

APRIL AND THE INFINITELY MANY PING PONG BALLS [1]

AMI MAMOLO

Chapter 1: ...We came in

"I can't accept that!" Jaime was almost shouting. He slammed his text shut and scowled. "The whole thing is impossible anyway. You could never move that fast."

The instructor looked back at Jaime a little helplessly. It was clear he hadn't expected such fierce responses to his question.

"The experiment is *definitely* impossible. It will never end!" replied Lora bluntly.

Pietr gave a shy smile and said, "OK, this is definitely outside the realm of possibility—but if we just assume the experiment *does* end—" Lora sat forward, ready for a fight. Jaime snorted loudly. Pietr lost his nerve but Awasin took him up.

"Well," he began carefully, "it is supposed to be a thought experiment—" the instructor smiled gratefully at this and began nodding his head vigorously and looking around at the other students. April sighed and put her head on her desk, resting it on her arms. Students from around the room were shouting out different ideas. It was difficult to follow much of any of it. *How many?* She thought vaguely, looking down to her notes at what she had copied from the board (See Figure 1, [2]). *How many?* She thought again, full of uncertainty.

The instructor was trying desperately to calm the conversation down and interject some ideas. Awasin and Pietr seemed to be the only ones ready to listen. Jaime, who was getting increasingly frustrated, asked what the point was anyway. Someone asked if this was going to be on the exam and April saw the instructor's shoulders sink. Jaime forced an audible yawn, muttered that he thought the problem was ridiculous and impossible, took out his phone and started playing games.

Lora looked at him with disdain and said curtly, "Let's get past 'it's impossible' and talk about *why* it's impossible. The experiment will never end and so there will be infinitely many balls left!"

"How d'you figure?" asked Awasin.

"Because," came the reply, "the sixty seconds will last forever, so the barrel will always be full of ping pong balls."

April raised an eyebrow, and Lora elaborated:

"What I mean is: the process is impossible since the time interval is halved infinitely many times, so the sixty seconds never ends. Since you're putting balls in the barrel for eternity, it will always be full."

"Eternity," mused Awasin, "didn't Aristotle describe infinity as *inexhaustible*? [3] Aristotle would claim the experiment must last forever, because the process of putting in and removing balls could never stop. I guess Aristotle might agree with you Lora."

"That's right!" cried Lora. "Even with one second left we can still divide this amount of time into infinitely many small amounts of time (ignoring physics as usual). Therefore, the experiment will continue into eternity and the number of balls in the barrel will be infinite."

"I *partially* agree," smiled the instructor, "but remember $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$. So 'the end' of the experiment does happen at one minute."

Lora scratched her nose, "Ok, yeah, I remember that from Calculus. So, is this a Calculus problem? And, are you saying that Aristotle was wrong?!"

There was nervous laughter.

"Aristotle wasn't wrong," said Prof. Quine, who had been silently listening at the door and couldn't help but intervene. "When resolving certain paradoxes of infinity, a fallacy can emerge—the mistaken notion that an infinite succession of intervals of time has to add up to all eternity" [4]. Aristotle was thinking of one kind of infinity—*potential* infinity, but this paradox involves a different type—*actual* infinity. The paradox is *falsidical*. It seems false because it is false." And with a wry grin, Quine disappeared down the hall.

"*Actual* infinity—a completed entity which encompasses the potential—," commented the Instructor.

Awasin leaned over to April and whispered: "Hold infinity in the palm of your hand, and eternity in an hour." [5] She smiled "Or a minute, apparently."

"I don't buy it," interjected Jaime stubbornly. "I get what Professor Quine is talking about, but that's only *in theory*. The experiment couldn't *actually* end. It's impossible."

"Thought experiment—" the instructor repeated gently. "So, if we can imagine such an experiment—if it is *possible* to imagine it—and to let go of physical restraints and engage— [6]" Jaime rolled his eyes and turned back to his phone.

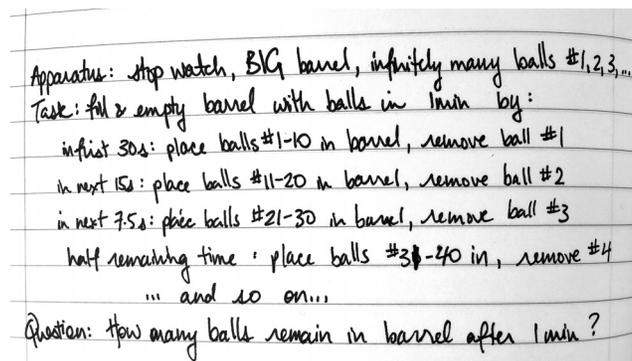


Figure 1. April's rendition of the ping pong ball problem.

"I think the answer is zero," came a soft voice from the corner; it was almost a question. Everyone turned to look. Clearing her throat with a cough, Zia continued, "Any thing minus that thing is always zero."

April considered this. Zia had a point, but hadn't they learnt that subtracting infinity was undefined? Awasin seemed to be thinking along similar lines.

"I don't know if we can think about it that way," he said reasonably. "There are infinitely many time periods, therefore infinitely many times during which 10 balls are put in and one thrown out. So—there are an infinite number of balls in the basket as well as an infinite number thrown out. It doesn't make sense to have this result, but there it is."

"I can't agree with zero balls remaining," objected Lora. "You put in more balls than you take out. I still think my original answer is correct."

The conversation started to get heated again. Lora and Zia argued back and forth while Jaime made smart remarks to irritate the lot. April was trying to take everything in. Lora looked as though she had just eaten something sour; Zia was speaking with more confidence. Awasin seemed to be toggling between the two ideas; Pietr was staring at the ceiling silently moving his lips.

Awasin turned to April and asked, "You as confused as I am?" She laughed and nodded, "my head hurts!"

The class was about the end and the instructor interrupted the discussion to have everyone write out their ideas and hand them in. They would take them up next week, after everyone had a chance to think things through and formalize a solution.

"Is he asking for a proof?" Zia asked.

"I guess so," replied Lora. The two left arm-in-arm.

After everyone else had left the room, April walked up to the instructor's desk to hand in her work. Still uncertain, she peeked at what others had written (see Figure 2). *Oh well*, she thought, looking down at her own uncertain response and turning it in (see Figure 3). "Is there really an answer to this problem?" she asked the instructor.

He looked up, smiled and said, "I'm hoping you'll tell me. See you next week."

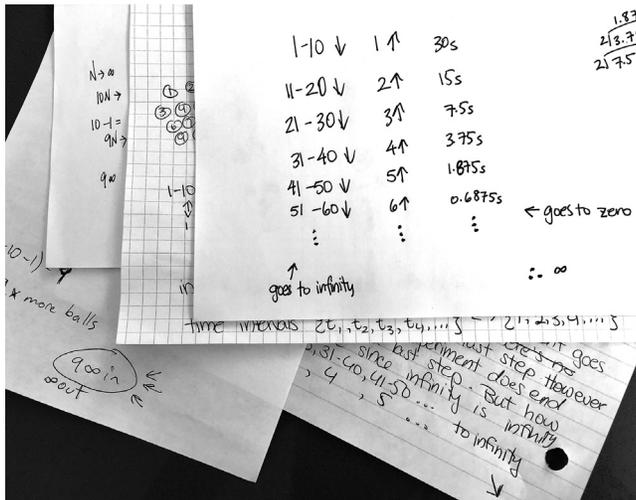


Figure 2. A peek at responses.

Chapter 2: Parcheesi and a cup of tea

April left class with a headache and no better sense of the problem. She, Awasin, and Pietr walked toward Café Blue Parrot still talking about it. The guys had booked a squash court near the café where April was meeting a friend. Pietr, who was not usually one to speak up in class, was explaining what he had submitted.

"Every time the remaining time is halved, the equivalent change (+10 - 1) = 9 balls are added. So there will be an infinite of balls in the basket. Some may say that an infinite amount of balls have been taken out of the basket, which is true, but it is not an equivalent infinity to what is put in—. There will be nine times as many in the basket as you took out."

"What do you mean by 'not an equivalent infinity'?" asked April. "Can there be different infinities?"

"Well," replied Pietr slowly, "I'm not really sure— it's just that more balls seem to go into the barrel at any given time than come out of it. So, if more go in at each step in the experiment, then at the end of the experiment (assuming it ends), there should be more in the barrel than out."

April frowned, "I don't see how you can have more than one infinity. 'Nine times infinity' is still infinity, isn't it? And what about the possibility that the barrel is empty? What if that's the solution?" The three friends walked the rest of the way to the café in silence. As they parted, April wished them a good game. Awasin looked at his feet as Pietr grinned, "I'm undefeated".

When April entered Café Blue, her friend Jana was already seated, nose buried in a book called *Wabi sabi*. Looking up, Jana asked innocently enough, "How was your morning?"

With a chuckle, April sank down and related the problem to her friend. A bit to April's surprise, Jana dove right in.

"Wow—" said Jana, starting to sketch out the problem. She drew out a barrel, with beautifully detailed knotted wood, and a collection of ping pong balls spiralling off the page. April admired her friend's artistic style and asked, "Is that helping you solve the problem?"

"Well," Jana chuckled, "at first, I'm just thinking about a massive collection of white ping pong balls. And an actual wooden barrel. Clearly thinking about actually performing the experiment but— you could never have a barrel big enough. There isn't enough space on earth for such a large

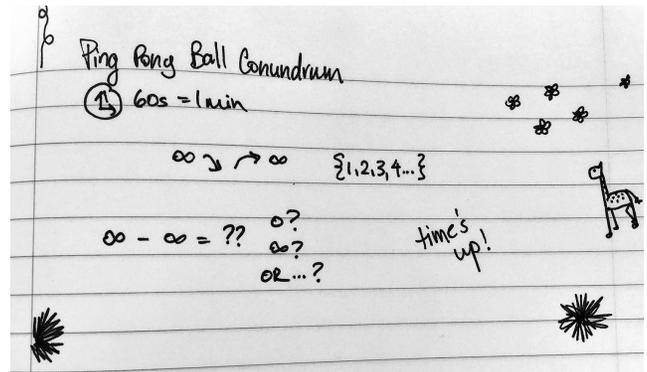


Figure 3. April's uncertainty.

barrel.” April smiled, and thought *definitely outside the realm of possibility*.

“And then,” Jana went on, “I’m thinking there is no way I can actually move that fast in real life anyway—even if I could work at light speed—the experiment will last for eternity. But I don’t want to do that with my life!”

April laughed, “I know what you mean!” Wanting to change the subject, she asked, “What are you doing with all these books and sketches?” Jana’s eyes lit up as she described the aquascape she was designing. She pulled out more books and flipped through pictures of what she said “captured the impermanent and imperfect beauty of life.”

“You know,” Jana said after a while, “your problem kind of reminds me of something I read in this book—” she flipped some pages “about the ‘tendency to leave the unexplainable unexplained’ [7]. But I’m afraid that won’t help you with your class!”

The two friends chatted, comparing and contrasting: art, mathematics; impermanence, in-transience; *wabi sabi*, infinity. April liked the notion of *an aesthetic pleasure that lies beyond conventional beauty* [8]—it seemed to somehow capture her feelings about the ping pong ball problem. It was certainly an unconventional appeal, frustrations and all.

As April sipped her third cup of tea, she noticed a particularly competitive game of Parcheesi was underway at the back of the café. “I think that’s Prof Galileo,” she said to Jana. “He’s kind of a rockstar— as far as mathematicians go.”

“Who’s he with?” asked Jana.

“Not sure, but let’s see if they can help with the problem.”

April and Jana approached the pair and timidly presented the paradox and nature of their problem. Galileo introduced his friend Bolzano, who smiled, happy for the distraction. Bolzano replied to the problem first, “Certainly most of the paradoxical statements encountered in the mathematical domain [...] are propositions which either immediately contain the idea of the infinite, or at least in some way or other depend upon that idea for their attempted proof” [9].

“Paradoxical indeed,” replied Galileo “I have spent some time myself working on Zeno’s paradoxes—of which this seems to be a sort of extension.”

“First, we must consider that the infinite is more than ‘that which has no end,’”[10] Bolzano went on. “We could compare the sets by coupling, yet I have a better idea, which I think will put an end to your struggles April and Jana.” Galileo’s right eyebrow twitched slightly. “Let us consider a similar problem, and you can then apply the reasoning in resolving your ping pong problem. Consider the sets of rational numbers $A = [0, 5]$ and $B = [0, 12]$. If we construct a map $5y = 12x$ for x in A and y in B , then we may couple each element in A with exactly one element in B , and vice versa.”

“So, you’ve created a bijection,” said April. “Does this mean the sets are equinumerous?”

Both men shook their heads.

“Although every quantity in A or B allows of coupling with one and only one in B or A , yet the set of quantities in B is other and greater than in A , since the distance between the two quantities in B is other and greater than the distance between the corresponding quantities in A ” [11].

“Here a difficulty presents itself which appears to me insoluble,” interjected Galileo. “Since it is clear that we may

have one line greater than another, each containing an infinite number of points, we are forced to admit that, within one and the same class, we may have something greater than infinity, because the infinity of points in the long line is greater than the infinity of points in the short line. This assigning to an infinite quantity a value greater than infinity is quite beyond my comprehension” [12].

Bolzano insisted, “two sets can still stand in a relation of inequality, in the sense that—” [13].

Galileo cut in firmly, “the attributes ‘equal’, ‘greater’, and ‘less’, are not applicable to infinite, but only to finite quantities” [14].

April and Jana were beginning to regret their decision.

Galileo continued, “let us consider a different example: the set of natural numbers and the set of perfect squares.”

“Clearly, one is found to be a whole and the other a part of that whole” chimed Bolzano.

“And yet,” replied Galileo, “there are as many squares as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square” [15].

“Ah, but my friend, one-to-one correspondence never justifies us,” Bolzano reasoned, “in inferring the *equality of the two sets, in the event of their being infinite*, with respect to the multiplicity of their members—that is, when we abstract from all individual differences— two sets can still stand in a relation of inequality, in the sense that the one is found to be a whole and the other a part of that whole” [16].

“So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former”[17]. At this point, Galileo flipped the Parcheesi board and the two men succumbed to a shouting match, leaving poor April and Jana no nearer to a solution.

“I think I’m starting to understand what it means to find beauty in the irresolvable,” Jana said with a wink. April laughed, “At least one of us is closer to understanding something!”

Chapter 3: The usefulness of small napkins

A few days later, April still had not made further progress on the ping pong problem. With a full course load, there was enough other work to keep her busy. This morning, she was feeling rather miserable, sitting in the computer lab with wet feet waiting for class to start. It had been raining nonstop for days. Awasin and Zia slipped into class just as it was getting underway and sat next to April. Awasin pushed his wet hair out of his face and grinned sheepishly. “Busted umbrella,” he said. “Any closer to solving—”

“Not even,” April replied, and she filled them in on the argument between Galileo and Bolzano.

Pieter was sitting nearby and overheard. He joined in, “I still think it’s got to be infinitely full, but coming up with a proof is a different story. I wonder if we could write a program to tackle this—” He trailed off and turned his attention back to his computer, clearly on the hunt. It didn’t seem like anyone was listening to the professor, who seemed aware of this but unsure what to do about it.

April, who was now fully focused on ping pong balls, mused “I think there’s something useful in the idea of a one-to-one correspondence, but neither Galileo nor Bolzano felt that was sufficient to solve the problem.”

Awasin chewed on this for a while and then said, “Well, even though there is a one-to-one correspondence between the sets $\{1, 2, 3, 4, \dots\}$, $\{9, 18, 27, 36, \dots\}$, the rate at which you are putting in is more than you are taking out. So even if there are just as many numbers in each set, they will never even out, because the process continues infinitely and you continue to put more in than you take out.”

“OK, but—” April began hesitantly, after thinking for a while. “But, *which* balls? I mean, for every numbered ball, we know the exact time interval that the ball was removed. So, if there are infinitely many balls left in the barrel, then which ones are still in there?”

“That’s a good question,” replied Awasin, chewing his lip.

Zia cut in. “It’s impossible to name a specific ball,” she said firmly. April and Awasin waited for her to continue. Zia took her time, thinking things through as she spoke. “It’s impossible to name a specific ball left in the barrel. As soon as a number is chosen, it is possible to determine the exact time that ball was removed— I can’t name a numbered ball that remains but then I also couldn’t tell you how many balls we began with because there were infinity.”

Class had ended by now and they packed up their bags to leave.

“Pietr, did you make any progress?” Awasin asked. Pietr shrugged in answer. Zia rolled her eyes, smiled, and waved goodbye as she headed to her next class. The other three made their way out of the building. Trying as much as possible to avoid the rain, they darted from awning to awning. Whenever they could catch a breath, they resumed talking about the problem.

As they turned a particularly windy corner and took shelter, two older gentlemen walking along in silence, with

heads seemingly in outer space and totally unaware of the bad weather, passed by. On overhearing their conversation, one of the gentlemen perked up and interjected.

“Questions of infinity on your minds?” he asked eagerly. “Allow me to share some of my thinking on the subject.”

“Be careful young ones,” warned his companion “once my friend Georg starts talking of infinity he tends to go on and on—”

“Oh Albert,” sighed the man named Georg, “as if I’ve never heard *that one* before.”

The three students, not quite sure what to think, decided to hear him out.

“I’ve written about this extensively in my theory of transfinite numbers” he began. “To compare the cardinalities of infinite sets we must do so in a consistent way, and for that, we need a few definitions. The first is that a set is considered infinite if and only if it can be put into a one-to-one correspondence with one of its proper subsets. The second is that two infinite sets are considered to have the same cardinality if and only if they can be put in one-to-one correspondence with one another”. [18]

“So, to answer your question of ‘how many’, and whether one set is more numerous than the other, we must look to the existence (or not) of a one-to-one correspondence.” At which point Georg took out a crumpled but unused napkin from his pocket and began to scribble (see Figure 4). Mumbling mostly to himself, with Albert looking over his shoulder, Georg jotted down some sparse ideas, threw the napkin to Awasin, exclaimed “Ha,ahaha! That should get you started” and left with a skip in his step and his friend smiling close behind.

“Ohhhhh Kaaaaayyyyy—” Awasin said bemusedly. April weighed what was written, and Pietr said, “Thank goodness for small napkins.”

The three of them stared at the napkin and Pietr asked:

“So, what—”

At that exact moment, the wind changed directions and started soaking them with rain. In a panic they sprinted for the nearest open building.

“Protect the napkin!” shouted April. The rain was pelting them from all angles, splashing up from the ground as they ran. Pietr was favouring one leg and was slow to catch up to Awasin, who had run effortlessly ahead. Out of breath and finally out of the rain, Pietr panted, “So, what shall we do with this?”

After some thought, April shrugged, “I’m not sure, let’s see what the others think.” They decided to meet in the library Sunday afternoon with whomever else from the class wanted to join.

Chapter 4: Beyond conventional

The weekend disappeared in a blink, and before April knew it, it was Sunday afternoon. The rain was relentless and created a gloomy atmosphere in the nearly empty library. When April arrived, her friends were already there, even Jaime. “Awasin dragged me here” he said. April asked where Pietr was, and Zia replied, “Wandering the stacks. I think he’s been here all weekend.”

On cue, Pietr appeared with an armful of books. “Background reading,” he smiled and sat down, pulling out the crumpled napkin. “I think I’m starting to get a handle on this.”

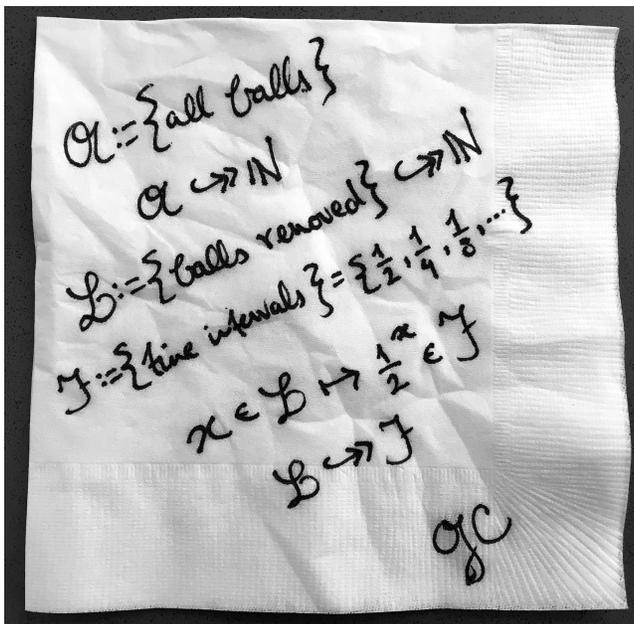


Figure 4. Good ideas on small napkins.

April and Awasin started filling everyone in on their encounter. Lora laughed good humouredly at the account and began deciphering the scribbles for herself. Frowning slightly, she ultimately concluded “I cannot agree with this napkin.”

“I have my doubts too, but I’m sure it makes sense if you’re comfortable with the concept of infinity” said April.

Pietr passed around some books and started explaining his thinking, eventually ending with: “So, I think this means that the barrel is empty at the end of the experiment.” Lora looked as though she had been personally insulted; Awasin sat back appreciatively, while April, still uncertain, doodled in his notebook as she thought.

Zia was the first to respond, “That fits with my thinking. Infinity is infinity is infinity—it’s all the same. And, anything minus that same thing is going to be zero.”

Pietr perked up and showed Zia a book he was holding. “Not exactly,” he said, “There are different infinities! Ones that aren’t equivalent. But it’s not quite like how I was thinking about it before—”. He pointed to a diagram that showed an array of numbers with the diagonal circled. Zia blinked and Lora leaned over as Pietr described what he had been reading. After a while, a voice chimed in from a nearby table: “And that’s not even counting *geometric* infinities!” April recognized Quade, whose head was peaking over a pile of molecular models.

“Let’s stick to one infinity at a time please,” Jaime said rudely. Quade’s head disappeared back into his work, as April said to Jaime “Well, that seems silly.”

Lora brought them back to the problem, “I will not accept a logical argument that the basket is empty. Such an argument would be flawed.”

The group debated this for a long time; the idea of an empty barrel was certainly not popular. There was much discussion. Every so often, a shadowy figure would appear from the stacks, listen in on the debate, and then quickly scuttle off. “I see him here a lot,” said Pietr, “I think he’s a prof.” As Pietr went in search of more books, April and Awasin decided to pay Quade a visit. Zia and Lora stayed focused on the ping pong balls, coming up with another twist on the problem. April could hear Zia saying, “You just go back and remove ball 2 after the last step—” and Lora countering, “It’s infinity—there is no last step.” After some time, Jamie excused himself, satisfied with the solution he had. The clouds in the sky were so dark that no one noticed the sun setting. As they worked, their conversation diverged away from infinity now and then, but always wound its way back.

In due course, Awasin, the cautious one, reflected: “There are conflicting views and now I am not sure whether there is none or infinite balls in the barrel. My gut feeling seems to want to say that there are an infinite number but there seems to be none as well.”

Pietr nodded, “Intuitively it seems that the number of balls *should* blow up to infinity—though intuition frequently fails us when it comes to the infinite.”

Lora, still sceptical, conceded: “I can now entertain the idea that there are no balls in the barrel, but I don’t like it.”

As the group started writing up their solution, a familiar face approached. “Thought I might find you here,” Jana said

to April. “You never showed up for dinner!”

“Oh no! I’m so sorry! I totally lost track of time.” April exclaimed.

“I think we all did,” replied Zia, looking at her watch.

“Well,” asked Jana, “was it worth it? Did you find the answer?”

“*The answer?*” the shadowy figure emerged from the stacks, bouncing on the balls of his feet and grinning mysteriously. Awasin and April exchanged a look. *What now?* She wondered.

Pietr looked up and asked keenly: “Can there be more than one answer?” The two had clearly chatted before.

“That depends on how you are thinking about infinity.”

Awasin let out a weary sigh, and Lora said bluntly: “We’ve already been through that—”

“Yes, I heard your rather standard approach. I was thinking of a more *nonstandard* [19] one.” He blew his nose on a handkerchief he pulled from his front pocket. It was monogrammed with fancy letters: **AR**

Jana looked over at April. “This should be interesting,” she whispered with a smile. **AR** began looking for a pen and paper.

Jana reached into her bag for some and swore loudly. Blushing, she said: “Sorry— but everything’s soaked!” Her bag was full of mostly books and all were dripping wet. April, who had over the years dropped more than one book in the lake, offered to help dry them. The two began diligently drying, page-by-page.

“What are you writing?” Awasin asked, straining to read what **AR** was jotting down in Lora’s notebook (see Figure 5).

They gathered around to have a look at the notebook, and after some thought, Lora asked, “What does it mean to have W greater than any integer?”

“Well,” **AR** replied, eyes alight, “ W is an element of the *hyperintegers*, a subset of the *hyperreals*. *Hyperreal numbers* generalize properties of real numbers to infinitely large or small numbers—and you don’t need to think of them as cardinalities!” There were a few raised eyebrows at this and that was enough to encourage him to continue.

“Sets A and B can be measured by the span of the intervals $[1, 10W]$ and $[1, W]$, respectively. So the measure—or size, if you want—of $[1, W]$ would be $W-1$, which is also an

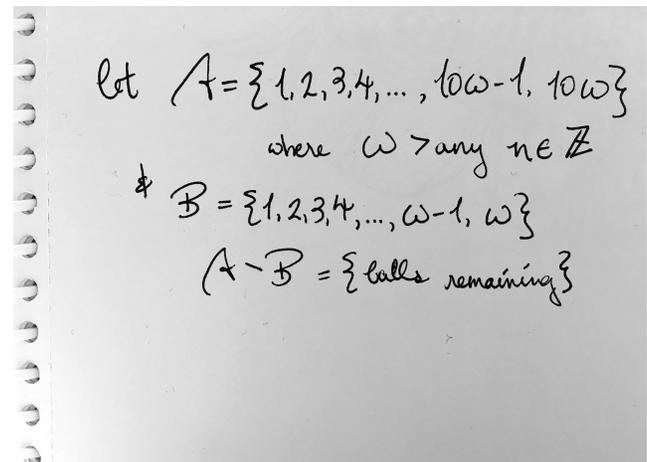


Figure 5. A nonstandard approach.

Let $A = \{1, 2, 3, 4, \dots, 10\omega - 1, 10\omega\}$
 where $\omega > \text{any } n \in \mathbb{Z}$
 $B = \{1, 2, 3, 4, \dots, \omega - 1, \omega\}$
 $A - B = \{\text{balls remaining}\}$
 $A - B = \{W+1, \dots, 2W, 2W+1, \dots, 10W\}$
 $\Rightarrow 10W - W = 9W$

Figure 6. An infinitely full set.

infinite number, and one that is less than W . In fact, with hyperintegers, what you might expect to be true is!”

“Rrrreally?” asked Lora suspiciously.

“The *transfer principle* is important,” AR said, plucking a book out of nowhere and passing it to Pietr, who immediately started skimming.

“So—” Pietr asked, “does this mean we can add and subtract hyperintegers in the usual way?” AR nodded happily. The group sat quietly, tossing the ideas around for themselves.

“The trick,” said AR “is to apply a different interpretation of quantity. You can vary the context of the problem, without changing the problem, and you might be pleasantly surprised by the result!” He sat for a few moments tapping a rhythm on his knees and then, without saying another word, he got up and left.

“Wait a minute! So, if we set up the problem in this way, does this mean we could end up with infinitely many balls in the barrel?” asked Lora, sitting straight up.

“How d’you figure?” asked Awasin. Lora added her idea to the page (see Figure 6).

Zia pursed her lips, “So, you found the span of the set of remaining balls— and it’s an infinite number?”

“Is this span what Bolzano was thinking of as distance?” wondered April.

“I suddenly feel like a game of Parcheesi,” Jana muttered back.

“But,” replied Awasin, “if you’re counting from $W+1$ to $10W$ —”

“Hang on,” interrupted Pietr, holding up a book. “We’re not counting— we’re measuring. So we have to measure the intervals $[0, 10W]$, $[0, W]$, and $[W, 10W]$ to see what they respectively span.” He passed the text around for each to read.

“So, we pick some arbitrary hyperintegers that allow us to set up the problem so that there will be ‘9 infinity’ balls remaining in the barrel. The interval $[W, 10W]$ has measure $9W$, which means a bigger infinite number remains in the barrel than was removed from it!”

It took time for this to sink in.

“Convenient,” remarked April.

“One thing that’s interesting,” mulled Awasin, pointing

to the page, “is that we’ve now changed the number of steps taken—we’re actually removing more balls in this version. There’s one step for each hyperinteger up to W , rather than just one for each natural number!”

“Not only that,” agreed Lora, “but it also takes less time— infinitesimally less time. The experiment will end 2^{-W} seconds before the end of a minute!”

An announcement over the public address system told them that the library would be closing soon. It was welcome news as everyone was both exhausted and wide awake, and quite ready to put the problem to rest at last. And then—

“Hang on, I just realized something,” said April. “We now have *two* solutions—and they both seem to be correct but they give totally opposite answers! Can they both be correct?”

The group wasn’t sure what to make of this question.

“I guess both are correct. They both have proofs after all,” was Pietr’s reply.

“But is that OK?” asked Zia. “We’re supposed to submit a solution—”

“Have we really spent this whole time debating only to come up with two resolutions and isn’t that in effect, no resolution?” Lora sounded exasperated. “Have we just been wasting our time?”

Jana, who was drying the pages of her last wet book, read aloud as a kind of response: “Playing with abandon I lose track of time. People passing by point and laugh asking, ‘What is the reason for such foolishness?’ I respond only with a deep bow, for even if I answered it would be beyond their understanding, look around, there is nothing more than this.” [20]

Chapter 6: Isn’t this where...?

The rain was starting to let up as April and Awasin walked quietly home, each lost in separate thought. It was after midnight by the time they arrived at April’s house. The two friends lingered on the street for some time.

“I hardly expected we’d come up with one solution, let alone two.” April admitted. “But none of it feels quite right. Is it zero? Is it infinity? Is one truer than the other? And if they are both true, what does that say? Does it mean something?”

After a while, Awasin answered, “I guess you’re asking ‘What does it mean to be ‘true’?’ Maybe that’s more a philosophy question than a math one?” They looked up to the night sky.

“It does remind me though,” he continued, “of something James Newman wrote. “It is hard to know what you are talking about in mathematics, yet no one questions the validity of what you say. There is no other realm of discourse half so queer.” [21]

April thought this over and smiled, “Good night Awasin.”

Awasin smiled back, “Good morning April.”

Notes

[1] This story is a playful retelling of ideas related to infinity as they progressed over time by different players. The ideas are presented in a sort of historical fiction and reflect the thinking of research participants who addressed the ping pong ball conundrum, and where indicated, the actual individuals who contributed to modern formal understandings of infinity. Certainly, I am not the first to play with these ideas, but I offer this as a way of engaging with questions, controversies, ideas, and beliefs related

to infinity. The characters in the story are confronted with a situation that challenges the notion of an 'objective' truth. Through their musings about the ping pong ball conundrum, April and her friends stumble upon the contextually-dependent nature of mathematical truth and open the door to further conversation.

[2] This popular problem has been known by different names, such as the Ross-Littlewood paradox, the infinite ball problem, the tennis ball problem, and the ping pong ball conundrum.

[3] Moore, A.W. (1995) A brief history of infinity. *Scientific American* 272(4), p. 114.

[4] Quine, W. (1966) *Ways of Paradox and Other Essays*. New York: Random House, p. 5.

[5] Blake, W. (1863) *Auguries of Innocence*.

[6] The physical impossibility of supertasks has been discussed and debated from philosophical points of view in (e.g.) Oppy, G. (2006) *Philosophical Perspectives on Infinity*. Cambridge: Cambridge University Press; Van Bendegem, J.P. (1994) Ross' paradox is an impossible super-task. *British Journal for the Philosophy of Science* 45, 743-748; Allis, V. & Koetsier, T. (1995) On some paradoxes of the infinite II. *The British Journal for the Philosophy of Science* 46, 235-247.

[7] Juniper, A. (2003) *Wabi Sabi: The Japanese Art of Impermanence*.

North Clarendon, VT: Tuttle, p. ix.

[8] Juniper, 2003, p. 111

[9] Bolzano, B. (1950) *Paradoxes of the Infinite* (translated by Prihonsky, F.). London: Routledge and Kegan Paul, p. 75

[10] Bolzano, 1950, p. 82

[11] Bolzano, 1950, p. 100

[12] Galilei, G. (1638) *Dialogues Concerning Two New Sciences* (translated by Crew, H. and de Salvio, A., reprinted 1914). New York: Dover, p. 32.

[13] Bolzano, 1950, p. 98

[14] Galilei, 1638, pp. 32-3

[15] Galilei, 1638, p. 32

[16] Bolzano, 1950, p. 98

[17] Galilei, 1638, p. 32

[18] Cantor, G. (1915) *Contributions to the Founding of the Theory of Transfinite Numbers* (translated by Jourdain, P., reprinted 1955). New York: Dover.

[19] Robinson, A. (1974) *Non-standard analysis*, revised ed. Amsterdam: North-Holland.

[20] Juniper, 2003, p. 157

[21] Newman, J.R. (1956) *The World of Mathematics*. New York: Simon and Schuster, p. 1614.

Much of our teaching efforts are directed toward making mathematics a finite game for our students. We emphasize the mastering of finite skills and our students naturally come to see mathematics as a finite game. Of course, the great mathematicians of history saw mathematics as the quintessential infinite game. That was precisely its attraction. This is the paradox of mathematics teaching. How can students be expected to recognize and value the finiteness, important as it is, that is embedded in the more essential infinitude of mathematics?

When mathematics is thought of as a *finite* game it is played for the purpose of finding an ending. Successfully finding an ending in doing mathematics is winning the finite game. In contrast, mathematics as an *infinite* game is played for the purpose of continuing the play. As an infinite game mathematics is potentially recursive. A finite piece of mathematics becomes the input for mathematics at a higher level thus insuring its continuance. It must be thus for otherwise it would have died long ago of over-familiarity.

Alton T. Olson (1990)
Mathematics as an infinite game
For the Learning of Mathematics 10(2), p. 29
