Three Hungry Men and Strategies for Problem Solving

JANE M. WATSON

Sometimes a problem starts out innocently, like the proverbial mustard seed; but then it starts to grow and, Zap, you are engulfed by a mustard tree! So it was with the Three Hungry Men problem:

> Three tired and hungry men went to sleep with a bag of apples. One man woke up, ate 1/3 of the apples, then went back to sleep Later a second man woke up and ate 1/3 of the remaining apples, then went back to sleep Finally, the third man woke up and ate 1/3 of the remaining apples. When he was finished there were 8 apples left How many apples were in the bag originally?

Although variations on the problem occur in many places, this one came from the *Mathematics Teacher* calendar, via Bennett and Nelson [1985, p 341]. Its main charm as a problem is that it offers so many avenues for possible solution. Perhaps most readers will see the solution immediately as an application of elementary algebra. If you have not seen the problem before, I would encourage you to pause now and write out what you consider to be an elegant solution, before reading further It may help to appreciate the efforts which follow.

The problem was originally set for preservice primary school teachers, many of whom had not been successful at high school algebra and hence did not have that tool available for finding a solution. In fact the problem was meant as an extension activity during the study of fractions As one might suspect,the errors ranged from misuse of the equal sign, to the horrors of adding the denominators of two fractions. At the other extreme were some students who claimed that they could not solve the problem *without* algebra. When asked how they might approach the problem with primary children, they had no reply or thought the problem was too hard. Having seen the solutions provided by the preservice teachers, I wondered what types of solutions would be given by primary children themselves. I also wondered whether older students who had access to algebra skills would use them to solve the problem. Hence I began to seek local teachers to ask their students from Grade 3 to third year university From there the seed started to sprout vigorously. The strategies employed for solution were so varied that this one problem could be used to illustrate nearly every strategy suggested in most books on problem solving.

The purpose of this article is to present a selection of solutions and then to comment on some of the implications for teaching, of using a problem such as the Three Hungry Men in the classroom. The solutions will be presented as illustrations of various strategies rather than as examples of increasing sophistication with age. The grade from which each solution came, however, will be noted Of course many attempts show a mixture of strategies. As well, it is possible to speculate on the causes of errors which stopped some strategies from working.

Strategies which worked backword

Without fractions. The solution in Figure 1 is the result of a team effort by four bright Grade 3 girls who had not had any formal work with fractions. The teacher asked if they knew what "a third" was and then *told* them the problem (hence a plate rather than a bag). The girls found it more difficult to explain their solution than to do it. All ages produced variations on this solution, many with better explanations

There were 8 left when the men had had their serve So the last man had 12 on the plate before he had his third. A third of 12 is 4 so the amount left on the plate after that man had his serve 12 plus 6 = 18 so after the man before him had had his serve there were 18 left So the first man had 9 which makes

27

Figure 1

With fractions Figure 2 shows a typical solution using fractions The poor use of the equal sign was not only evident in Grade 6 (this solution) but in every grade, including the preservice teachers. The lack of willingness to use words in explaining the solution to a mathematics problem was typical of many solutions

Draw a picture or diagram A preservice teacher drew and labelled the solution in Figure 3. She may have had in mind the need to give a concrete explanation to primary children. A Grade 8 student presented the diagram in Figure 4 as part of her solution.

For the Learning of Mathematics 8, 3, (November 1988)
FLM Publishing Association, Montreal, Quebec Canada

$8 = \frac{2}{3}$

$\frac{1}{3} = \frac{1}{2}$ of 8

$\frac{1}{2}$ of $8 = 4$

$8 + 4 = 12$

$12 = \frac{2}{3}$

$\frac{1}{3} = \frac{1}{2}$ of 12

$\frac{1}{2}$ of $12 = 6$

$12 + 6 = 18$

$18 = \frac{2}{3}$

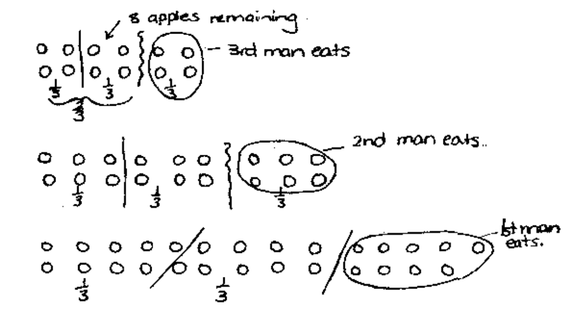
$\frac{1}{3} = \frac{1}{2}$ of 18

$\frac{1}{2}$ of $18 = 9$

$18 + 9 = 27$

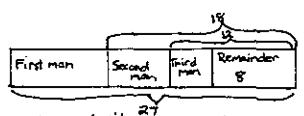
27 = no. of apples in bag.

Figure 2



∴ there were 27 apples in the bag originally.

Figure 3



Inside my mind, I like to see a picture of how the apples were eaten

Figure 4

Algebra, working systematically There were many variations on the Grade 11 solution shown in Figure 5. Some used three variables (x, y and z) rather than one. Only a very few students below Grade 10 were able to use algebra successfully.

$$\frac{2}{3}x = 8$$
$$\therefore 2x = 24$$
$$\therefore x = 12 \text{ APPLES}$$

$$\frac{2}{3}x = 12$$
$$\therefore 2x = 36$$
$$\therefore x = 18 \text{ APPLES}$$

$$\frac{2}{3}x = 18$$
$$\therefore 2x = 54$$
$$\therefore x = 27 \text{ APPLES}$$

∴ THERE WERE 27 APPLES IN THE BAG ORIGINALLY.

Figure 5

Reciprocal or ratio A third year university student used the ratio concept to write equations using the reciprocal of 2/3 in a three step procedure to find the solution given in Figure 6. No one produced a complete solution using a proportional representation similar to

$$2/3 = 8/x$$

followed by

$$2/3 = 12/x, \text{ etc.}$$

$8 \times \frac{3}{2} = 12$
after 3rd after 2nd

$12 \times \frac{3}{2} = 18$ = after 1st

$18 \times 3/2 = 27$ = before 1st

$\therefore$ 27 apples.

Figure 6

Errors in backward strategies

Equal thirds. Although the solution in Figure 7 was from Grade 6, the difficulty of assuming that the men each ate the same number of apples persisted commonly through Grade 9. The number of apples varied widely; many students appeared to feel happy with making up whatever seemed to be a suitable number, the largest being 15 for each man.

There were 14 apples
the 1st man ate 2
the 2nd man ate 2
the 3rd man ate 2
and there were 8
left.

Figure 7

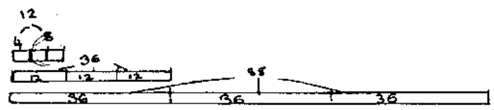
Ignoring remainder. The error shown in Figure 8, by a Grade 8 student, occurred at every level up to third year university. Often it was written in algebraic form. Nearly all of these students were unconcerned with the implications of the first man eating 72 apples!

$8 \div 1/3 = 24$
$24 \div 1/3 = 72$
$72 \div 1/3 = 216$

216 apples were in the bag originally.

Figure 8

Distinguishing 1/3 and 2/3. From Grade 8 upward difficulties were encountered in keeping track of whether 1/3 or 2/3 was the appropriate fraction to use to represent the information given in the problem. Figure 9 shows a potentially successful diagramatic approach which appreciated the correct relationship of thirds in the problem but after stage one forgot the remaining two thirds. There is also a multiplication error in the solution.



there were 66 apples in the bag originally.

Figure 9

Algebraic representation. In trying to use algebra, the Grade 8 student who wrote the solution in Figure 10, appeared not to grasp the nature of the problem with the "1/3" of remaining parts being eaten". The first attempt (crossed out) showed that the student realized a straight multiplication by 1/3 each time was not sufficient; but the dilemma was not resolved correctly. There were also difficulties in multiplying whole numbers by fractions and in addition. The student who wrote the solution in Figure 11 realized that 8 was some part of those apples left when the third man woke up—a fraction greater than one being needed to express this relationship in terms of the number left. Instead of choosing 3/2, the inverse of 2/3, 4/3 was chosen. As well, an additive procedure was employed to find the final answer when the "fraction greater than one" concept would have given the answer directly. An addition error also occurred at the end of the solution.

Let the number of apples be x

$\cancel{x = (\frac{1}{3} \times 8) + (\frac{1}{3} \times 8)}$
$x = (8 \times \frac{1}{3}) + (8 \times \frac{2}{3}) + (8 \times \frac{3}{3}) + 8$
$= 8\frac{1}{3} + 8\frac{2}{3} + 8 + 8$
$= 8\frac{1}{3} + 8\frac{2}{3} + 16$
$= 19 + 16$
$= 37$

Figure 10

Let the number be x

$x = 8 + (1\frac{1}{3} \times 8) + (1\frac{1}{3} \times 8 \times 1\frac{1}{3}) + (1\frac{1}{3} \times 8 \times 1\frac{1}{3} \times 1\frac{1}{3})$
$= 8 + 10\frac{2}{3} + 14\frac{2}{9} + 18\frac{26}{27}$
$= 8 + 10\frac{18}{27} + 14\frac{6}{27} + 18\frac{26}{27}$
$= 51\ \frac{23}{27}$

Figure 11

Strategies which worked forward

Guess and check This strategy was popular in Grades 6 through 9, particularly for those with access to calculators. The producer of the solution shown in Figure 12 was thorough in completing the checking procedure and in being able to know whether the next guess should be larger or smaller. Many guess and check approaches stopped at any attempt when the answer was not a whole number.

Working systematically. From Grade 11, students were more willing to attempt the problem working forward. The solution in Figure 13 shows the use of fractions and an understanding of the situation, although the use of the equal sign is poor. Other solutions used algebra in the same step by step fashion, similar to the working backward example in Figure 5. Working forward, however, involved twice as many steps, since subtraction was involved each time.

I used the guess and check method

25 apples $\frac{1}{3} \times 25 = 8\ 3$

-8.3

$16.7 \div 3 = 5\ 56$

-5.56

$11\ 14 \div 3 = 3.713$

-3.713

$7\ 427$ This answer is too small

so I tried

$27 \div 3 = 9$

-9

$18 \div 3 = 6$

6

$12 \div 3 = 4$

-4

8 this is the right answer.

There were 27 apples in the bag

Figure 12

$\frac{1}{3} \times 1 = \frac{1}{3}$ $\quad \frac{3}{3} - \frac{1}{3} = \frac{2}{3}.$

$\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$ $\quad \frac{2}{3} - \frac{2}{9} = \frac{4}{9}.$

$\frac{4}{9} \times \frac{1}{3} = \frac{4}{27}$ $\quad \frac{4}{9} - \frac{4}{27} = \frac{8}{27}.$

$8 = \frac{8}{27}$ apples.

$\therefore 1 = \frac{1}{27}$ apples.

$\therefore 27 =$ apples.

Figure 13

Setting up one algebraic equation. No one below university level successfully set up one algebraic equation to solve the problem. The solution in Figure 14 shows the correct but quite laborious efforts of a preservice primary teacher to achieve one equation and then successfully solve it. This solution set up the equation in terms of the number of apples left (8). The partial solution in Figure 15 shows a similar approach in terms of the total number of apples in the bag (x). Not until third year university, however, was it common for students to present a solution in terms of the remainder as shown in Figure 16. Some solutions were more fully explained than this one

Let total apples $= x$.

First man ate $\frac{1}{3}$ of the apples $= \frac{1}{3}x$.

$\therefore$ Apples left $= x - \frac{1}{3}x$

Second man ate $\frac{1}{3}$ of apples left $= \frac{1}{3} \times (x - \frac{1}{3}x)$.

apples left after first man

$\therefore$ Apples left after 2^{nd} man $= x - \frac{1}{3}x - \frac{1}{3}(x - \frac{1}{3}x)$

The third man ate $\frac{1}{3}$ of apples left after 2^{nd} man

$= \frac{1}{3}\left[x - \frac{1}{3}x - \frac{1}{3}(x - \frac{1}{3}x)\right]$

$\therefore x - \frac{1}{3}x - \frac{1}{3}(x - \frac{1}{3}x) - \frac{1}{3}\left[x - \frac{1}{3}x - \frac{1}{3}(x - \frac{1}{3}x)\right] = 8$

$x - \frac{1}{3}x - \frac{1}{3}x + \frac{1}{9}x - \frac{1}{3}x + \frac{1}{9x} + \frac{1}{9x} - \frac{1}{27}x = 8$

$x - \frac{3}{3}x + \frac{3}{9}x - \frac{1}{27}x = 8$

$\frac{3}{9}x - \frac{1}{27}x = 8$

$\frac{9-1}{27}x = 8$

$\frac{8}{27}x = 8$

$8x = 8 \times 27$

$8x = 216.$

$\therefore x = 27.$

Figure 14

Let x = the Total number of apples in the bag.

$x = \frac{x}{3} + \frac{2x}{3} \times \frac{1}{3} + (x - \frac{x}{3} - \frac{2x}{9}) \times \frac{1}{3} + 8$

1st man $\frac{1}{3}$ of apples

Figure 15

$x \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = 8$

i.e. $\frac{8x}{27} = 8$

$\therefore x = 27$

Figure 16

Errors in forward strategies

Concept of 1/3. Up to the Grade 6 level, the solution in Figure 17 was typical of a few students who understood the problem to mean that each man ate a third of *an* apple. A Grade 11 student who tried to use algebra, produced the same answer although it was impossible to tell whether the difficulty was a result of misunderstanding the language of the problem or the way to represent the information in algebraic form

$\frac{1}{3} \times 3 = 1$ whole

1 whole + 8 whole

= 9 apple

Figure 17

Distinguishing 1/3 and 2/3. Figure 18 shows a solution typical of older students who tried to use algebra and set the problem up as one equation working forward; the result is the same as that in Figure 8, working backward. Figure 19 shows a solution in which the student tried to be systematic but got confused after the first step. Figure 20 shows another single equation solution where only in the last component of the equation did the student fail to take away 1/3 of the remainder.

let original amount = x

$\therefore x/3 \times \frac{1}{3} \times \frac{1}{3} = 8$

$\therefore \frac{x}{27} = 8$

$\therefore x = 216$? (very big)

check $\frac{216}{3} = 72$, $72/3 = 24$, $24/3 = 8$

$\therefore$ 216 apples in bag originally.

Figure 18

bag apples with x apples init

after 1st man $\frac{2x}{3}$ apples left

after 2nd man $\frac{1}{3} \times \frac{2x}{3} = \frac{2x}{9}$ apples left

after 3rd man $\frac{1}{3} \times \frac{2x}{9} = \frac{2x}{27}$ apples left

$\frac{2x}{27} = 8$ (because $\frac{2x}{27}$ apples remaining; 8 given that 8 apples left)

$2x = 216$

$\therefore x = 108$

$\therefore$ There were 108 apples originally.

Figure 19

Let original no of apples be x

Now $x - \frac{1}{3}x - \frac{1}{3}(x - \frac{1}{3}x) - \frac{1}{3} \cdot \frac{1}{3}(x - \frac{1}{3}x) = 8$

$x - \frac{x}{3} - \frac{x}{3} + \frac{x}{9} - \frac{x}{9} + \frac{x}{27} = 8$

$27x - 9x - 9x + 3x - 3x + x = 8 \times 27$

$10x = 216$

$x = 21.6$ apples

Figure 20

Algebraic representation of 1/3. Many students, from Grade 8 to the preservice teacher education program, who tried to use algebra, had difficulty representing 1/3. The solution Figure 21 shows the attempt of one preservice teacher, which also displayed other manipulative difficulties with fractions and a general lack of flair!

$\frac{1}{3}$ of x

$\frac{1}{3}$ of $[x - \frac{1}{3}]$

$\frac{1}{3}$ of $[x - \frac{1}{3} - \frac{1}{3}] = 8$

$= \frac{1}{3} \times x = 8 + \frac{1}{3} + \frac{1}{3}$

$= \frac{1}{3}x = 8 + \frac{1}{3} + \frac{1}{3}$

$= x = \dfrac{8 + \frac{1}{3} + \frac{1}{3}}{\frac{1}{3}}$

$= \dfrac{8 + \frac{2}{3}}{\frac{1}{3}}$

$= 8\frac{2}{3} \times 3$

$= \frac{26}{\cancel{3}} \times \cancel{3}$

$\therefore x = 26$

$\therefore$ There were 26 apples altogether

Figure 21

Diagram but inability to complete. The diagram in Figure 22 shows a forward attempt to set up the problem but then no further progress Many students below Grade 10 drew similar pictures, or caricatures of the Three Hungry Men themselves, but could not complete the problem

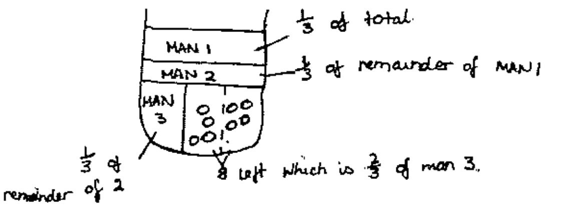


Figure 22

Strategies which worked forward and backward

Explaining carefully A few students, particularly at the Grade 8 and 9 level, carefully explained how they interpreted the problem from the beginning and solved it from the end, sometimes with pictures The solution in Figure 23 is also interesting because it used an additive approach at the end of the solution, probably consistent with the student's interest in the story line.

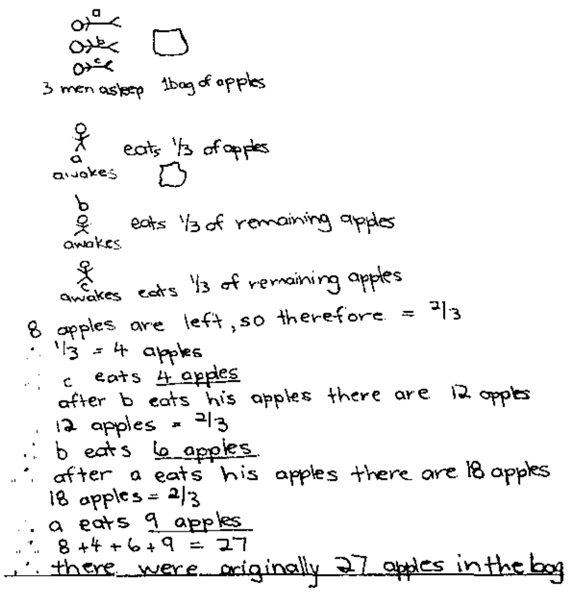


Figure 23

Accounting for all three men Several Grade 5 and 6 students could get started but appeared to forget how many men were involved One such solution is shown in Figure 24.

1st Man ate 1/3
2nd Man ate 1/3 of remaining apples
3rd Man ate 1/3 of remaining apples leaving 8
8 = 2/3
12 = 2/3
18 = apples started with

Figure 24

Implications

In fact there were many other incorrect strategies and variations on correct ones, but this selection ought to be sufficient to demonstrate that the Three Hungry Men is an excellent problem The implications of problems like this for the teaching of mathematics will be discussed from several different angles.

The teaching of problem solving itself. First of all, just considering the number of different correct strategies, the Three Hungry Men problem provides an excellent teaching device for those who wish to explore problem-solving strategies with their students. Particularly if used at the high school level, nearly all students should be able to comprehend and appreciate the problem itself. Even some of the strategies which were not explicitly stated here with the written solutions can be stressed by teachers in the discussion of solutions For example, "reading the problem carefully" can be mentioned in situations where students interpret the problem as saying "1/3 of *an* apple" or forget the remainder. Similarly "making a model" with actual objects could be appropriate with primary children, or "making a table" can be a way of recording the information when either a forward or a backward strategy is used. Checking answers will be mentioned after but of course is an important aspect of solving problems.

No suggestion was made throughout the presentation of solutions that some were "better" than others. As teachers we are probably thrilled to see such variety and ingenuity in the solutions shown here. This variety is a starting point for discussion and comparison of strategies in the classroom An exploration of the length and "elegance" of solutions may suggest short cuts to interested problem solvers, particularly those who have been exposed to algebra but see little use for it.

Difficulties of expression. The misuse of the equal sign was widespread even in solutions which obtained the correct solution. This raises the question of our initial teaching of this most widely used of mathematical symbols. One issue involved rivals that of the use of correct grammar in English expression. Students who write run-on sentences are usually asked to delete "and's", and insert punctuation marks and capital letters to clarify meaning. But do we ask students who have used the equal sign as in Figure 22 to rewrite their solutions to avoid ambiguity, particularly if the answer happens to be correct? The transitivity of equality in mathematics requires that students must check each time they write on either side of it. Too many students treat the equal sign as a connective (like "and") between the various steps of their solutions.

Another difficulty with the equal sign is shown in Figure 2, where a student claimed, $8 = (2/3)$ Obviously the student understood what was written in the context of the problem but it certainly was not correctly written mathematics. Here the insertion of a few words, such as "2/3 of the apples left", would clarify what was meant It is possible to imagine a rewarding classroom discussion built around the solution in Figure 2 and how to improve it.

The preceding example leads into several comments about the importance of language in mathematics. Many students need to be encouraged to use language more effectively in writing their solutions. This would probably help in their analyses of problems themselves and it certainly would help the teacher in finding where the strategies have gone astray. It appears that many students think the introduction of algebra alleviates the need to use words. Perhaps this follows from a mystique we as teachers perpetuate that algebra is the mathematicians' special language that solves problems. One thing these solutions demonstrate, then, is that certainly for quite a few years, we are leading students astray in this belief. Very few purely algebraic solutions were successful below the university level.

Incorrect solutions. What do we learn by looking at the incorrect solutions? Surely the variety must again be surprising to most readers. To think that as teachers we need to be aware of so many ways that students can go wrong is nearly mind-boggling. But of course, this quality of the Three Hungry Men problem to elicit so many possibilities for problem solving is exactly what makes it a good problem. Imagine the great classroom discussions which could take place as students explained and defended their strategies. Imagine how much some students could learn as they explored the differences between their solutions and a careful rereading of the problem.

It is interesting to note that very few unrealistic answers were given at the primary school level. Even those who just guessed would generally say something like, "I think there were 40 apples". It was the older students, who felt the need to use more sophisticated methods involving algebra, who tended to present unrealistic answers without questioning. The answers shown in Figures 8, 11, 19, and 20 are examples of this difficulty. The student who wrote the solution in Figure 18 was the only one to question 216 and in the end he did not go back to the original problem to check its reasonableness. Asking students to check their answers to see if they are reasonable against the original problem is perhaps one of the most important aspects of teaching problem solving strategies.

No attempt has been made here to say that some mistakes are more serious than others. Readers probably will have formed their own opinions on this. It seems certain, however, that some strategies were closer to achieving success than others. It was also interesting to note during the collation of incorrect solutions that the same incorrect answer did not necessarily imply that the same incorrect strategy was applied. The answers 108 and 216 were obtained in several different ways. We must be careful as teachers not to categorize students too quickly on the basis of their answers only. A big responsibility appears to rest on our shoulders in the remediation of incorrect problem solving strategies—a big red "X" is not sufficient.

The teaching of algebra. Is there a relationship between the Three Hungry Men problem and the teaching of algebra? Obviously algebra is sufficient (if used correctly!) but not necessary to solve the problem. Many students who had been taught algebra did not choose to use it. Perhaps this was because the algebra tool had not been used often enough yet to make it appear as a help rather than a hindrance. Generally it was not until the Grade 11 level, well past the introduction of algebra, that students appeared comfortable with the tool. This is perhaps what we should expect—when faced with a new problem we use our most familiar techniques rather than new techniques with which we may have less confidence. Intuitively, I feel it must be a long time, if ever for some students, before algebra becomes and unconscious tool, like multiplying or dividing, in a problem solving situation. This does not really bother me so long as students have other techniques at their disposal for problem solving.

On the other side of the algebra coin is the situation where some of the preservice teachers could easily come up with algebraic solutions but claimed to be unable to think of any other way to solve the problem. This also occurred in a primary school where an adult proudly announced solving the probem but then sheepishly admitted that algebra had had to be used. It almost seems as if, for some people, we have been too successful in making algebra a tool for problem solving. To put teachers with only these algebra skills in a primary class would be very sad, for either students would not be exposed to problems such as the Three Hungry Men at all or they might be forced to accept algebraic solutions which they could not yet comprehend. Hence in training preservice teachers we need not only persuade those without algrebra skills that other problem solving skills should be developed, but also we need to stress these skills with students who can use algebra.

Multiple-choice format. What if the Three Hungry Men problem had been set as a multiple-choice question? For Grade 6 students the best alternatives for answers would have been: 9, 24, 27 and 32. For high school and older, the best alternatives would have been: 27, 72, 108 and 216. To think of using this problem in the multiple-choice format means limiting greatly the scope for answers and it makes me wonder how often we do this with other problems that we put in that format. The longer I teach the more I move away from multiple-choice questions except for eliciting very low level recall-type responses.

Conclusion

Finally, looking in detail at a problem like the Three Hungry Men may help us to get away from the idea that the answer is the most important thing in problem solving. Perhaps we should encourage students to try out more strategies. We as teachers often tend to see *our* way of solution as best and lead students toward that method, especially when they are stuck. One lesson this exercise has taught me is to consider setting fewer problems and rewarding students for finding different ways of obtaining the answer.

At the beginning of this article, I alluded to the similarity of the Three Hungry Men problem and the proverbial mustard seed. If nothing else, I think it has been demonstrated that the problem produced solutions which grew into long, sometimes entangled branches, producing a large crop of varying answers. Perhaps some of the seeds produced by the matured problem will grow for others and encourage them to nurture problems of their own to see what growth can take place.

Reference

Bennett, A.B., Jr & Nelson, L.T. *Mathematics: an informal approach,* 2nd edn. Boston: Allyn and Bacon, 1985