interpreting machine calculations.

It is my belief that much less attention should be paid to general results on the convergence of sequences and series, and much more on questions related to the rapidity of convergence and the stability of the limit. This applies even more to the tertiary level However, at the secondary level, we should be emphasizing iterative procedures since these are so well adapted to computer programming. Perhaps the most important result—full of interesting applications—is that a sequence $\{x_n\}$, satisfying $x_{n+1} = ax_n + b$, converges to $b/(1-a)$ if $|a| < 1$ and diverges if $|a| > 1$. (For one application see [4].) It is probable that the whole notion of proof and definition by induction should be recast in "machine" language for today's student.

The next recommendation is integrative in nature, yet it refers to a change which is long overdue. Fractions start life as parts of wholes and, at a certain stage, come to represent amounts or measurements and therefore numbers. However, they are not *themselves* numbers; the numbers they represent are rational numbers Of course, one comes to speak of them as numbers, but this should only happen when one has earned the right to be sloppy by understanding the precise nature of fractions (see [5]). If rational numbers are explicitly introduced, then it becomes unnecessary to treat ratios as new and distinct quantities. Rates also may then be understood in the context of ratios and dimensional analysis However, there is a further aspect of the notion of rate which it is important to include at the secondary level I refer to average rate of change and, in particular, average speed. The principles of grammatical construction suggest that, in order to understand the composite term "average speed" one must understand the constituent terms "average" and "speed". This is quite false; the term "average speed" is much more elementary than either of the terms "average", "speed", and is not, in fact, their composite. A discussion of the abstractions "average" and "speed" at the secondary level would be valuable in itself and an excellent preparation for the differential and integral calculus

Related to the notion of average is, of course, that of arithmetic mean I strongly urge that there be, at the secondary level, a very full discussion of the arithmetic, geometric and harmonic means and of the relations between them. The fact that the arithmetic mean of the the non-negative quantities $a_1, a_2, \ldots, a_n$ is never less than their geometric mean and that equality occurs precisely when $a_1 = a_2 = \ldots = a_n$, may be used to obtain many maximum or minimum results which are traditionally treated as applications of the differential calculus of several variables—a point made very effectively in a recent book by Ivan Niven.

Traditionally, Euclidean geometry has been held to justify its place in the secondary curriculum on the grounds that it teaches the student logical reasoning. This may have been true in some Platonic academy What we can observe empirically today is that it survives in our curriculum in virtually total isolation from the rest of mathematics; that it is not pursued at the university; and that it instills, in all but the very few, not a flair for logical reasoning but distaste for geometry, a feeling of pointlessness, and a familiarity with failure. Again, it would take a separate article (at the very least) to do justice to the intricate question of the role of synthetic geometry in the curriculum. Here, I wish to propose that its hypothetical role can be assumed by a study of elementary number theory, where the axiomatic system is so much less complex than that of plane Euclidean geometry Moreover, the integers are very "real" to the student and, potentially, fascinating Results can be obtained by disciplined thought, in a few lines, that no high-speed computer could obtain, without the benefit of human analysis, in the student's lifetime

$$\text{e.g. } (7^{10^6})^{12} \equiv 1 \bmod 13$$

Of course, logical reasoning should also enter into other parts of the curriculum; of course, too, synthetic proofs of geometrical propositions should continue to play a part in the teaching of geometry, but not at the expense of the principal role of geometry as a source of intuition and inspiration and as a means of interpreting and understanding algebraic expressions.

My final recommendation is also directed to the need for providing stimulus for thought. Here I understand, by a paradox, a result which conflicts with conventional thinking, not a result which is self-contradictory. A consequence of an effective mathematical education should be the inculcation of a healthy scepticism which protects the individual against the blandishments of self-serving propagandists, be they purveyors of perfumes, toothpastes, or politics. In this sense a consideration of paradoxes fully deserves to be classified as applicable mathematics! An example of a paradox would be the following: Students A and B must submit to twenty tests during the school term Up to half term, student A had submitted to twelve tests and passed three, while student B had submitted to six tests and passed one. Thus, for the first half of the term, A's average was superior to B's. In the second half of the term, A passed all the remaining eight tests, while B passed twelve of the remaining fourteen Thus, for the second half of the term, A's average was also superior to B's Over the whole term, A passed eleven tests out of twenty, while B passed thirteen tests out of twenty, giving B a substantially better average than A.

The elementary school curriculum

This article (like the talk itself!) is already inordinately long. Thus I will permit myself to be much briefer with my commentary than in the discussion of the secondary curriculum, believing that the rationale for my recommendations will be clear in the light of the preceding discussion and the reader's own experience. I will again organize the discussion on the basis of the "In" and "Out" format beginning with the "Out" list

OUT (Elementary Level)

1. TEACHING STRATEGIES
   - Just as for the secondary level.
   - Emphasis on accuracy.
2. TOPICS
   - Emphasis on hand algorithms
   - Emphasis on addition, subtraction, division and the order relation with fractions
   - Improper work with decimals.

Commentary

The remarks about teaching strategies are, if anything, even more important at the elementary level than the secondary level. For the damage done by the adoption of objectionable teaching strategies at the elementary level is usually ineradicable, and creates the mass phenomenon of "math avoidance" so conspicuous in present-day society On the other hand, one might optimistically hope that the student who has received an enlightened elementary mathematical education and has an understanding and an experience of what mathematics can and should be like may be better able to survive the rigors of a traditional secondary instruction if unfortunate enough to be called upon to do so, and realize that it is not the bizarre nature of mathematics itself which is responsible for his, or her, alienation from the subject as taught.

With regard to the topics, I draw attention to the primacy of multiplication as the fundamental arithmetical operation with fractions For the notion of fractions is embedded in our language and thus leads naturally to that of a fraction of a fraction The arithmetical operation which we perform to calculate, say, 3/5 of 1/4 we *define* to be the product of the fractions concerned. Some work should be done with the addition of elementary fractions, but only with the beginning of a fairly systematic study of elementary probability theory should addition be given much prominence. Incidentally, it is worth remarking that in the latter context, we generally have to add fractions which have the same denominator—unless we have been conditioned by prior training mindlessly to reduce any fraction which comes into our hands

Improper work with decimals is of two kinds. First, I deplore problems of the kind $13.7 + 6.83$, which invite error by misalignment Decimals represent measurements; if two measurements are to be added, they must be in the same units, and the two measurements would have been made to the same degree of accuracy Thus the proper problem would have been $13.70 + 6.83$, and no difficulty would have been encountered Second, I deplore problems of the kind 16.1×3.7, where the intended answer is 59.57. In no reasonable circumstances can an answer to two places of decimals be justified; indeed all one can say is that the answer should be between 58.58 and 60.56. Such spurious accuracy is misleading and counterproductive. It is probably encouraged by the usual algorithm given for multiplying decimals (in particular, for locating the decimal point by counting digits to the right of the decimal point); it would be far better to place the decimal point by estimation.

Again, we turn to the positive side

IN (Elementary Level)

1. TEACHING STRATEGIES
   - As for the secondary level.
   - Employment of confident, capable and enthusiastic teachers.
2. TOPICS
   - Numbers for counting *and* measurement—the two arithmetics.
   - Division as a mathematical model in various contexts.
   - Approximation and estimation.
   - Averages and statistics.
   - Practical, informal geometry
   - Geometry and mensuration; geometry and probability (Monte Carlo method).
   - Geometry and simple equations and inequalities
   - Negative numbers in measurement, vector addition.
   - Fractions and elementary probability theory.
   - Notion of finite algorithm and recursive definition (informal).

Commentary

Some may object to our inclusion of the teacher requirement among the "teaching strategies"—others may perhaps object to its omission at the secondary level! We find it appropriate, indeed necessary, to include this desideratum, not only to stress how absolutely essential the good teacher is to success at the elementary level, but also to indicate our disagreement with the proposition, often propounded today, that it is possible, e.g. with computer-aided instruction, to design a "teacher-proof" curriculum.

The good, capable teacher can never be replaced; unfortunately, certain certification procedures in the United States do not reflect the prime importance of mathematical competence in the armoury of the good elementary teacher

We close with a few brief remarks on the topics listed. It is an extraordinary triumph of human thought that the same system can be used for counting and measurement—but the two arithmetics diverge in essential respects—of course, in many problems both arithmetics are involved. Measurements are *inherently* imprecise, so that the arithmetic of measurement is the arithmetic of approximation. Yes, $2 + 2 = 4$ in counting arithmetic; but $2 + 2 = 4$ with a probability of 3/4 if we are dealing with measurement *

The separation of division from its context is an appalling feature of traditional drill arithmetic. This topic has been discussed elsewhere [7]; here let it suffice that the solution to the division problem $1000 \div 12$ should depend on the context of the problem and not the grade of the student.

Geometry should be a thread running through the student's entire mathematical education—we have stressed this at the secondary level. Here we show how geometry and graphing can and should be linked with key parts of elementary mathematics. We recommend plenty of experience with actual materials (e.g., folding strips of paper to make regular polygons and polyhedra), but very little in the way of geometric proof. Hence we recommend practical, informal geometry, within an integrated curriculum

We claim it is easy and natural to introduce negative numbers, and to teach the addition and subtraction of integers—motivation abounds. The multiplication of negative numbers (like the addition of fractions) can and should be postponed.

As we have said, multiplication is the primary arithmetical operation on fractions. The other operations should be dealt with in context—and probability theory provides an excellent context for the addition of fractions. It is however, not legitimate to drag a context in to give apparent justification for the inclusion, already decided on, of a given topic.

The idea of a finite algorithm, and that of a recursive definition, are central to computer programming. Such ideas will need to be clarified in the mathematics classroom, since nowhere else in the school will the responsibility be taken. However, it is reasonable to hope that today's students will have become familiar with the conceptual aspects of the computer in their daily lives—unless commercial interests succeed in presenting the microcomputer as primarily the source of arcade games.

But this is just one aspect of the general malaise of our contemporary society, and deserves a much more thorough treatment than we can give it here. It is time to rest my case

* If $AB = 2$ ins., and $BC = 2$ ins., each to the nearest inch, then $AC = 4$ ins to the nearest inch with a probability of 3/4.

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