# GEOMETRICAL VISUALISATION – EPISTEMIC AND EMOTIONAL

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"… the theorem is the object of a vision …" (Thom)

When animals look around in their environment, their vision enables them to learn about that environment, what features may be useful and how reliably those features exist. Predicting how those features relate to each other and to the animal itself is a survival mechanism and to prime such a mechanism, emotions are engaged. As human animals, we too stravaig a feature-filled environment. Our senses draw (us to) objects, things and the relationships between them and our sense of sight plays a dominant role. [1] As Dick Tahta stated, "we cannot not do geometry". Yet how does this lived geometry of sight relate to mathematical knowledge of theorems? In this essay, my aim is to explain how, in the context of elementary Euclidean geometry [2], visualisation is related to the visualiser's affective state and *can* be epistemic: knowledge-granting.

For example, figure 1 shows a Euclidean geometry stimulus (parallel lines and right angles enclose a rectangle). You may be able to see at-a-glance that the two shaded rectangles have the same area (and as Proulx and Pimm, 2008, argued, area is indeed a geometric notion). If you have seen the truth at a glance, certain visual processing pathways will have been active and not blocked by negative affect such as panic or lethargy. If you did see-at-a-glance the shaded rectangles' areas as equal, you have experienced an example of a *theorem* being *the object of a vision* (paraphrasing Thom's quotation above). The meaning of 'visualisation' I am using in this article captures that experience. Visualising in this sense comes quickly (though often after 'incubation'), preceding verbalisation or further symbolic representation; and it can be lost too – the exact same diagram or stimulus can fail to elicit the same experience on another occasion (even mere moments later). Such visualisation is educationally interesting for many reasons (which the reader may contemplate). These reasons include responding to a student's questions, such as:

(1) 'Can I get to know some geometrical fact [3] from just seeing?'

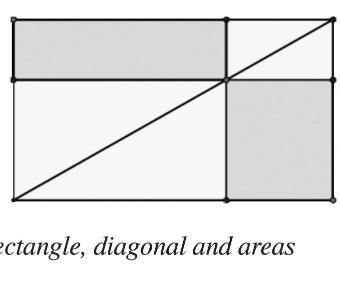

*Figure 1. Rectangle, diagonal and areas*

Philosophically this question can be phrased as: 'Is visualisation epistemic?'

(2) 'Why can't I see it (the theorem) now?'

Assuming no sight impairment or other physical barrier, this cry suggests a 'mental block'; in other words there is an affective aspect involved in having/realising this geometrical knowledge.

The crux of this article is that there are good reasons to assert that propositions of elementary Euclidean geometry can be 'seen as true', yet for a given person at a given time, affect affords or denies this seeing. The argument that visualisation is 'epistemic and emotional' uses the work of the philosopher, Marcus Giaquinto (*e.g.,* Giaquinto 2007) to make sense of 'epistemic' in the context of visual reasoning; the affect-dependency of visualisation is explained through the use of mathematics education examples, tasks (as above) and reports. And because 'visualisation' is a well-used word, other usages are discussed within the paper, as are cognate words such as 'insight'. In a sense the 'visualisation' investigated here has a special status as, potentially, it is a route to knowledge without standard justification methods (*i.e.,* written proof) and, indeed, part of Giaquinto's aim is to establish the existence of the *synthetic a priori* which Kant asserted. My iteration of Giaquinto's claim is to argue that the geometrical *synthetic a priori* that Giaquinto asserts involves affect too: thus visualisation as 'epistemic and emotional'. And because 'epistemic and emotional' appears paradoxical – knowledge is lasting but feelings are fleeting – I conclude by embedding my argument in a Gibsonian (*e.g.,* Gibson, 1977) eco-realist framework where the environment that affords experience is significant and that *environment* can be conceptualised in terms of relationships between actors and aspects within it. The role of the next section is to conceptualise this geometrical environment.

## The environment of perceptual space and Euclidean geometry

Ever since the entrance of non-Euclidean geometries, people have enquired about the geometric structure of humans' perceptual space, as well as that of the physical world. In 1977, the philosopher Patrick Suppes wrote an article which reviewed work on the question "Is perceptual space Euclidean?" (Suppes, 1977). He included reviews of empirical work, as well as philosophical studies, and noted that the answers 'yes', 'no' and 'maybe' were each represented in the literature as answers to that question. The debate as to the nature of perceptual space is still ongoing. Neuroscientific

*Figure 2. The 'Kanizsa square'*

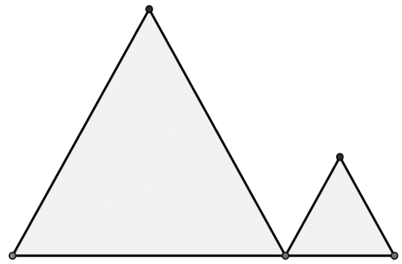

*Figure 3. Two equilateral triangles on a line segment with a point in common.*

and statistical techniques are used to interpret data from tests on people's perceptions. Results include, for example, a 2009 article by Fernandez and Farell who posited perceptual space to be locally general-Riemannian (and so, typically not flat, as Euclidean space is, but having various curvatures). Another recent paper (Todd *et al.,* 2001) posits an affine structure. As far as I have been able to find out, these recent empirical studies are concerned with perceptual three-dimensional space and the 'near-far' dimension, from a viewer's perspective, is the one that is particularly prone to distortion from the flat (and thus Euclidean). Indeed, Colin Ware, a software designer, refers to visual space as 2.05 dimensional with sideways and up-down as the two dominant dimensions, near-far, being minimal. Ware does not discuss curvature or metric, but notes:

> Pattern-processing resources in the brain are mostly devoted to information in the image plane, as opposed to depth. These two dimensional patterns are fundamentally important for two reasons. First, they are the pre-cursors of objects. Second a pattern also can be a relationship between objects. In some ways, pattern finding is the very essence of visual thinking, and often to perceive a pattern is to solve a problem. (Ware, 2008)

Perception of pattern and problem solving are central to geometrical visualisation. In the context of the school geometry being discussed here, two dimensions dominate and the question of seeing relationships comes down to what is on the viewer's image plane. Is this image plane Euclidean flat?

Evidence for flatness: primates' eyes are hardwired to see verticals and horizontals. [4] Such figures are paradigmatically Euclidean and give visual root to concepts of right angles and parallels. This is the environment for 2-D geometrical visualisation.

Furthermore, a flat plane that is perceived by sight as well as touch is developed in learning. The environments in which animals grow and develop contribute to neural pruning through prioritising certain salient types of shape (*e.g.,* those with right angles). This is part of, in Gibson's (1977) terms, an *affordance* between animal and environment. A typical 12-year-old mathematics student's triangle-affordances include her flat desk, pencil, paper and ruler, various uninterrogated relationships like, triangles have an inside and outside, primed relationships like they have angle sum of two right angles, and also a visual relationship as when a picture of a triangle is seen at a glance as 'triangle'. By the time a typical student has passed through a primary education using desks and rulers, doing drawings and looking at shapes, s/he will see parallels on the exercise book (as) not (ever) meeting and shapes rotated, reflected or translated staying the same. However, it is not possible to predict how or when a given person will put features of a given stimulus together whatever the environmental (including cultural) support. For example, in the figure, two equilateral triangles are constructed on the same side of a line segment with one point in common, as shown (the point cuts the segment at an arbitrary point).

Draw in the two line segments that join the end points of the segment to the top vertex of the opposite equilateral triangle (see fig. 3). Two new triangles are formed that have these new segments, part of the initial base segment and a side of an equilateral triangle respectively.

What pops out?

Maybe nothing. Maybe you see their congruence. Maybe next time.

The second time I worked with this diagram with the segments construction, I set the students preliminary exercises rotating triangles they had traced and they saw quite quickly a rotation-for-congruence in the task above. The first time I worked with this diagram I did not set up such environmental priming – and, in general, the theorem was not seen by the students.

## "Visualization" in philosophical writings

Unlike Gibson, whose theory does not separate an animal from its environment, in other philosophical traditions, the knower is idealized and operates on 'the' environment. From the latter viewpoint, an 'epistemic' process – a way of getting knowledge – is in some sense person-independent, given some assumptions about cognitive capacity. In the context of proofs and proving, this person-independency is an issue that I have addressed before (Rodd, 2000); here, I am just concerned with seeing-a-truth of a geometric theorem and the nature of this occurrence. In this section, I focus on Marcus Giaquinto's work on 'visualization' as he is a philosopher who is concerned with visual thinking in mathematics and has for many years been refining his claim that visual appraisal can produce knowledge. Giaquinto himself does not connect visualisation to the emotions. However, in his recent book *(op. cit.),* Giaquinto has engaged with recent neuroscientific results, as well as more traditional philosophical arguments to reinforce the claim that geometrical truths can be directly perceived. This reliance on neuroscience ties in with the relational nature of Gibson's eco-realism. My argument is that Giaquinto's claim, which integrates neuroscientific findings into analysis of the nature of visualisation, implicitly includes affective aspects of visualisation, because affect cannot be prised off from a neuroscientific understanding of perception and cognition. [5] Hence, accepting Giaquito's argument that neuroscience gives us knowledge about perception that enables us to claim that some basic geometrical knowledge could come from visualisation, has a consequence – which is not discussed by Giaquinto – that emotion is also intrinsic to the process of visualisation.

**Giaquinto on "visualizing": an outline and example**

Giaquinto (2007) claims that 'reliable justifications of belief' can come from direct visual appraisal. He explains the process of getting this 'pure geometrical knowledge' as follows:

> Our initial geometrical concepts of basic shapes depend on the way we perceive those shapes. In having geometrical concepts for shapes, we have certain belief forming dispositions. These dispositions can be triggered by experiences of seeing or visual imagining, and when that happens we acquire geometrical beliefs. The beliefs acquired in this way constitute knowledge, in fact synthetic a priori knowledge, provided that the belief-forming dispositions are reliable. (p. 12)

As Giaquinto uses 'disposition' as an undefined term, (the quotation above gives the first use of the word in his 2007 book). I referred to the Stanford on-line encyclopaedia of philosophy where 'dispositions' are explained through discussion of inanimate objects – disposition of a rubber band to stretch or glass to shatter on being dropped. So, analogously, according to this view, a person's disposition to form beliefs about geometrical shapes comes, like a rubber band's stretch, from the way humans' brains encode visual stimuli (like verticals and horizontals being wired in, which was cited by Giaquinto). It is in our nature.

So while I agree with Giaquinto's plan to accept the scientists' reports and use as fact that primates' visual systems have evolved to reliably pick out symmetries, horizontals and verticals – these features of visual systems that can be said to be dispositional in the philosophical sense – I feel that if neuroscience has been used to claim these features of perception, there is no reason to stop there as it can be used to uncover more about such dispositions. In particular, neuroscience gives details about the processing of perception that shows, through experimental tracking of processing pathways in the brain, that emotion and perception are closely linked. No surprise. See a ferocious beast coming for you and fear comes quickly. The point for this argument is that if some of the nature of belief-forming dispositions can be revealed by neuroscientific findings, as neuroscientific findings are not restricted to perception but include results on affect, (*e.g.,* on emotion-perception relationships), so, the belief-forming dispositions that enable geometric visualisation are (naturally) affect-laden. A consequence of their affect-laden nature is that they are not static, they can change with mood, emotion and feeling and develop accordingly (positively or negatively).

Giaquinto's argument for the epistemic status of geometric visualisation involves his synthesising five criteria for a belief to be justified through "visualization":

a. One feels that it is not the case that there might turn out to be a future counterexample, and this feeling is not weakened by recognising the fallibility of inductive generalisation;

b. The putative evidence of sense experience is meagre, but conviction is strong;

c. The belief … is not undermined by recognising that the putative evidence of perceptual experience is of a kind that could not warrant that belief;

d. the phenomenology of scrutinising one's experience and noticing some feature is absent;

e. one has a feeling of certainty … that is not undermined by recognising the great fallibility of inner observation. (*ibid.,* p. 63)

Understanding these criteria is central to Giaquinto's argument. As these criteria seem set up for specific instances of geometrical visual problem solving, applying them to the 'rectangles' stimulus should help to make them meaningful:

a. How could the areas of the shaded rectangles not be equal? It really does not matter where the point on the diagonal is (it can even be outside the original rectangle);

b. The shaded rectangles (can) look pretty different but their areas have to be the same;

c. I can't be sure the areas of the shaded rectangles are the same just by looking at them;

d. Once the 'penny dropped', I did not worry that I'd missed something;

e. The unshaded triangles are always going to be in two congruent pairs, one of each pair on opposite sides of the area-halving diagonal; I can't think differently about the situation.

***Another application***

Consider, now, another stimulus for a theorem, which will be used to delve into the affective aspects a little more: an arbitrary triangle with equilateral triangles constructed, as shown, on each side (see fig. 4).

What can be seen as true?

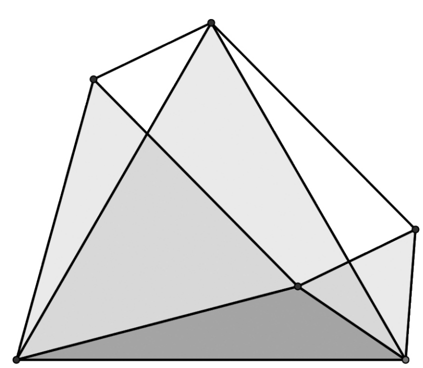

*Figure 4. Triangle with equilaterals forming a quadrilateral.*

I can see at a glance as true that the quadrilateral formed by joining the furthest-from-the-base vertices of the triangles is a parallelogram. I do not mean that it looks like a parallelogram merely, but that it is such a shape because of line segments that I see as necessarily equal. [6]

What did it take for me to 'see this at a glance' ? A prepared mind and a relaxed mood that arises from education, development of belief-forming dispositions into a particular visual affordance. Given this visual stimulus [7], I spent time playing with the figure, which was represented, respectively, by pencil and paper drawings, a little model and on DGS; I had also gazed at figure 3 ('equilateral triangles on a line segment with a point in common'). After having done this work, I can see the result, I am thinking visually. And a character of such a way of thinking is that it does not come sequentially, it is at-a-glance, it "pops out" (Ware, *op. cit.,* p. 27). [8] And, I can "lose it". When I lose the visualisation, then the figure does not hold the structural meaning, ABCD merely *looks like* a parallelogram. Panic.

From an educational perspective, it is important to recognise that competence is not monotonic. Visualisations can be lost as well as gained; emotions both contribute to and are stimulated by these losses and gains. Visual reasoning is delicate. [9]

Returning to Giaquinto's criteria, in the light of the experience of being able to visualise the truth that the quadrilateral is a parallelogram, I note that while I feel I can satisfy the conditions, their satisfaction did not happen simultaneously. In particular, the experiences of creating a DGS representation of the diagram satisfied (a) and of tracking congruent triangles (c), and in this case, "feelings of certainty" (e) come and go.

To summarise, underlying Giaquinto's claim that visualisation can be epistemic is that geometrical visualisation affordance arises from a dispositional stance that is neurologically primed. I broadly agree with this, but our nature is also driven by subliminal emotions, personal feelings and social propensities and these affects bias attention so perception is not passively received. Dispositions are not independent of person, time and cultural space.

## Dove on diagrams

Another related philosophical issue compares the status of a direct visual appraisal asserting a truth compared with a symbolic argument. According to Ian Dove (2002) visual reasoning with pictures should be no less reliable than symbol-based reasoning and visualisation can satisfy 'reliable justifications of belief' required for a knowledge claim. His key point has to do with the representativeness of the diagram used as stimulus. Dove uses Maxwell's 1959 'proof' that all triangles are isosceles [10] to discuss how figures represent mathematical concepts. Though not explicitly part of Dove's argument, his discussion highlights that a diagram that holds the geometrical meaning needs to be 'tweakable' and interrogable (*i.e.,* the putative visualiser is in relationship with the figure, rather than a passive viewer [11]) which accords with the Gibsonian eco-realist view.

## Other visions

Because 'visualisation' has different meanings, I discuss a few educational uses of the word and cognate terms in order to help clarify its use here. (This is not a comprehensive review of 'the literature'.)

### *Visualization, intuition and insight*

A well known use of 'visualization' in mathematics education is that of van Hiele who developed a theory of geometric learning, inspired by Piaget's levels of biological development, that conceptualised learning geometry as occurring in five levels, the first three of which can be summed up as the processes of recognising, describing, and reasoning about geometric objects/configurations/situations based on the Euclidean notion of the systemisation of space (1986, p. 53). [12] In the twentieth century, particularly in the mid to latter part when there was a boon for the set-theoretic foundationalist approach to mathematics, there was a privileging of the systematised over the experiential. Hence, as a child of that time, Van Heile's theory of learning about geometry is a formally structured system that does not attend to visual reasoning. Instead his theory conceptualises that a learner discards as reliable first the visual, then the descriptive as s/he goes up the levels of geometric proficiency, arriving at a symbolic-logical way to think about geometry.

Piaget, who was of an earlier generation, was less sucked into the mid-twentieth century formalist paradigm and, with Inhelder, retains a way of speaking about a child's relationship with her environment through the use of the word intuition:

> the radical separation of intuition from logic or axiomatics has never been achieved in practice; and, in fact is unattainable in principle. … in geometrical reasoning there always remains some link with intuitive experience. (Piaget and Inhelder, 1948/English edition, 1956, p. 448)

Van Hiele was a teacher as well as a theorist. As a teacher, van Hiele recognised the phenomenon he termed 'insight' which is related to, but different from, 'visualisation' as used here. Van Hiele (1986) observes, "the best examples of insight happen unexpectedly; they are not brought about by planning" (p. 154). In other words, something else is happening besides rational cognition when the seeing of a geometrical truth occurs. For pupils who experience this 'insight', van Hiele suggests that the process is repeatable and testable. An example of a task that is explicitly designed to "test insight" is "do the bisectors of the angles of a quadrangle pass through one point?" Van Hiele's example of a answer demonstrating insight is a pupil who replies "no, for quadrangles do not have an inscribed circle'. What I am try-

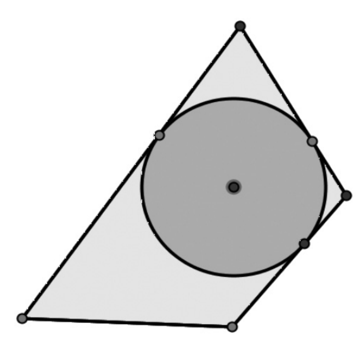

*Figure 5. A quadrilateral has no unique inscribed circle.*

ing to capture is visual reasoning, like that represented by figure 5, rather a post hoc verbal description.

***Registers***

A semiotic notion of visualisation has been developed by, among others, Raymond Duval (*e.g.,* Duval 1999). In this way of thinking there is a "gap between vision and visualisation" (p. 2). 'Vision' here is understood as raw perception while 'visualisation' is understood as a cognitive skill that operates on visual variables like graphs or triangles. Furthermore, such visual variables are themselves constituents of different visual registers. For example, learning about straight line graphs involves, or should involve, a facility to discriminate between $y = 2x$ and $y = x + 2$ as visual entities (therefore relative to axes) and to move between this visual register and the algebraic register that uses the representations '$y = 2x$' and '$y = x + 2$'. There is a visual register of congruence too and part of a good understanding of congruence includes facility with another visual register – that of isometric transformations.

In order to discuss the multi-register nature of some geometric problems, Duval considers the well-known puzzle: an 8-by-8 square is cut up into two 3-by-8 right angled triangles and two identical trapezia whose parallel sides are 3 and 5 and these pieces are reassembled to 'make' a 5-by-13 rectangle, thus 'showing' 64 = 65. Duval explains why the trick works by distinguishing between the relevant registers (which a student is not so likely to do). So because the 'operative' visualisation (cut up the bits) register is competing with the 'discursive' register of measures (areas) and is further muddied by 'perceptual' appraisal (the reconfigured shape appears as a rectangle) it is not surprising that the false result (64 = 65) seems to have been demonstrated!

To facilitate visualisation in the sense being developed in this paper, create a DGS file of a square with a variable point on a side corresponding to the ratio 3:5 (as a special case). Then through this point 'cut' the square into two rectangles and perform the rest of the dissections and the subsequent transformations of the puzzle. The parallelogram that winks at you from the centre of the oblong when you drag the variable point provides an experience for a knowledge- providing visualisation in the sense of Giaquinto's conditions (given an affect-infused 'disposition'). Duval dismisses as a (mere) "philosophical question": "is there any vision which could perform the epistemological function?" (*ibid.,* p. 15) He leaves this philosophical question alone, but that is precisely the meaning of 'visualisation' I am exploring here.

***Imagery and affect***

Norma Presmeg and Patricia Balderas-Canas (2001) investigated "Visualisation and affect in nonroutine problem solving" with some of their graduate students from their mathematics education seminar course. They adapted Gerald Goldin's (*e.g.,* Goldin, 2000) framework of affect for mathematics learning and tracked four individuals' affect and visualizations while solving high school type maths problems. Their meaning of the word 'visualization' is more like 'imagery' than suggesting a connotation of knowledge-producing. Indeed they suggest that 'visualization' might have an aim, in the sense that 'visualizing' is a strategy,

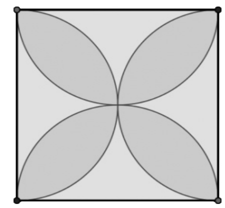

*Figure 6. Eileen's diagram*

elicited perhaps by Polya-like heuristics: 'draw a diagram'. They provide explicit examples of people engaged in attempting mathematical tasks where 'visualization' was primed as being of interest and where the people's affective states were tracked. Their case studies suggest the hypothesis that negative affect militates against holistic problem-solving, which is confirmed in some psychological studies (*e.g.,* Linnenbrink and Pintrich, 2002) and also in Eileen's report, presented below.

Eileen gives a first person account of how visualisation as affect-dependent and knowledge producing was experienced.

## Visualisation as emotional and epistemic – Eileen's story

The text below is an extract from an assignment from a course. [13] Eileen (pseudonym) was studying for her master's degree. She expresses how emotion impinges into experienced capacities for seeing/not seeing and being sure:

**Eileen's story of solving the geometrical problem: what is the darker shaded area?**

*When I first attempted a solution, within a few minutes I believed that I had succeeded in calculating the answer mentally and I did not bother to write anything down. Two nights later, however, when I needed to compile a presentation for a seminar that was to take place the next morning, I could neither remember my original mental solution nor embark upon a new written one. After nearly a half of an hour of rising panic, during which time I tried to trace my memories back to two nights before, I decided to start afresh and break up the diagram. That did not help. After several drafts and cups of tea which were spread out over an hour at the very least – I was becoming too angry to continue looking at the clock – I finally managed to produce a clear logical solution.... But in all honesty my belated achievement did very little to alleviate my dismay and frustration at still not being able to "see" my solution.*

*And then something happened which I had never experienced before. I made myself one last cup of tea, settled back in my chair and listened to some jazz on the radio. After ten to fifteen minutes or so I ruefully had one last look at the problem sheet: I immediately saw, [the solution] with no effort at all. I did this all mentally in a few seconds; this time I rushed to write down the solution!*

*I noticed that … whilst at first I needed to verbalize what I had been asked to calculate in the form of short simple word equations, my powers of visual perception seemed to merge with my other cognitive abilities so that I no longer* *separated the act of "seeing" a shape from characterizing it in linguistic terms. I continued to write word equations only out of pure conscientiousness and because I have always insisted that my pupils do so, but in fact I felt that I was in the process of dispensing with internal language altogether, or at least with the necessity to use language to interpret what I "saw".*

What's going on when, as in Eileen's story, a 'truth is seen'? Such geometrical visualisation is more than a cognitive process (although the visualiser must have had experience of the particular shapes and at least some of their properties before being able to put together her 'new truth'), it involves holistic appraisals that involve affect and environment.

Therefore, geometrical knowledge can come from visual reasoning yet is subject to affective influences.

## A paradox

The paradigmatic reliable means for proving a theorem of Euclidean geometry is by an argument that uses deductive reasoning of a computable simplicity on a finite set of axioms, primitives and previously proved theorems. The theorem, established in this fashion, is not contingent and, in particular, is independent of any particular person. However, a theorem established through visualisation is not person-independent, as it is affect dependent, therefore it is contingent. But this is a paradox if it is accepted that visualisation can be accepted as a reliable means to justify beliefs about two dimensional shapes and configurations. Giaquinto's use of the abstract human is designed to avoid this problem, but as discussed above, he implicitly does personalize his claim as individuals' dispositions and beliefs are influenced by their affective make up and are not constant through time.

### Paradox resolution

The problem with most paradoxes is that different discourses are being used. In the case here: person-free academic philosophy and person-centred talk of feeling. Some meta-discourse is needed to enable both these ways of thinking to be comprehensible to each other and to be able to tap into the power of each of them. A candidate for such a meta-discourse is provided by Gibson's (1977) environmental psychology in which animals and their environments are in relationships of co-dependency. As mentioned above, the animal (*e.g.,* a student of geometry) is tutored by its environment and propensities, initially 'hard-wired' by evolution, become honed over time through living in that environment. In the geometrical context, the environment is not merely physical artefacts that approximate parallels and other geometrical concepts, but established truths and ways of thinking passed down through human cultures; learners, teachers and practitioners of geometry with their human quest for meaning are part of this environment. The environment can be dangerous (*e.g.,* failure and rejection are classroom dangers) and also exciting (a student once referred to this as having 'a mathematical high'); animals' emotions and other affects, evolved from survival mechanisms, also co-exist in the environment. Without the environmental emotional stimulus from 'pictures in the sand', questions such as "is that always true?", "are those areas the same?", would not occur. Self-referentiality is integral to this ecological realist way of thinking as it is an inevitable aspect of non-dualism.

Returning to the paradox: visualisation can warrant geometric truths but as the process of visualisation is affect dependent the truths are contingent so not truths.

The paradox can be resolved as follows:

1. Shapes, representations of shapes (*e.g.,* in diagrams), hypotheses about these shapes or their representations and the hypothesisers all co-exist in a geometrical environment together with tools that people have fashioned to capture, manipulate, and communicate about aspects of this environment (like shapes and movement).
2. A visualiser of theorems is part of this environment: their eyes and their interests are environmental. This includes the biological propensities to pick out shapes and their properties reliably and the social and cultural encouragements and tools to make sense of these environmental features.
3. Visualizing has evolved as a tool for thinking – though is more delicate than writing as it is more subject to environmental pressures due to affect than knowledge 'set in stone'. Environmental tutoring can refine this human capacity, just as it can refine writing. While languages of visualisation can be developed, visualisation itself is experiential and can never be wholly captured by representations. Though it is possible to try to prompt a person to visualize with stimuli like models, diagrams and verbal advice as to where to direct attention, this process is never causal, unlike the if-then nature of formal deduction written with fixed symbols.
4. Visualisation does not require the machinery of formal deduction like the conceptual tools, technical language and presentation rituals; in a raw state, the process is available without language due to biological propensities for seeing features of the world reliably.
5. Therefore, geometrical visualisation is reliable, but in different ways to formal deduction: on one hand, visualisation requires less technical expertise – results are apparent more immediately than a proof that requires decoding, on the other hand visualisation is more subject to loss through affective pressures than a formal proof (even though 'seeing' a formal proof is also person dependent). Both visualisation and formal deduction are routes to being sure within the geometrical environment.

### Learning geometry, experiencing visualisation

Since 1988, there have been several changes in the mathematics National Curriculum in England, (which is where I am based), including a period where 'geometry' was dropped altogether in favour of a "shape and space" curriculum that emphasised measurement and naming of shapes rather than a theorem-orientated justification of properties.

Reasons for this curriculum-design decision were related to the perception that proving is 'too hard' for average students, but this view was argued against and now, since 2001, "geometry" has been reinstated and, furthermore, highlighted as an area of the curriculum through which deductive ("step-by-step") reasoning can be learned and assessed.

However, an issue, not unique to England, is that in an assessment-driven curriculum, there is little opportunity for learners to develop their visualisation skills that they can employ in justifying relational aspects of geometry. Geometry – in the late geometer S. S. Chern's phrase, "the jewel of mathematics" – is under threat. This was recognised by the Canadian historian of mathematics W.S. Aglin who observed dryly, "Euclidean geometry used to be taught in secondary schools, but it required imagination and insight, and so the average student (not to mention the average teacher), refused to do it, and it had to be replaced by a subject called 'memorisation of Algebraic Formulas for Rote Application on the Test'" (Aglin, 1994, p. 83).

## Notes

[1] The discussion here is focussed on sighted peoples' access to geometry. The UK-based *Maths, Stats and OR Network* (http://www.ltsn.gla.ac.uk) has published several articles recently on accessing mathematics for blind or sight-impaired learners.

[2] The focus is on geometry which is referred to as 'Euclidean' because it is clearer what 'epistemic' means in the Euclidean context, (where there are axioms, rules of inference and primitives), than in a more general 'geometric' context.

[3] Indeed, turning this question round, is the 'fact' geometric if it is known through algebra? Pythagoras's theorem is a good example of a fact that can be known geometrically (*e.g.,* shear the small squares and rotate them to fit together in the hypotenuse's square) or algebraically (*e.g.,* a special case of the *cos* fomula). Is the latter geometry?!

[4] From a developmental point of view, recent neuroscientific research (Csibra *et al.* 2000) has found that normal babies have the capacity to see a square (in the Kanizsa square pictured) at about eight months old (but not at six months).

[5] Damasio (2003), for example, gives explanations concerning the interaction of perception and cognition that necessarily includes affect. He explains how emotion can arise subconsciously given perception of an 'emotionally competent stimulus' (ECS). Objects of mathematics together with their environments for experience count as ECS as "few, if any objects are emotionally neutral" (*ibid.,* p. 56). Damasio also says that a function of education is to shape emotional responses and "bring them in line with the requirements of a given culture" (p. 54) thus, through guided evaluation of ECS, perception is in relationship with cognitive capacity mediated by/with affect.

[6] This is an example of the difference between van Hiele's 'visualization' (see below) and Giaquinto's notion.

[7] By Dietmar Küchemann.

[8] For teaching, I need to be able to sequentialise too. A problem that has happened several times is that when I start talking – in order to explain to a student– the geometric result can stop 'popping out' and I lose the vision. This phenomenon will be discussed in note [9] below.

[9] The idea that visualisation is 'delicate' has a root in my own experience and as a teacher of geometry; Eileen's story, below, contributes to this set of data that gives rise to the idea that visualisation is, or at least can be, delicate. The natural question to ask then is: why should visualisation of geometrical theorems be delicate? The key reason, as far as I understand it thus far, is that language and vision are processed in different systems or pathways in the brain and producing language activates a processing pathway that interferes with or blocks the taking in of visual wholes. Recently at a drawing class, we students were encouraged not to talk *about* what we could see but make marks *of* what we could see. James Austin, in his book about insight, cites a study that found that "out-loud verbalisation distracts from finding optimal solutions in problem-solving" (Austin, 2009 p. 150). Process-driven mathematics, like algebra or algorithmic work, feels more like a linguistic endeavour than geometrical visualisation. And while some cultures may prize silent reflection more than others, and therefore participants in such cultures may well find it easier to experience holistic witnessing than participants of cultures that privilege talk, it seems from what I can glean of the neuroscience literature, that the human brain has evolved to process information in such a way that language and visual appraisal are complementary.

[10] Dove's reconstruction includes an error on pp. 325–326 where he erroneously says that the intersection point of the vertex angle bisector and the base's perpendicular bisector must meet at O, which is inside the triangle for the false 'proof' to work – not so! I've demonstrated the false proof either way, the point is that one perpendicular from O to a non-base leg meets on the leg produced and the other one does not.

[11] DGS provides an environment that affords such tweaking, although DGS cannot be used for an 'all triangles are isosceles' performance: asking students to produce the construction used in a 'all triangles are isosceles' performance with DGS usually produces enlightenment, *i.e.,* why my construction on the board was fake.

[12] Van Hiele's levels four and five refer to formal logic and the nature of logical laws respectively and are not directly pertinent to this discussion.

[13] 'Geometry for teaching' module for an MA in mathematics education co-taught with Dietmar Küchemann at the Institute of Education, University of London.

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