ADVANCING A TEACHER-CENTERED PERSPECTIVE ON SUPPORT-FOR-CLAIMS TERMINOLOGY

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Consider the conversation below amongst a group of high school mathematics teachers [1]. Haley is the district mathematics coach; Ben, Mitch and Mary are all teachers.

Haley	What is the right language to use for the instructions? Do we mean prove? Justification? What language do we want to be using?
Mitch	First thing I jumped to is "prove"
Ben	I did too, but I want "justify" to work because proof is such a specific-
Mitch	It is but it isn't. I don't know that I was really thinking any fancy form.
Ben	I want them to know what a good justification is.
Mary	If it's a good justification, it is a proof.
Ben	So a good justification really tends towards a proof, or at least some [*pause*] even in paragraph type. It's not a formal proof.
Mitch	I wouldn't expect a two-column proof necessarily.
Mary	No, but I feel like when we say "justify" in a mathematical setting it really is what we call a proof. Justify in another setting is not necessarily-
Mitch	So what makes it what we call a proof?

Mathematical argumentation, justification, and proof are practices at the heart of mathematics *and* mathematics teaching and learning, but 'argumentation', 'justification', and 'proof' are often defined differently or may even be used interchangeably or not defined at all (Stylianides, Stylianides & Weber, 2017). This lack of consistency poses significant challenges for educators and education systems that are working toward the goal of supporting student proficiency with these practices. When the landscape is not clear, and there is no common, shared language, little collective progress can be made toward a specific desired goal (McDonald, Kazemi & Kavanagh, 2013). This situation can create challenges for teachers as they try to match the mathematical activity they hope to prompt from students with the language used to describe the activity.

Teachers voice their concerns through questions such as: "How do I provide enough direction so students give me the mathematical argument I want? What *do* I want?" "Is this derivation of the quadratic formula a proof?" To begin to address some of these practice-based questions, we propose here a view on defining support-for-claims terminology, specifically, 'argumentation', 'justification', and 'proof', that focuses on the daily work K–12 [2] teachers and students do so that such questions might be addressed within localized teacher communities. Our work is grounded in the assumption that deliberate, thoughtful, clear articulation of these terms, in ways that resonate with teachers and connect with their work of teaching children mathematics, is necessary to support professional conversations, inquiry, and student learning.

Note that we do not intend to assert a definitive stance regarding the terms or definitions that should be used. Rather, we claim that (local) coherence in terminology and corresponding descriptions can support teacher learning through collaborative discussions with colleagues, and ultimately advance teaching practice to support student learning. We hope to prompt other mathematics educators to be deliberate and make explicit their criteria, conversations and choices.

Taking the math teacher perspective

To date, the dialogue and healthy debate about definitions and purposes of mathematical argumentation, justification, and proof has played out primarily in mathematics and mathematics education researcher circles. While researchers have considered teachers' perspectives on proof, argumentation, and justification, these studies generally focus on teacher knowledge related to these support-for-claims activities. Serious attention to teachers' perspectives on how the terminology and corresponding definitions surrounding these activities support or thwart their work with students has yet to influence conversations. Of crucial importance is the issue of coherence in definitions and usages. Currently, it is generally assumed that teachers will follow mathematicians' definitions of these practices. At the same time, there is some acknowledgement that, in education, applying those definitions must be done in an age-appropriate way (CCSSM, 2010; Stylianides, 2007) that attends to term usage in everyday language (Shinno *et al.,* 2018) and to the pedagogical value of these practices (Hanna, 2000).

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In response, both authors have engaged practicing and prospective teachers in making sense of these 'support-for-claims' practices as part of university courses, grant-funded projects, and other professional development activities. In each setting, we have sought to advance teachers' proficiency in engaging students in argumentation, justification and proof, and related practices, such as critiquing and communicating. We have tried to wrestle these terms into comprehendible forms that are meaningful and useful for teachers as they try to meet the demands of their curricula and their daily work with students. This past work has shaped the questions we try to address in this article. Thus, our approach begins with questions central to teachers' professional work: What are the challenges teachers face when teaching for proficiency with argumentation, justification and proof? How can the demands of teachers' work be accounted for in shaping the terminology and definitions used? What choices related to terminology and definitions support teachers in meeting their professional obligations with students? We note that our work has been carried out in the context of the US public education system. Though the challenges and choices will vary by context, we contend there is something universally valuable about considering the math teachers' perspectives, and we offer our approach as a potential approach to others. We start here with challenges we see teachers facing in US schools. As appropriate, we point to other documents and studies that suggest such challenges exist in other educational systems as well.

Challenges

We have seen three common challenges teachers face when designing instruction to support students in argumentation, justification, and proof: (1) Determining what to expect or accept from students at different grade levels based on standards documents and students' developmental needs; (2) Communicating expectations to students so they are able to provide appropriate levels or types of support for claims; and (3) Articulating how the nature of the expected mathematical activity is aligned (or not) with similarly named activity in other disciplines.

National standards and curricula

The lack of agreement on the meanings and usage of argumentation, justification, and proof in the mathematics community is reflected in the imprecise use of terms within national curriculum or standards documents (Larnell & Smith, 2011). The majority of US teachers and students are currently held accountable to the Common Core State Standards (CCSSM, 2010). In CCSSM, *constructing viable arguments* is promoted as one of eight mathematical practices in which students in grades K–12 should be proficient. Specifically, students are expected to:

> understand and use stated assumptions, definitions, and previously established results in constructing *arguments*. They make conjectures and build a *logical progression of statements* to explore the truth of their conjectures. […] They *justify* their conclusions, communicate them to others, and respond to the *arguments* of others. (pp. 6–7, italics added)

In the grade-level content standards, students are asked to *"explain a proof"* in Grade 8, and in high school (Grades 9–12) students are expected to *prove* geometric theorems, *prove and apply* trigonometric identities and *prove* that linear functions grow by equal differences over equal intervals. In the elementary grades, students are asked to *justify formulas* (Grade 6) or *justify conclusions* about general properties of numbers or operations (Grade 3 and 4). How might teachers interpret these goals and learning expectations?

Similar mixed or imprecise language can be found in other national documents. For example, one of the aims of the national mathematics curriculum in England is to ensure that all pupils: "reason mathematically by following a line of enquiry, conjecturing relationships and generalisations, and developing an *argument, justification or proof* using mathematical language" [3]. A comparable statement from Australia's national curricula [4] calls for students to engage in mathematical reasoning, which includes "analysing, *proving*, evaluating, explaining, inferring, *justifying* and generalising" (italics added). Underspecified across these policy documents, however, is what the various practices might look like and how they relate to one another.

Communicating expectations

Teachers are consistently challenged to address gaps between the explanations, arguments, or justifications that they hope students will provide and those that students actually produce. As evidenced in the opening excerpt, some of the disconnect between what the teacher expects and what students produce might be attributed to the particular language used with students. A lack of agreement or consistency in the terms used during instruction or on assessments can lead to miscommunication or even frustration. In short, teachers and students need a common way to talk about argumentation, justification and proof so that expectations are clear. Clear expectations provide students and teachers with common targets and enable effective feedback from teachers to support students as they progress toward those targets.

We also note that it is unlikely that one term will suffice in capturing the range of support-for-claims activities that teachers wish to support in classrooms. Consequently, our goal is not to select a single best term, but rather to consider the nature of desired activity and outline an approach to identifying appropriate terminology that allows teachers and students to not only name, but also distinguish among these various activities.

Mathematics and other disciplines

Teachers are challenged to articulate commonalities and differences in how support-for-claims practices are used to establish knowledge in mathematics with how these practices are used in other disciplines. This challenge is illustrated by the continued conversation among the group of high school mathematics teachers shared earlier.

> *Mitch* I feel like the way we are using proof, it's a mathematically specific idea that fits into the rules and game of mathematics. You have these givens, and there are these properties and these theorems. My gut feeling when I hear the word proof is it's a math specific-

Mary That is a good point *[pause]* so justification *[pause]* and they do this in English class, right? They make them write papers where they have to justify and support, right? They have to provide support, using examples. The mathematical side of justification is proof because we have to provide justification and support through properties and rules.

In the US, national standards in both science and English language arts articulate expectations for students to engage in knowledge-establishing practices, such as argumentation, that are aligned with norms specific to the respective discipline (Lee, 2017). In England, the national curriculum for mathematics includes "justifying inferences with evidence." [3] While there are clear advantages to using similar language across content areas, there are differing expectations and epistemological stances toward the resulting disciplinary claims and certainty of evidentiary statements that can result in confusion (Lawrence & Crespo, 2016). To support students inside and outside of the mathematics class, teachers need to understand and be able to clearly communicate how practices such as argumentation and justification differ across disciplines. Relatedly, teachers need to be aware of how the use of similar terms in everyday discourse might be at odds with academic usage.

Criteria and rationale for term usage

Table 1 shows five criteria that we assert should be considered when selecting the term(s) to use with teachers and students that captures the idea of producing a logical chain of evidence by which a claim is supported. We have numbered them for ease of reference (not to suggest priorities or ranking among them). While adhering to these criteria alone cannot 'solve' the challenges described above, the set in Table 1 provides potential guidance when making decisions about support-for-claim terminology so that teachers can come to a shared understanding of what their respective standards documents are trying to convey, communicate effectively to students about these expectations, and articulate how these expectations relate to other disciplines.

This set of criteria is derived from reflections on our work as teacher educators, the research literature, and the goals of K–12 education. We acknowledge that there could be additional criteria other groups of educators may wish to consider. However, our interactions with teachers and other teacher educators indicate these criteria are useful and capture teachers' priorities. We do not know if these criteria are universally useful for guiding the selection and usages of support-for-claims terminology. We invite others in the field to use our criteria as a starting point, and to reflect and adjust as other criteria present themselves as useful for such work.

These criteria were selected based on our commitments to: foregrounding teachers' perspectives and the work teachers do with students in their professional contexts (Criteria 1 and 2), remaining true to mathematics and ensuring access to rigorous instruction for all students (Criterion 3 primarily, as well as 2 and 4) and supporting student learning (Criteria 4 and 5).

Centralizing argumentation

In our own work in K–12 settings, we have chosen to centralize the term argumentation and the idea of producing mathematical arguments. The purpose of mathematical *argumentation* is to examine the veracity of a (well formed) mathematical claim. A mathematical *argument* is a sequence of statements and reasons—supported by appropriate evidence—that collectively provides a rationale or basis for accepting or rejecting a mathematical claim. Mathematical *argumentation* then is the community practice by which a mathematical statement is formulated, represented, vetted and subjected to critical analysis, for the purpose of determining the veracity of the claim (Conner, Singletary, Smith, Wagner & Francisco, 2014).

This term, and our use of it, fits the criteria outlined above. First, teachers generally view argumentation as an

Table 1. Criteria to guide selection and usage of terms.

Criteria	Descriptions
1. Resonates with teachers	The term and its usage make sense intellectually and emotionally; the term is accepted by teachers as something they could and would work toward.
2. Is consistent with calls in public documents (*e.g.,* national standards or curriculum)	Teachers can directly tie the term and its usage to content and practices for which they are accountable.
3. Is consistent with the discipline	The term and its usage are compatible with definitions in the mathematics and math education communities. It is *intellectually honest* (Stylianides, 2007).
4. Can apply to mathematical work for *all* grade levels	The term usage is coherent from the learners' perspective as they progress through the grades. The term does not 'expire' (Karp, Bush & Dougherty, 2014), and works across contexts, including different topics, courses, and grade levels.
5. Facilitates learning across disciplines	The term and its usage support communication of commonalities and distinctions between how knowledge or 'truths' are established in mathematics versus other disciplines [5].

accepted goal (Criterion 1). The term itself is approachable and commonly used by teachers with whom we have worked. This easy acceptance is perhaps linked to the fact that argumentation is connected to specific language in CCSSM (our teachers' context), as mentioned with Challenge 1. Moreover, the term is consistent with valued twenty-first-century skills (*e.g.,* critical thinking and communication) (Bellanca, 2010), another valued goal in the US. Thus, it is consistent with calls in public documents (Criterion 2).

Turning to Criterion 3, our definition is consistent with the discipline's core practices. Any of the terms ('argumentation', 'justification', and 'proof') satisfy this criterion if defined appropriately. A core practice in mathematics is logical reasoning. In mathematics, we chain together statements and reasons to ultimately establish something as true that originally was not known to be true. Our definition of argumentation centralizes deductive reasoning (or inductive reasoning when warranted by the particular claim being addressed) and establishing new knowledge or local truths based on the mathematics one already has mastered.

The term 'argumentation' satisfies Criterion 4, as it is a term that can be used across grade levels and attends to students' developmental needs. CCSSM is explicit that this mathematical practice *(i.e., construct viable arguments and critique the reasoning of others)* applies to all grade levels. Proof, however, requires an appreciation of the fact that one is working within an axiomatic system, with assumptions, specific definitions, rules of inference, *etc.* Students, even up through college, may not be in a position to appreciate this level of deduction (Harel & Sowder, 2007). Students, however, can offer claims with reasons at any grade level (see, *e.g.,* Reid, 2002).

'Argumentation' is also a term used across disciplines. In the US for example, the term appears in Next Generation Science Standards (NGSS Lead States, 2013) and CCSS English-Language Arts, so has the potential to facilitate learning across disciplines—Criterion 5. The use of argumentation provides the opportunity for teachers and students to have explicit discussions around what is appropriate in different disciplines, including: what counts as evidence in science, history, language arts, or mathematics; what constitutes a valid argument in mathematics versus science; and how the status of claims differs across disciplines (*e.g.,* the infallible nature of a mathematical proof versus a scientific theorem). Moreover, the use of a common term helps both teachers and students see how all disciplines have a process by which ideas are debated, and new knowledge established. Ultimately the goal is for students to understand these interconnections, and also what is unique in each discipline.

To fully understand our choice to centralize the term argumentation, it is also necessary to demonstrate how the terms 'justification' and 'proof' would then be used in a complementary fashion to capture other types of important mathematical activity. We recognize that others may have differing opinions, or that other choices may make more sense in a different context. We share one approach we have found useful in our work with teachers to foster ongoing discussion and encourage other educators to reflect on their particular context and students' needs as they make determinations about terms and usages.

Relationship between argumentation and proof

In accord with Balacheff (2008), we view argumentation as compatible with proof but distinct from it, as mathematical proof or demonstration (to use Balacheff's terms) has an underlying aim of contributing to the formalization or systematization of mathematical knowledge. Thus the practice of proving pays careful attention to definitions, previously established results, and ensuring all logical connections are accounted for.

One reason for not centralizing proof in our work is that proof is a term that is distancing for many practitioners, or seen as appropriate only for certain students (Knuth, 2002) (*i.e.,* it violates Criterion 1). Teachers can hold very formalistic views of proof and their experiences with and understanding of its role, value, purpose, historical positioning *etc.,* may be limited. This brief exchange during teachers' professional development illustrates how some teachers see proof as relegated to formalistic activity done in geometry class.

Ben From an Algebra I point of view, *prove* is not something they've *[students]* run into much, formally.

Mary *[agreeing]* We don't talk about it- it's a geometry thing

Ben So prove means something different to them, maybe even from a geometry student that just went through.

Worse, students themselves may also be conditioned to think proof is not accessible—an idea captured in Mary's comment:

Mary Do you think that kids would be less freaked out about geometry proofs if we called them geometric justifications? Because it is a loaded word, if I say the word proof kids go "argh, proof," but if you ask them to justify something they'll do it.

Finally, as noted in the third challenge above, teachers generally agree that proof, as understood in mathematics, has no legitimate corollary in other domains. Unlike in mathematics wherein a mathematical theorem, once proven, is infallible, a scientific theory is never 'proven' and indeed may later be revised as contradictory evidence emerges. Thus, while the term 'proof' meets Criterion 3 (see Table 1)—it is clearly situated within the domain of mathematicians—and in many respects it satisfies Criterion 2, it falls short with regard to Criteria 1 and 4. Given that 'proof' is a term that is not relevant to the knowledge claims of other disciplines, though can be used in the everyday sense, we note that using the term at some point in math class to clarify the discipline's unique approach to establishing knowledge could support learning about mathematics, though not likely learning across the disciplines (violates Criterion 5).

Relationship between argumentation and justification

We use the term 'justification' in a manner that is complementary to 'argumentation'. The term is used extensively in everyday language to indicate one is offering a rationale for an assertion or belief (Rodd, 2000). It is a general practice of providing some support for one's claim, often a choice or preference, and is not limited to only claims with truth values. Justification, then, can be understood as the support one offers for a decision or a choice, or to motivate one's commitment to a particular stance or idea. While these decisions or choices are mathematical in nature, justification can apply in situations outside of those in which one is deciding the truth of a claim. For example, one can justify one's choice of method ('I used elimination instead of substitution because-') or justify a model ('My model is well suited to this problem situation because-'). This kind of activity is mathematically valuable. The nature of the activity, however, is different from chaining together statements in powerful ways to compel the truth of a mathematical claim [6].

The use of these two terms—'justification' and 'argumentation'—with these differing definitions allows us to value justification but also distinguish it from the other valuable mathematical activity focused on deciding the truth of statements. Such a distinction is important to make from a pedagogical standpoint as students come to understand the processes of establishing disciplinary knowledge.

The nature of the mathematical work in classroom settings

How might these different types of support-for-claims activities be related to our term usage of 'argumentation', 'justification' and 'proof'? Students at the K–12 level are asked to support claims of different kinds, which require different kinds of reasoning. To support the development of students' mathematical proficiency, we believe it is imperative that teacher educators, teachers, and ultimately students have language that distinguishes among these different support-for-claims activities.

Support for claims that are choices

In a classroom, teachers regularly engage students in offering some kind of support for their choice of method or strategy, example, or representation. These claims—*e.g.,* 'I chose to use elimination to solve the system'; 'I was able to compare by getting a common denominator'—are not claims with a truth value. A teacher may want students to justify why they chose elimination over substitution, but the choice is not something that can be determined true or false. Rather, one can compel the choice, reveal one's decision-making, and show how it was reasonable and productive, but one would not *prove* that elimination is the most appropriate method. Consequently, we want students to distinguish between justifying a *choice* and providing a mathematical argument that supports (or refutes) the mathematical truth of a claim. Along these same lines, students are often asked to justify their problem solving strategies or conjectures. For example, as students work on a problem, teachers might ask them why it made sense to them to try a particular method or what led them to think some pattern was likely to hold. The teacher is hoping to explore the genesis and logic behind the idea as a starting point or next step, and not the truth of the idea. Students (and teachers) should recognize this type of support-for-claims activity as different from activities where the truth of the statement is in question, and acknowledge that such activity requires different support or evidentiary basis.

Support for claims that are answers

Another type of support-for-claims work that happens in math classrooms is in relation to finding a solution to a specific problem. Teachers regularly ask students to provide some kind of support or logical chain of reasoning to demonstrate that their solution is correct. For example, a student might assert, 'I figured out each pen is 25 cents and each eraser is 10 cents'. This mathematical work—finding a solution to a specific problem—is different from the mathematical work that establishes a new mathematical 'fact' that can be built upon in the future, for example, a student assertion that, 'the sum of the lengths of two sides of any triangle must be longer than the third'. The appropriate application of mathematical ideas and processes that results in an 'answer' is a different kind of knowledge production than the appropriate chaining together of mathematical ideas to produce new knowledge about mathematical objects. We suggest that providing support for a claim that is an answer (relevant for a particular context) is better described by 'argumentation', whereas providing support for a claim that is a mathematically important fact (relevant for all math) in a manner that establishes the fact as part of a systematized knowledge base is better described by 'proving', as we now elaborate.

Support for claims that are mathematically consequential

A final type of support-for-claims activity is the work done to establish a new mathematical truth about mathematical objects. Often this work is done while paying careful attention to the axiomatic system one is working in, but it can also be done less formally. We reserve the use of the term 'proof' for the establishment of mathematically important claims that contribute to the knowledge base of the discipline, which pays attention to definitions, previously established results, and rules of inference. Statements that are proven can then be used as the basis for proofs of other claims. We offered one example above—the triangle inequality theorem. As a second example, one could prove that the quadratic formula can be used to determine the roots of any second degree polynomial. We acknowledge that what constitutes a proof is a decision made by the community in which the proof is to function (Stylianides, 2007), and thus not something universally agreed upon.

We emphasize here that proof is reserved for mathematically consequential claims, but just because the claim is mathematically consequential does not mean that the support-for-claims offered is a proof. A proof must also attend carefully to definitions and previously established ideas. Notably, argumentation can also be applied to instances in which the claim is mathematically consequential, but the

Table 2. Descriptions and examples of different support-for-claims activities.

Support-for-claims Activity	Nature of claim	Nature of the support for the claim	Example/sample task prompt
Justification	Claim can be a choice, decision, conjecture, *etc.* Claim need not have a truth value.	Range of types of evidence and reasoning. Reveals thinking and what 'evidence' was noticed that led to choice or assertion.	Solve each quadratic. Justify your choice of method (factoring, quadratic formula, completing the square).
Argumentation	Claim has a truth value. Claim need not be of mathematical significance (*e.g.,* the answer is 3/5).	Support should include warrants and evidence. May not explicitly attend to definitions or previously established results.	John solves $x^2 - 9$ and gets 3. Jane solves $x^2 - 9$ and gets 3 and -3.Who is correct? Provide an argument to support your assertion.
Proof	Claim has a truth value. Claim is of mathematical significance (*e.g.,* about mathematical objects/concepts).	Support should include warrants and evidence. Includes careful attention to definitions and priors.	Use completing the square on a quadratic in standard form to prove the quadratic formula.

chain of reasoning used to establish the claim may be less formal. The argument may not explicitly attend to definitions and previously established results, or position the new result within an axiomatic system.

Each of these different types of claims (choices, answers, properties of mathematics and its objects) is essential in a mathematics classroom and each has an important role to play in the teaching and learning of mathematics. This perspective—that attends to the nature of the claims made in classrooms, as well as the nature of the support—illustrates why argumentation is a practice that can permeate K–12 mathematics classrooms. More importantly, foregrounding the nature of the mathematical activity positions teachers to make consistent, appropriate choices regarding term use with students.

Table 2 summarizes these ideas and provides examples, highlighting differences in the nature of claims made in mathematics classrooms as well as the type of evidence and chain of reasoning that can be offered in support of these claims.

An illustrative example

To further illustrate and differentiate among the three types of support-for-claims work that would fit our descriptions of justification, argumentation, and proof, we present three ways in which teachers and students might engage in the following math task:

> Many counting numbers can be written as the sum of consecutive positive integers. Which ones? How many sums does each counting number have? And what is the length of each sum? (That is, how many addends are in each sum?)

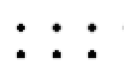
Working on this task, students may explore and develop various conjectures, and ultimately be asked to establish some of their conjectures. A common assertion that students propose is: 'All odd numbers can be written as the sum of two consecutive numbers'. Based on our definitions and descriptions of argumentation, justification and proof presented above, let us look at the differences between a teacher asking students to *justify* this conjecture, offer an *argument* to establish the conjecture, and *prove* the conjecture. The sample student responses are not taken from actual student work samples, but are composite responses to illustrate and help differentiate the types of mathematical work being requested and undertaken.

> *Justify your conjecture.* This means: explain to the class what motivated your conjecture and why, based on your mathematical activity, your conjecture is reasonable. Show the data that led you to it.
>
> *Example student response:* I think all odds can be written as the sum of two consecutive numbers because I made a table and noticed that all of the sums of two consecutive numbers were odd. Then I tested some other numbers, like 23+24 and 500+ 501. I always got odd numbers for the sums so I think you can make them all.

This chain of reasoning renders the conjecture reasonable. The conjecture has been motivated or justified (as a plausible conjecture), as one can see how the proposal or the conjecture follows from the mathematical work the student did and the student's observations of that work.

> *Create a mathematical argument for why your conjecture is true.* This means: provide a logical argument that shows our community (the class) why all odd numbers can be written as the sum of two consecutive numbers. A teacher might prompt such activity by asking, "show how you know this is always true."
>
> *Example student response:* Any odd can be shown like this—with two rows, and one row is one dot longer than the other. You always have one dot that does not have a pair.

or



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> When you try to divide the odd number in half, you can't do it exactly—you have one row that is one longer than the other row. So, to write the odd number as a sum of two consecutive numbers, you add the number that's one below half and one above half, and that is your consecutive sum.

This chain of reasoning compels the claim, which is a general, mathematically consequential statement with a truth value. The reasoning, as provided, does not attend to the definition of odd and even numbers explicitly, though an implied definition is being used. The language attends to all cases (all odd numbers), but the representation does not fully—as students do not necessarily yet have the representational acumen to symbolically represent all cases. The result that *all* odd numbers can be written as the sum of two consecutive numbers has been established within the given community. This chain of reasoning is appropriately described as an argument (but not a proof).

> *Prove your conjecture.* This means: Using previously established results, and shared definitions, provide an argument that leads from those definitions and prior results to a conclusion that all odd numbers can be written as the sum of two consecutive numbers.
>
> *Example student response:* By definition, odd numbers are integers that are not evenly divisible by 2. Thus, they can be represented as $2n+1$ where n is a positive integer. Using n=0, 1, 2, 3, …, we can generate all odd numbers, starting with 1.
>
> Consider
>
> $2n+1 = n+n+1 = n + (n+1)$
>
> Note that n and $n+1$ are consecutive integers. Thus, all odd numbers ($2n+1$) can be written as the sum of two consecutive integers, namely, n and $n+1$.

This chain of reasoning compels the result, and rests on an explicit definition of an odd number. It is fully general, covering all cases. The claim pertains to an important mathematical object—odd numbers—and, once substantiated, can be used in further arguments pertaining to parity or sums of consecutive integers. This chain of reasoning we would call a proof.

We contend that these three different support-for-claims activities, in relation to this same conjecture, are all valued mathematical activities that teachers support in classrooms. Thus, teachers need to be able to name, distinguish, and request these varied activities from students in their daily work. Having language that distinguishes among these activities can help teachers and students recognize when one or the other type of activity is most appropriate to achieve a specific learning goal, and to assess (or self-assess) progress or proficiency with regard to argumentation, justification and proof. Distinguishing these activities also helps students better understand how mathematics as a discipline establishes knowledge. Teachers can begin this process by providing students with explicit language and examples, such as those presented above, to describe these complementary mathematical practices.

Conclusion

We have offered our ideas regarding terminology and definitions, along with our rationales, in order to open opportunities for discussion, analysis, and reflection. We do not promote that the field decide to use a single term, or a single definition for any particular term. However, we urge those making these choices (*e.g.,* groups of teachers, district leaders, teacher educators, curriculum designers) to attend to explicit criteria and to the nature of mathematical activity teachers must support in classrooms. The criteria we have offered may provide an adequate set, or those criteria may need to be adjusted depending on the values, commitments or cultural context of the community. Regardless, consistent use of terminology can help teacher educators and teachers to distinguish differences among student performances and productions, and help students understand the nature of the mathematical work being asked of them. Establishing a general direction for the terms and definitions is an essential first step. It sets the stage for tackling the three challenges we noted, although further work is necessary to fully address these challenges, and others that K–12 teachers face.

We hope this article has advanced the idea that centralizing teachers' work, needs, concerns, and goals, is essential, as it is teachers' work with students that ultimately makes the difference in student learning. Moving forward, we recognize that extensive, in-depth conversations among mathematics educators and mathematics teacher educators are needed to 'make good on' these ideas and have them positively impact classroom instruction and student learning. Each term must be accompanied by a shared understanding, which then supports capacity-expanding conversations. Having these distinct terms is necessary but not sufficient for improving teaching and learning with respect to argumentation, justification, and proof—for teachers and for K–12 students.

Notes

[1] Teachers were participating in Mathematics Studio, a professional learning experience in which the second author was a participant-researcher. All names are pseudonyms.

[2] 'K–12 teachers' encompasses teachers of the US grade Kindergarten (age 5) through Grade 12 (age 18).

[3] Online at https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study/national-curriculum-in-england-mathematics-programmes-of-study

[4] Online at https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/key-ideas/

[5] Note that teachers also must attend to the connection between language use related to support-for-claims activities and terminology use in everyday language. Given the space available for our discussion, we do not take up that point in this article. Discussions about this can be found elsewhere in the literature (*e.g.,* Pimm, 1987).

[6] We realize our decision to interpret or limit justification in this way may run counter to definitions proposed by others. For example, Lannin, Ellis, and Elliott (2011) discuss the process of justifying and refuting mathematical claims and define mathematical justification as "a logical argument based on already-understood ideas." (p. 12)

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