EXPLICATING INTERPRETATIONS OF EQUIVALENCE IN MEASUREMENT CONTEXTS

JOHN PAUL COOK, PAUL DAWKINS, ZACKERY REED

> The literature abounds in such phrases as 'A is equivalent to B,' which, unless properly defined, are often meaningless or misleading (Burington, 1948, p. 1).

We have come across an odd distinction in students' reasoning about proportional situations. Consider the responses students might propose to isomorphic proportionality tasks shown in Figure 1.

Task A is a version of the oft-studied Student/Teachers problem. Much of the research on this scenario has investigated issues related to students' attempts to devise the correct formula in Line 3. Instead, our discussion in this paper centers on three questions involving nuanced uses of equivalence that are representative of those asked by students we interact with [1]:

> In Line 1, given that these are isomorphic tasks, how can it be valid that '28 days = 1 lunar cycle' but not valid that '28 students = 1 teacher'?
>
> How can it be that '28 students = 1 teacher' is not valid, but reasoning with this equation leads to the correct answer in Line 2?
>
> Given that Line 1 might be abbreviated as $28D = L$ or $28S = T$, how are we to resolve the apparent contradiction with $D = 28L$ and $S = 28T$, the (correct) equations in Line 3?

In this article, we answer these questions through the lens of a *conceptual analysis* (Thompson, 2002) that explicates three interpretations of equivalence, revealing more nuance pertaining to the notions of equivalence at play than might be expected. We use the key ideas that emerge to call attention to the value for both educators and students in explicitly attending to and facilitating reflection on these interpretations.

Task A: In Glenn Junior High School, there are 28 students for every 1 teacher. Determine the number of students if there are 37 teachers. Then write an equation that generally relates the number of teachers, *T*, to the number of students, *S*, in Glenn Junior High.

Solution:
1) 28 students = 1 teacher
2) $S = 28 \cdot 37 = 1036$ students
3) $S = 28 \cdot T$

Task B: There are 28 days in a lunar cycle. Determine the number of days that pass in 37 lunar cycles. Then write an equation that generally relates a number of lunar cycles, *L*, and the number of days that pass during those lunar cycles, *D*.

Solution:
1) 28 days = 1 lunar cycle
2) $D = 28 \cdot 37 = 1036$ days
3) $D = 28 \cdot L$

Figure 1. Possible responses to two isomorphic proportionality tasks (solution lines numbered for convenience).

Three interpretations of equivalence

Some classic and insightful research on how students think about the equal sign distinguishes between students treating it as a connector between steps in a computational process (*operational view*) and as an indication that the objects it links are equivalent and are thus in some way the same (*relational view*) (*e.g.,* Knuth, Stephens, McNeil & Alibali, 2006). This fundamental distinction helps to explain why some students are able to correctly solve algebraic equations and develop a capacity for algebraic thinking while others experience considerable difficulties in their attempts to do so. To answer the questions we have posed, however, we argue that it is useful to further refine the notion of a relational view of the equal sign.

We begin by explicating a common interpretation of equivalence for two common algebraic objects: expressions that are *numerical* (no variables) and *algebraic* (may have a variable [2]). Numerical expressions are equivalent when they have the same numerical value, and, similarly, algebraic expressions are equivalent when "for any admissible number that replaces [the variable], each of the expressions gives the same value" (Saldanha & Kieran, 2005, p. 955). Notice that both of these characterizations center on the numerical [3] value(s) of the relevant expressions. This is a common interpretation of equivalence that is indispensable for productive algebraic reasoning—indeed, part of what makes algebraic manipulation so powerful is the realization that certain manipulations preserve the numerical value of the expression so that the expressions that are generated are interchangeable. Additionally, a numerical interpretation is key to what Iszák and Beckmann (2019) proposed as a coherent view of multiplication: the *number* of base units in one group times the *number* of groups equals the *number* of base units in the product (p. 91). We observe, however, that it has some limitations (and thus by itself is insufficient) in measurement contexts. Vergnaud (1994), for example, pointed out that, "many ways of reasoning concern relationships between magnitudes or quantities, rather than pure numbers" (p. 45). Consider a task posed by Vergnaud (1983) himself: given that one cake costs 15 cents, you must determine how much 4 cakes cost. He pointed out that, even though it might be clear that the answer is 60 cents (the numerical value of $4 \cdot 15$ coupled with the relevant monetary unit), it might not be obvious "why 4 cakes × 15 cents yields cents and not cakes" (p. 129). Notice that this scenario centers on the fact that the numerical equality is indeed preserved but does not directly account for the structure of the underlying measurement scenario (the units of the answer). Our introductory examples illuminate another limitation of numerical equivalence in such contexts, but in a different way: the expressions in Task B, Line 1 are not numerically equivalent—after all, 28 is not equal to 1—yet we would all agree that (1) 28 days is, in fact, equivalent to 1 lunar cycle, and (2) the reason involves the structure of the underlying measurement scenario. Our point here is not that a numerical interpretation of equivalence is undesirable—far from it—but simply that additional interpretations are needed to reason about measurement contexts. The literature appropriately supplies two others.

A *transformational* (*e.g.,* Solares & Kieran, 2013; Prediger & Zwetzschler, 2013) interpretation of equivalence involves viewing the relationship between objects primarily in terms of the sequence of actions by which one can be changed into another using some set of acceptable properties. This can involve, for example, viewing an equation in terms of the operations and properties by which the expression on one side is changed into the expression on the other (what we will call *expression to expression*), or viewing the entire equation itself as an object to be operated on and changed (*equation to equation*). Transformational reasoning plays a critical role in algebra because it helps students "develop a sense of the actions needed in order to reach a desired algebraic form" (Harel, 2008, p. 15), which plays a key role in, for example, the equation solving algorithm. A key component of transformational reasoning in measurement contexts centers on how one views the behavior of units under (expression to expression) transformations. Vergnaud (1983) detailed two such views that are particularly relevant for our purposes here. First, continuing the cakes scenario from above, one might attempt to resolve the apparent discrepancy in units by reasoning that, since $4 \cdot 1$ *cake* = 4 *cakes*, then the total cost is $4 \cdot 15$ *cents* = 60 *cents*. This scalar transformation actually preserves the units. Alternatively, one might reason that (4 cakes) · (15 cents per cake) = 60 cents. Here, the units of each factor are combined via multiplication to create a new kind of unit: the units of the cost of 1 cake is viewed as a quotient of the original units in order to facilitate the transformation of 'cakes' into 'cents'. This approach can be (and, in our experience, often is) regarded in terms of the cancellation of units—that is,

$$(4\ \cancel{cakes}) \cdot \left(15 \frac{cents}{\cancel{cake}}\right) = 4 \cdot 15\ cents = 60\ cents$$

Lastly, a *descriptive* (*e.g.,* Prediger & Zwetzschler, 2013) interpretation of equivalence treats two objects as equivalent if they describe, represent, or model the same quantity or situation. Though descriptive equivalence has many possible uses, we use it here to explore measurements of the same magnitude with different units (Thompson, Carlson, Byerley & Hatfield, 2014). The *magnitude* of a quantity A is the size of that quantity measured with respect to a unit **u**—symbolically, $|A| = m_u(A) \cdot |u|$, where $|A|$ is the magnitude, $m_u(A)$ is the measure of A in unit u, and $|u|$ is the magnitude of the unit. This characterization of magnitude is helpful for two reasons. First, it gives us language to distinguish and relate a *magnitude* (the amount of a quantity) and its *measure* relative to a given unit. This can be understood as a multiplicative comparison between the magnitude measured and the magnitude of the unit. Second, by decoupling magnitude and measure we can relate different measures of the same magnitude. Doing so "makes explicit the fact that the magnitude of a quantity is invariant with respect to a change of unit" (Thompson *et al.,* 2014, p. 5). That is, a magnitude is the same regardless of the unit used to measure it.

Using the three interpretations of equivalence to gain insight into Tasks A and B

We now turn our attention to using these three interpretations to answer the three questions we posed above. We note that our intention is not to outline all possible insights afforded by these three interpretations but rather to use them as a lens through which to (1) provide a rational frame of reference for the responses of the (epistemic) students we described in the introduction, and (2) provide plausible answers to the associated questions we posed.

Answering Questions 1 and 2

We again note that a *numerical* interpretation affords little insight into Line 1 because 28 is not equal to 1. Reasoning descriptively, however, '28 days' is immediately seen to be equivalent to '1 lunar cycle' because it expresses one magnitude (the duration of one lunar cycle, |lunar cycle|) using the other (the duration of one day, |day|) as a unit, yielding |lunar cycle| = 28 · |day|). This interpretation is less clear, however, for the student-teacher scenario in Task B, as 'student' and 'teacher' are not readily seen as compatible units with which to measure either each other or some other magnitude. This explains why '28 days = 1 lunar cycle' is conventionally regarded as valid but '28 students = 1 teacher' is not (Question 1 above).

We propose that it is enlightening—both pedagogically and mathematically—to instead express '28 students = 1 teacher' using a colon, indicating a proportional relationship. Pedagogically, being clear and intentional about expressing certain proportional relationships with a colon and some with an equal sign calls attention to the fact that *there is something to be distinguished.* Mathematically, the ratio maintains the structure that is needed to answer the question at hand. Indeed, one reason these kinds of equations appear and persist is because some actions on equations are also valid transformations on a proportional pair. Similar to how transforming an equation by multiplying both of its sides by the same scalar will produce another true equation, multiplying both entries in a proportional pair (whether expressed as an equation or ratio) by the same scalar will produce another proportional pair maintaining the same ratio. This is an example of Vergnaud's (1983) scalar transformation that preserves (rather than changes) units. For example, since 28 students : 1 teacher, then 37 · 28 students : 37 · 1 teacher; one also achieves the same correct answer when starting from the premise that 28 students = 1 teacher. This is the key to our response to Question 2: the invalid equation '28 students = 1 teacher' can be used productively because it preserves the same proportional relationship when multiplicatively transformed. This transformation is appropriate on equations like '28 days = 1 lunar cycle' and ratios like '28 students : 1 teacher'. Table 1 summarizes the insights about Line 1 afforded by the three interpretations of equivalence.

Answering Question 3

We focus our response to Question 3 on Task B. We first note that the equation '$D = 28L$' in Line 3 can be easily interpreted from the numerical perspective: if you consider any corresponding number of days (D) and number of lunar cycles (L) and substitute them into this equation, both sides of the equation will be numerically equal. This interpretation could also underlie the reasoning in Line 2: as the number of lunar cycles times 28 is the same as the number of days, then the number of days in 37 lunar cycles is $37 \cdot 28$ days. However, as Line 1 did not immediately lend itself to a meaningful numerical interpretation, we find it useful to consult the other two interpretations to identify coherence between Lines 1 and 3.

Recall our descriptive interpretation of Line 1: the duration of one lunar cycle is 28 times as long as the duration of one day, or $|\text{lunar cycle}| = 28 \cdot |\text{day}|$. From this perspective it is easy to see that in Line 1 the letters L and D are abbreviations representing the magnitudes $|\text{lunar cycle}|$ and $|\text{day}|$. More generally, in Line 3 L and D are variables representing the number of lunar cycles and the number of days. Thus, measuring a more general duration of time (denoted by $|t|$) using the duration of one day as a unit involves iterating by the number of days—that is, $|t| = D \cdot |\text{day}|$. Measuring that same duration with the duration of one lunar cycle as a unit will yield a measurement of L—that is, $|t| = L \cdot |\text{lunar cycle}|$. Thus, D and L are both the results of a multiplicative comparison (Thompson *et al.,* 2014) of $|t|$ to $|\text{day}|$ and $|\text{lunar cycle}|$. This shows how Line 3 can also be understood descriptively: $D \cdot |\text{day}|$ and $L \cdot |\text{lunar cycle}|$ represent the same duration of time via two different measurement processes and are thus equal. It also underscores that the variables L and D are the *measurements* resulting from these multiplicative comparisons. Thus, one way in which we can coherently view Lines 1 and 3 is by interpreting the roles of the letters D and L as different yet interconnected components of the same measurement process: they represent *magnitudes* in Line 1 and *measurements* resulting from multiplicative comparisons using those magnitudes in Line 3.

Table 1. Insights into Line 1 afforded by the three interpretations of equivalence.

Numerical	Transformational	Descriptive
Not immediately useful in either task because 28 is not equal to 1.	Enables transforming both sides by the same scalar (*e.g.,* $37 \cdot 28$ students corresponds to $37 \cdot 1$ teachers), regardless of whether the student attends to the nuances of 28 students = 1 teacher vs. 28 students : 1 teacher.	Uses measurements and magnitudes as a coherent way to interpret and classify certain proportional relationships as equalities (*e.g.,* 28 days = 1 lunar cycle) and others as correspondences (*i.e.* 28 students = 1 teacher)

Reasoning transformationally affords additional insight into the relationship between Lines 1 and 3. A transformational interpretation of $D = 28L$ involves focusing on how multiplying by 28 changes the number of lunar cycles into the number of days (an *expression to expression* transformation). We view such reasoning as an instance of Vergnaud's (1983) unit transformation (as opposed to the preservation of units via scaling). Accordingly, we find that the units of the number 28 can be interpreted as the units needed to transform 'lunar cycles' into 'days,' perhaps via cancellation— for example,

$$28 \frac{days}{\cancel{lunar\ cycle}} \cdot 37\ \cancel{lunar\ cycles} = 28 \cdot 37\ days$$

There is, however, another useful view of the number 28 that uses transformations to build upon the descriptive interpretation set forth in the previous paragraph. Specifically, we call attention to the (*equation to equation*) transformation in which both sides of $D \cdot |day| = L \cdot |\text{lunar cycle}|$ are divided by the (nonzero) magnitude $|day|$. This yields

$$D = \frac{|lunar\ cycle|}{|day|} \cdot L$$

As we know that $|lunar\ cycle| = 28 \cdot |day|$ from our descriptive interpretation of Line 1, this yields $D = 28L$, the equation in Line 3. These various interpretations of $D = 28L$ are summarized in Table 2.

The fruitfulness of explicating and coordinating interpretations of equivalence

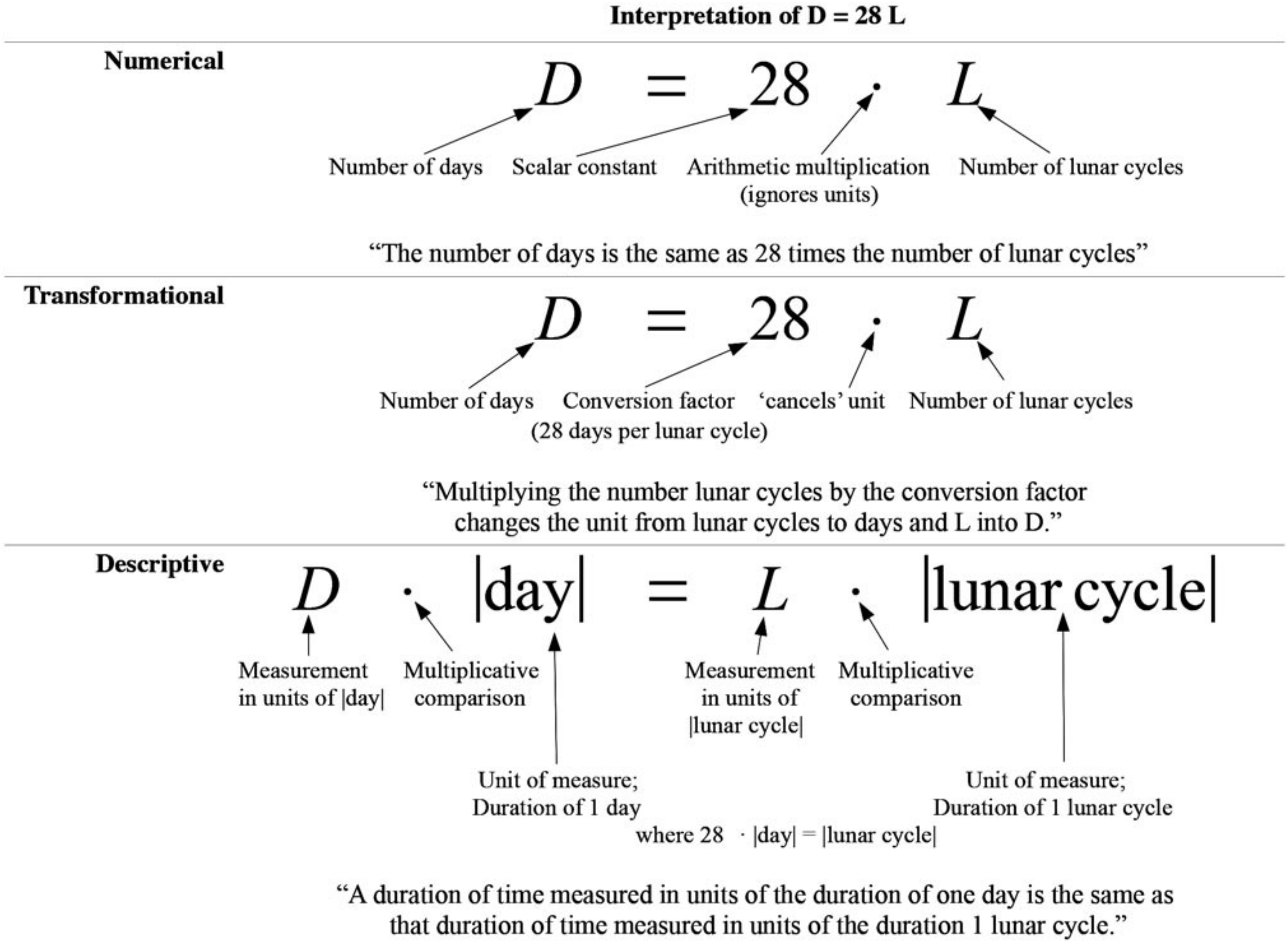
The framework we have outlined is an example of a *conceptual analysis* (Thompson, 2002) of the concept of equivalence because it explicates "what it is students might understand when they know a particular idea" (p. 196). Here we discuss the potential uses for this conceptual analysis and outline the various contributions it makes to the literature.

Conceptually grounded conversations with students

Thompson (2002) pointed out that a conceptual analysis is necessarily grounded in students' experience and therefore "entails imagining students thinking about *something* in the context of *discussing* it" (p. 196). Accordingly, here we identify several points we raised in the previous section that could serve as the starting point for such conceptually grounded discussions with students.

One of our key points in our response to Question 1 was the distinction that '28 days = 1 lunar cycle' represented two measurements of the same magnitude using different units while '28 students = 1 teacher' could not readily be interpreted in this way. The descriptive interpretation thus renders the former valid, but not the latter. We propose that rendering binary judgements about the validity of such equations potentially misses an opportunity for rich discussion. Changing the question from "is it valid or invalid?" to "what is the essential relationship the students and teacher 'equation' expresses?" could foster a productive discussion. We hypothesize that such an activity could reinforce a greater awareness of why it is normative and conventional in the mathematical community to distinguish between proportional relationships

Table 2. Using the three interpretations to parse D = 28L.



that are best represented using equations (*e.g.,* 28 days = 1 lunar cycle) and those that are best represented using ratios (*e.g.,* 28 students : 1 teacher).

More generally, in our experience students often recognize the apparent paradoxes that manifest in the various equations they write (though sometimes we have to juxtapose examples of student work and invite some reflection). Most of the time the equations are not invalid if understood in the manner intended: students know what they mean by and can reason productively with '28 students = 1 teacher,' even if we could improve the notation. This is a great opportunity for us as educators to look for the coherent meaning in what is written before applying conventions to correct someone's work. Learners use equations first to express a thought process, and we hope educators attend to that thought process. Therein lies the power of conceptual analysis: it provides a theoretical tool by which we can avoid surface-level judgments about 'misconceptions' and 'errors,' enabling us instead to identify and explicate productive lines of reasoning.

Another rich point of discussion involves encouraging students to reflect on and recognize the inherent nuance in uses of equivalence that otherwise might seem trivial and overly familiar. Productively reasoning about such fundamental tasks as those featured in this article requires subtle shifts between equations that express multiplicative comparisons of units (*e.g.,* 28 days = 1 lunar cycle) and equations that express the relationship between measurements in these units (*e.g.,* $D = 28L$)—indeed, distinguishing between D and |day| is at the core of our answer to Question 3 above. We find that many of our own students (at universities in the United States of modest selectivity) are not accustomed to thinking about these subtleties, even when implicitly present in their work. We propose that engaging students in guided reflections on these kinds of paired tasks through the lens of the interpretations of equivalence can serve as an excellent means by which to encourage students to attend to these ideas.

Contributions to the literature

In addition to facilitating productive conversations with students, we believe the conceptual analysis set forth in this article contributes to the literature in three key ways. First, the three interpretations contribute to the literature on equivalence by providing an answer to the question: what exactly does a relational understanding of the equal sign entail?

While the three interpretations of equivalence we leverage here are all set forth in some form in prior literature, our analysis extends this work by explicitly operationalizing these interpretations through the lens of research on measurement and multiplicative reasoning (*e.g.,* Iszak & Beckmann, 2019; Thompson, *et al.,* 2014; Vergnaud, 1983). More generally, we see this framework as a powerful theoretical tool that could inform (and be refined by) subsequent studies of students' cognition and instructional design.

Second, our analysis contributes to the literature on multiplicative reasoning. Vergnaud (1988), for example, defined the *multiplicative conceptual field* to be the set of "all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide" (p. 141). We observe that such measurement equations as '28 days = 1 lunar cycle' are a facet of the multiplicative conceptual field that has not received clear attention in the literature. Our analysis reveals that understanding and flexibly reasoning with such equations—and how they relate to their algebraic counterparts like '$D = 28L$'—to be surprisingly nuanced and complex. This paper thus focuses on sensitizing both mathematics educators and the students we teach to this recurrent challenge while also providing a means of addressing it.

Lastly, our analysis underscores a need for more explicit attention to descriptive equivalence, an interpretation which afforded critical insights in our analysis but has generally received the least explicit attention in the literature. Consider the following topics across the K-16 spectrum that could be explored from an explicit descriptive focus:

> At the elementary level, for example, descriptive equivalence allows us to warrant equations like the distributive property in ways that complement the conventional transformational approach (see Figure 2).
>
> At the middle and secondary levels, descriptive equivalence can productively justify exponential laws. For instance, if a population is doubling every minute, then it is quadrupling every two minutes. By expressing the population after 2 minutes in these two ways, we can conclude $P \cdot 2^t = P \cdot 4^{t/2}$. Recognizing how the growth of the same population can be alternatively modeled in terms of doubling, tripling, quadrupling, *etc.* while keeping the correspondence between time and population invariant is quantitatively meaningful and productive.

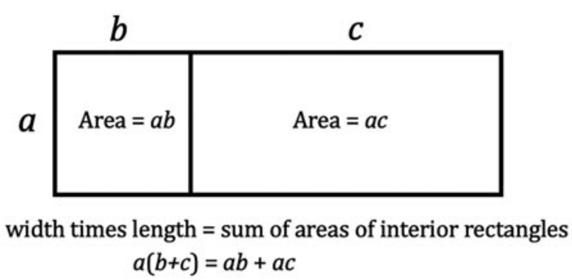


Figure 2. Warranting the distributive property using a descriptive interpretation of equivalence.

> At the undergraduate level, Lockwood, Caughman, and Weber (2020) have explored how combinatorial proofs differ fundamentally from other types of proof by virtue of the fact that they often warrant equivalence based on different counting processes for the same set of objects. This suggests that descriptive interpretations could be important beyond measurement contexts.

Our point is not that descriptive equivalence is *underutilized* but that it is *underemphasized*. The above examples spanning key topics across the school and university mathematical spectrum provide some indication of the scope and importance of descriptive equivalence and, accordingly, the need for researchers and educators to attend to it more explicitly. We hope that future work will continue to explore how students coordinate these various notions of equivalence in practice and how they can be harnessed and juxtaposed for rich sense-making. Indeed, we would love if more of our preservice teachers' work with equations was less guided by 'what is allowed' and more guided by 'what makes sense'.

Notes

[1] The students we refer to in this article are *epistemic*—that is, they are theoretical images of real students we adopt in order to explain their mathematical activity and render it sensible in some way (Thompson, 2002). Though our primary experiences in this regard are with undergraduate students, these tasks are commonly presented to middle school and high school algebra students as well; we anticipate those students exhibit similar lines of reasoning.

[2] For simplicity, we consider expressions with at most one variable.

[3] There is ambiguity with respect to some of these terms. First, we shall use 'numerical interpretation' to refer to the interpretation of equivalence and 'numerical expression' to refer to the algebraic object. Second, the interpretations of equivalence we describe appear under different names in the literature. Numerical equivalence has been called *insertion* (Prediger & Zwetzschler, 2013) or *substitution* (*e.g.,* Bishop, Lamb, Philipp, Whitacre & Schapelle, 2016) equivalence, *transformational equivalence* has been called *syntactic equivalence* (*e.g.,* Solares & Kieran, 2013), and *descriptive equivalence* has been called description equivalence (*e.g.,* Prediger & Zwetzschler, 2013).

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A clay envelope and tokens used for counting (Uruk period, 3200 BC to 2700 BC). Tokens were used to record the amount of a commodity, with different shapes of tokens standing for different numbers. The envelope ensured that the number and type of tokens could not be altered, but they could only be checked by breaking the envelope. To avoid this, images of the tokens were later indented into the outside of the envelope, and these marks eventually replaced the tokens *(see p. 24)**. Adapted from a photo by Marie-Lan Nguyen, CC-BY 2.5.*