RESEARCH ENDS AND TEACHING ENDS IN THE ANTHROPOLOGICAL THEORY OF THE DIDACTIC

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In Gascón and Nicolás (2017) we made a *first step* in a dialogue between scholars working in different theories in didactics of mathematics. The question that gave rise to this initial step was the following:

> Q_1: To what extent, how, under which conditions can (or must) didactics set value judgments and normative prescriptions in order to provide criteria about how to organise and manage study processes?

Our answer, given from the point of view of the *Anthropological Theory of the Didactic* (ATD), was unequivocal: didactic of mathematics, regarded as a science, is legitimised to present, as results of research, neither normative prescriptions nor value judgments of any kind.

At the end of the aforementioned paper there were some open questions to broaden the dialogue started with Q_1. To refer to them, we will use the labels Q_2 (i), Q_2 (ii) and Q_3 which convey that, in our view, the first two questions might constitute a *second step* of the dialogue, while Q_3 is different enough to open a *third step*.

Q_2(i) wonders about what kinds of results of research can be legitimately stated in didactics and, ultimately, which are the *research ends in didactics*.

> Q_2(i): Which are the research ends of each theory? In other words, which are the research problems considered, the didactic phenomena problematised? Concerning this, which are the research results regarded as admissible by each theory? Finally, which is the link between the research ends and the basic assumptions of each approach?

On the other hand, Q_2 (ii) focuses on *teaching ends:*

> Q_2(ii): Which is the relationship between the normative prescriptions, the teaching ends (implicitly) assumed and advocated by each theory, and the underlying epistemological model (that is to say, the way to conceptualize what is to be studied, and how)?

The main goal of this article is to answer, from ATD perspective, these two questions.

Basic assumptions and research problems of didactics according to ATD

We devote this section to question Q_2(i). This question deals with the relationships between the *research problems* considered by a certain theory and the *basic assumptions* of this theory. In our view, those basic assumptions already condition both the conception and the formulation of research problems. Thus, in the next seven subsections we sketch some relevant basic assumption of ATD, and in the last subsection we describe the kind of research problems addressed by ATD.

Every human activity can be described in terms of praxeologies

This basic assumption is the core of the *theory of human action* proposed by ATD. It strongly affects the statement of any research problem since any knowledge and any social manipulation of it would be expressed in terms of praxeologies. In a few words, a *praxeology* has four components: a set of *types of tasks*, a set of *techniques* to face those type of tasks, *technological* considerations about the techniques, devoted to giving a precise description of them, determining their scope, their reliability, *etc.,* and finally a *theoretical* discourse, which establishes the ontology assumed by the praxeology, namely, the basic objects and the relationships between them. In Chevallard (1999) one can find a fully fledged description both of the structure and the dynamics of praxeologies in the particular case of the mathematical-didactic activity, regarded as a human activity among others. The interested reader can find in Bosch & Gascón (2014) a detailed use of praxeologies.

Didactics as a science beyond disciplines

According to the latest works in ATD, didactics should become emancipated from the compartmentalisation of school knowledge into disciplines (mathematics, physics, history, *etc.*) that takes place at a specific historic moment (Chevallard, 2007). This comes from a broad view on didactics according to which it studies *all the social manipulations* (genesis, development, teaching-learning, diffusion and use) *of every kind of knowledge.* Under this view, *the didactic* is a founding dimension of human societies (Chevallard, 2013). The existence of this dimension, governed by *didactic laws* that cannot be properly analysed from other approaches (psychology, sociology, semiotic, *etc.*), explains the existence of didactics as a (relatively) autonomous science.

Didactic transposition and the institutional relativity of knowledge

The *theory of didactic transposition* is historically at the kernel of ATD. The idea of didactic transposition underlines that there is no privileged institution providing an absolute reference system from which a certain piece of knowledge could be observed, described, analysed and assessed. Hence, it is not possible to give an univocal characterisation of a piece of knowledge regardless of the institution where it lives and its functions therein. This fact brings to light the *institutional relativity of knowledge* and affects both the wording of research problems and methodology.

Many works in ATD contemplate didactic transpositions between other kinds of institutions (here we only mention academic and school institutions) in which some disciplines other than mathematics are taken into account. See for instance Castela (2016).

However, for the sake of simplicity, in this article we are mainly dealing with those transpositive processes in which a piece of mathematical knowledge (which can always be described in terms of praxeologies) is transposed from some *academic institution* I_1 to a *school institution* I_2 to be taught and learnt. The praxeologies imported to I_2 from I_1 undergo a *transposition process* to be adapted to the epistemological ecology of I_2. This process is never just a simplification but rather a complex process of reconstruction and reorganisation aimed at making mathematical knowledge into something suitable to be taught and safeguarding its usefulness. The extension and complexity of the praxeologies of I_1 are usually serious obstacles to this adaptation. Indeed, differences between 'academic' and 'taught' knowledge can be severe, but this gap cannot be openly set out, and it could even be concealed in I_2 to preserve the *epistemological legitimacy* of the taught knowledge (Chevallard, 1985/1991; Bosch & Gascón, 2006). Concerning this, there is a strong tension in the transposition process, as the knowledge transposed to I_2 must be: (a) close to the corresponding knowledge in I_1, (b) but also understandable in the institution I_2, (c) and not too close to common sense in order to avoid the feeling that this knowledge is already known and so irrelevant.

Unit of analysis in ATD and scale of the phenomena studied

Every experimental science uses, more or less explicitly, a unit of analysis which is both the basic theoretical construct and the elemental sphere for the empirical data. Therefore, the chosen unit of analysis is a key piece in the relationship between the theory and the empirical world, and it constitutes one of the distinctive features of the discipline. Indeed, when one determines a particular unit of analysis, the following are fixed: (a) the kind of empirical data to be taken into account (and so, the ones to be discarded), (b) the allowed ways of interpreting these data, (c) the kind of relationships to be considered between the constituent elements of the unit of analysis, and (d) the kind of problems to be tackled.

According to ATD, the *minimal unit of analysis of the didactic process* encompasses all the stages and all the institutions involved in the process of didactic transposition, which includes the institution *producing* knowledge and the so-called *noosphere*, as well as the *school* institution and the *study community* which is the main character in the didactic process (Bosch & Gascón, 2004). If we focus on mathematics, the unit of analysis should contain a *didactic praxeology* (concerning the study and the help to the study of mathematics) dealing with a *local and relatively complete* mathematical praxeology (Bosch, Fonseca & Gascón, 2004). Oversimplifying, we could say that the unit of analysis should not be a simple and isolated mathematics exercise, but a bunch of intertwined activities (different tasks, techniques, theoretical considerations, *etc.*) making a small *theory* around a certain mathematical issue.

Thus, ATD claims that, in its current state, didactics should prioritise *macro-phenomena,* namely, those involving a minimal unit of analysis in the sense stated above. ATD acknowledges the importance of studying *micro-phenomena*, but they can only be fully explained when regarded as a part of a *macro-phenomenon*.

The economy and the ecology of institutional praxeologies

We can say, briefly, that didactics is the science in charge of studying conditions and restrictions having an impact on the (institutional or personal) genesis, development, use and spreading of knowledge. ATD claims that there is a *dialectics between the personal and the institutional relationship to knowledge:* our *personal* relationship to knowledge (or our *personal praxeologies,* as ATD likes to say) is, to a great extent, a product of our past and present bonds with certain institutions, and reciprocally, *institutional praxeologies* emerge from the personal praxeologies of the individuals belonging to those institutions.

For methodological reasons, in its current state ATD *mainly* studies *institutional* praxeologies. More precisely, ATD is interested in aspects related to the *economy* and the *ecology* of institutional praxeologies (Gascón, 2011). When we speak of the *economy* we refer to the *nomos* or *nomoi* which rule the genesis and development of praxeologies (metaphorically viewed as living beings) dwelling in a certain institution (metaphorically viewed as an *eco*system). When we speak of the *ecology* we refer to the *logos* which explains why the praxeologies (metaphorically viewed as living beings) dwelling in a certain institution (metaphorically viewed as an *eco*system) are as they are, and what would be required in order to modify them in a certain sense.

Structure of the empirical field considered by ATD

The empirical field of ATD is the set of conditions having an impact on the genesis and development of praxeologies. This set of conditions is not regarded by ATD as a formless bunch of data, but rather structured according to the so-called *scale of levels of codeterminacy* (Figure 1). This scale consists of several interrelated levels from which didactic phenomena can be considered. At each level one finds specific conditions which, a priori, can affect any other level (Chevallard, 2002).

Therefore, according to ATD methodology, the study of didactic phenomena, regardless the level at which they appear, requires taking into account data coming from all the levels of the scale.

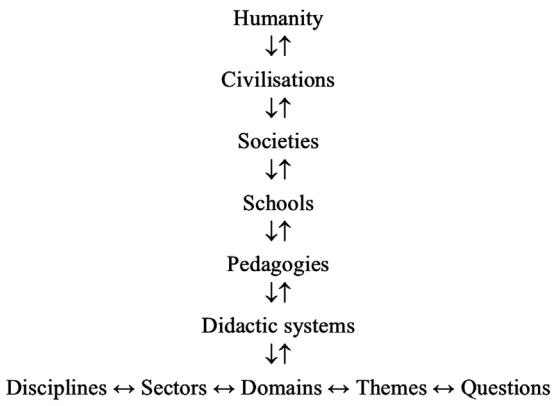

Figure 1. Levels of didactic codeterminacy.

Epistemological models in the transpositive analysis

The analysis of transpositive *adaptations* and *distortions* of knowledge play a key role in ATD research methodology, and it is a crucial tool to break away from the current epistemological models in the institutions under study. This analysis is carried out using a praxeological description of knowledge (this could be said to be the *transpositive-praxeological analysis*) and it is the entrance door to the analysis of the study processes in which this knowledge takes part (this could be said to be the *didactic analysis*).

To carry out these analyses, the researcher necessarily uses (even if implicitly) a reference *epistemological model* of the knowledge at stake, namely, a certain description of how this knowledge is. The more explicit the model, the better you will control your own research. One of the distinctive features of ATD research is that we always try to make explicit our epistemological models of the piece of knowledge we are dealing with. At the level of discipline, in the case of mathematics, the *reference epistemological models* (REM) built by ATD reconstruct the different sectors, domains, *etc.,* of school mathematics and the links between them. We could say that REM are crucial tools for the researcher to become *emancipated epistemologically* from the *prevailing epistemological models* of the institutions under study (Gascón, 2014). In any case, a REM should be regarded as a *working hypothesis* being an attempt of solving a mathematical-didactic problem and, as such, it should be *revised* in the light of relevant *experimental testing.*

Research problems and admissible results in ATD

Summarising what we have said above, we have a first answer to question Q_2 (i): the *primary* ATD research problems are linked to the *institutional* genesis, development, teaching-learning, use and spreading of any kind of knowledge, described in terms of *praxeologies*. Among these issues, the most esteemed are those related to the *economy* and the *ecology* of *macro-didactic phenomena*, for which we take into account data coming from all the stages of the didactic transposition and all the levels of didactic codeterminacy. The so-called *transpositive-praxeological analysis* of the knowledge at stake is the entrance door to the so-called didactic analysis, and it requires the building and the use of a REM of this knowledge. Hence, ATD belongs to the *epistemological programme of research in didactics* (Gascón, 1998, 2003) started by the *theory of didactic situations* (Brousseau, 1986).

In coherence with Max Weber's thesis on social sciences (Weber, 1917/2010), these problems can be stated in terms of *means-ends*. As Weber said, the question about the legitimacy of an end can not be tackled from a scientific point of view. Only the suitability of a certain *means* to attain a certain *end* constitutes a true scientific problem. Consequently, the allowed results in didactics can never contain judgment values or normative prescriptions about teaching, but only *explanations and laws concerning certain didactic phenomena,* and assessments of the efficacy of certain *means* to achieve a previously given *teaching end* (whose validity cannot be rationally stablished) (Gascón & Nicolás, 2017).

Teaching ends, corresponding paradigms and normative prescriptions

To answer question Q_2 (ii) the key notion is that of didactic paradigm, characterized by the underlying epistemological model and the advocated teaching ends (which, in turn, are stated in the terms provided by this epistemological model). First we will address the general case of any theory in didactics. Then we will focus on the particular case of ATD, analysing the teaching ends embraced by this theory both at the *pedagogical* and at the *disciplinar-mathematical* level. Finally, we will consider the role played by the *specific* teaching ends assumed by ATD at *sub-disciplinary* levels. For this, we will use some of the theoretical tools presented above such as that of the scale of the level of codeterminacy and the idea of the epistemological model of a piece of knowledge.

Social structure and the teaching ends embraced by a theory in didactics

We should distinguish between the teaching ends adopted, more or less explicitly, by a society or a certain institution, and the teaching ends embraced by a theory in didactics. In this last case, those ends play the role of *postulates* of this theory. But they are a special kind of postulates, as they do not aim to state a truth (which might be refuted in a future), but to point a direction for the research action. According to Weber (1917/2010), they are in the so-called 'value sphere' and cannot be rationally stablished.

Where do the teaching ends come from? According to Durkheim (1924/1991), they are determined by the current *social structure.* For a theory in didactics to become emancipated and to gain self-control, it is necessary to make explicit those ends and to examine them from the point of view of the inner logic of this theory. Indeed, a teaching end cannot be assessed from the point of view of the dichotomy true/false, but it can be evaluated with respect to its coherence with some other teaching ends or some other principles adopted by the theory.

In our view, the teaching ends shared by all the members of an institution (society, school, theory, *etc*.) live always inside something broader, that we call a *didactic paradigm*. It consists not only of *teaching ends,* but also of *means* to reach those ends, and a set of *didactic facts* to which the paradigm aims to react as they are regarded, somehow, as limitations to

the fixed ends. Indeed, the presence of didactic facts to be beaten is the only reasonable hypothesis to explain the consideration of new teaching ends. Otherwise, one would keep considering only already existing teaching ends. On the other hand, the inseparable presence of means also seems unavoidable. If a teaching end *E* (for instance, to learn mathematics as mathematicians learn in real life) is adopted by an institution, then this very institution will immediately focus on the problem finding suitable means to achieve *E*. In other words, if an institution considers a teaching end *E*, then this institution aims to surmount certain didactic facts and starts looking for suitable means to reach *E*. Of course, the existence of a didactic paradigm, to the extent that it states how things should be done, implies normative prescriptions.

Typically, the assumption of a certain didactic paradigm underlies something still more abstract: a certain way of conceptualise *knowledge* and *didactic processes.* This is what we call a *reference epistemological-didactic model.*

In sum, teaching ends corresponding to a socially constructed didactic paradigm: (a) have their origin in the current social structure; (b) play the role of implicit postulates; (c) affect the type of research problems addressed by the theories that assume this paradigm; and (d) give rise to normative prescriptions about how to teach.

Pedagogical paradigms in ATD

Since ATD assumes didactic paradigms specific to each level of codeterminacy (Figure 1), one expects to find normative prescription specific to each level.

At the *pedagogical level,* ATD advocates the so-called *paradigm of questioning the world* (PQW). This is a reaction to a bunch of didactic facts interpreted from ATD as a phenomenon called *monumentalism*. This phenomenon, in turn, can be regarded as a consequence of a previous pedagogical didactic paradigm, the so-called *paradigm of visiting works* which, somehow, prevails in school institutions nowadays (Chevallard, 2013). The main *teaching ends* of PQW are:

- To promote a new *cognitive ethos* characterised by a curious and 'problematising' attitude towards knowledge, able and eager to question well-stablished statements and to pose challenges concerning how the world is.
- To promote a learning based not only on the *study* of already made pieces of knowledge, but also on the *research*, as it is the case in scientific communities.

These ends, which are at the core of PQW, point to the role that knowledge and education should play in our society. Let us see now which are the corresponding *means* suggested by PQW.

PQW regards research as a process which starts with a certain *question Q* and continues with a kind of dialectics between more questions and attempted answers. In this process, the study community [*X, Y*], formed by a group of students (*X*) and one or several teachers (*Y*), constructs a *means M* which has all the tools required to construct a suitable answer $R^{\heartsuit}$ to *Q*. To express all this with a diagram we use the *Herbartian scheme* [1]:

$$[S\ (X, Y, Q) \rightarrow M] \rightarrow \mathrm{R}^{\heartsuit}$$

The construction of $\mathrm{R}^{\heartsuit}$ is achieved along a *study and research path* (SRP). The Herbartian scheme only describes the elements making the *structure* of the SRP, while the *dynamics* of those processes can be described by certain *dialectics* or *study movements*: questions-answers, individual-collective, media-*milieu, etc.* (Chevallard, 2007).

Teaching ends of the disciplinary paradigm of mathematical modelling

At the *disciplinary-mathematical* level ATD proposes the so-called *paradigm of mathematical modelling* (PMM). This paradigm is compatible with the PQW, and, actually, it translates at the discipline and sub-discipline level some of the pedagogical normative prescriptions attached to PQW.

The main teaching end of PMM is that mathematics must be constructed at school as a coherent conglomerate of models which allow mastering several portions of reality. Moreover, ATD regards *intra-mathematical modelling* (that is, mathematical modelling of mathematical systems) as an essential part of *extra-mathematical modelling* (that is, mathematical modelling of extra-mathematical systems). In other words, ATD includes intra-mathematical modelling as a particular and important component of mathematical modelling (MM). This enlargement of the traditional conception of MM is coherent with the historical development of mathematics and regards MM as a process of progressive 'mathematisation' of a system in which the first mathematical model goes over to the role of (mathematical) system in a new process of MM, and so on. This leads to work with models, models of models, models of models of models, *etc*. Thus, we find a clear *recursive* character of MM. There is also a *reflexive* character in this activity in that, at a certain point, the system can play the role of the model and vice-versa. A historical example of this can be found in the mutual modelling of Euclidean and Cartesian geometries. When suitably developed, this provides a whole *general epistemological model of mathematical activity* (García, Gascón, Ruiz Higueras & Bosch, 2006). The *means* considered by PMM to achieve this end are again the aforementioned SRP, which seem to be specially well adapted for the institutional genesis and development of mathematical modelling (Barquero, Bosch & Gascón, 2011).

Of course, PMM has an impact on the sub-discipline levels by proposing important changes in the study processes of the different sectors, domains and themes of school mathematics (Figure 1). It is important to point out that PMM does not determine completely how the teaching at sub-discipline levels must be organised. It just proposes a general teaching aim at the discipline level, globally regarded. The normative prescriptions leading the organisation of teaching at more specific levels rather depend on the teaching ends proposed at these levels.

Normative prescriptions associated to the teaching ends of sub-disciplinary paradigms

In ATD, the teaching ends of the paradigms proposed at sub-disciplinary levels always pursue to provide a new *raison d'être* to a certain piece of knowledge. This new rationale, and the corresponding means, are aimed at *avoiding* certain

undesirable didactic facts (undesirable, for instance, because they are not compatible with the teaching end of PMM or PQW).

Thus, for instance, in our research group we find works which try to overcome or to avoid what follows: the isolated and rigid character of school mathematics praxeology (Bosch, Fonseca & Gascón, 2004); considering negative numbers as 'arithmetic objects' (Cid, Bosch, Gascón & Ruiz-Munzón, 2017); the isolation of proportionality with respect to the other functional relations (García, Gascón, Ruiz Higueras & Bosch, 2006); regarding elementary algebra as a kind of generalised arithmetic (Bolea, Bosch & Gascón, 2001); the disconnection between elementary algebra and functional modelling (Ruiz-Munzón, Bosch & Gascón, 2015); the reductive identification of numeral systems with mere systems to name numbers (Sierra, 2006); the absence of a rationale for elementary differential calculus coherent with the role it plays in scientific activity (Lucas, 2015); or the disconnection between numbers and measurement of magnitudes (Licera, 2017). In each case, a specific REM was constructed in order to analyse the corresponding piece of knowledge in its current prevailing state and to reconstruct this knowledge to lead its teaching towards the desired end.

Even if, once the teaching ends are fixed, all have a strong normative flavour, the research problem of all these works can be rephrased as a try to *assess* the *efficacy* of certain *means* to achieve certain previously fixed *teaching ends.* Certainly, the empirical signs intended to measure this efficacy are not explicitly stated. Therefore, in a strict sense, this methodology could not be regarded as *scientific*. But still, in the current state of didactics, this burden seems unavoidable.

Conclusion

Many research problems considered by researchers in didactics are related to *means*. They are typically of the form: is this a good means of teaching? This research cannot be properly tackled without making explicit the teaching end for which this means is aimed to serve. We agree with Postman (1999) when he says that the emphasis needs to be on *teaching ends*. In our answer to Q_2 (ii) we state explicitly, at different levels of generality (pedagogical, disciplinary and sub-disciplinary), the didactic paradigms embraced by ATD. We can conclude that the didactic paradigm assumed by a theory is the link between the research carried out within this theory (the issue addressesd by Q_2 (i)) and the promotion of the teaching ends advocated by this theory (the issue addressed by Q_2 (ii)).

The basic assumptions of the different approaches in didactics having been revealed, there is further a question pending:

> Q_3: Are the basic assumptions of the different theories in didactics compatible? In particular, are the teaching ends compatible? Are the research ends compatible? If not, to what extent do the different theories work in the same discipline?

In other words, are the *didactic paradigms* assumed by different theories compatible? What about the *research programmes* in which these theories operate? Taking this into account is essential for networking, or even dialogue, between different approaches.

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Note

[1] This name for such a scheme was introduced by Chevallard, who comments:

> The model introduced so far is first enriched with the following formal description called the *reduced Herbartian schema:* $S(X; Y; Q) \rightarrow A$. (Beware! Here, the adjective *herbartian*, which refers to the German philosopher and pedagogue Johann Friedrich Herbart (1776-1841), is something of a misnomer.) Here, A is the answer to the question Q that the didactic system (or the research system) is expected to produce. It is usual to write the answer A with a heart ♥ in superscript: $S(X; Y; Q) \rightarrow A^{\heartsuit}$ {a upper heart}, a gentle reminder of the fact that, henceforth, this answer will be "at the heart" of the didactic system, of which it will allegedly be—at least for some time—the "authorised" answer to question Q. (Chevallard, 2016, ¶ 10.1 online at http://www.atd-tad.org/wp-content/uploads/2016/03/Chevallard_TAD-5_TexteCoference_EN.pdf)

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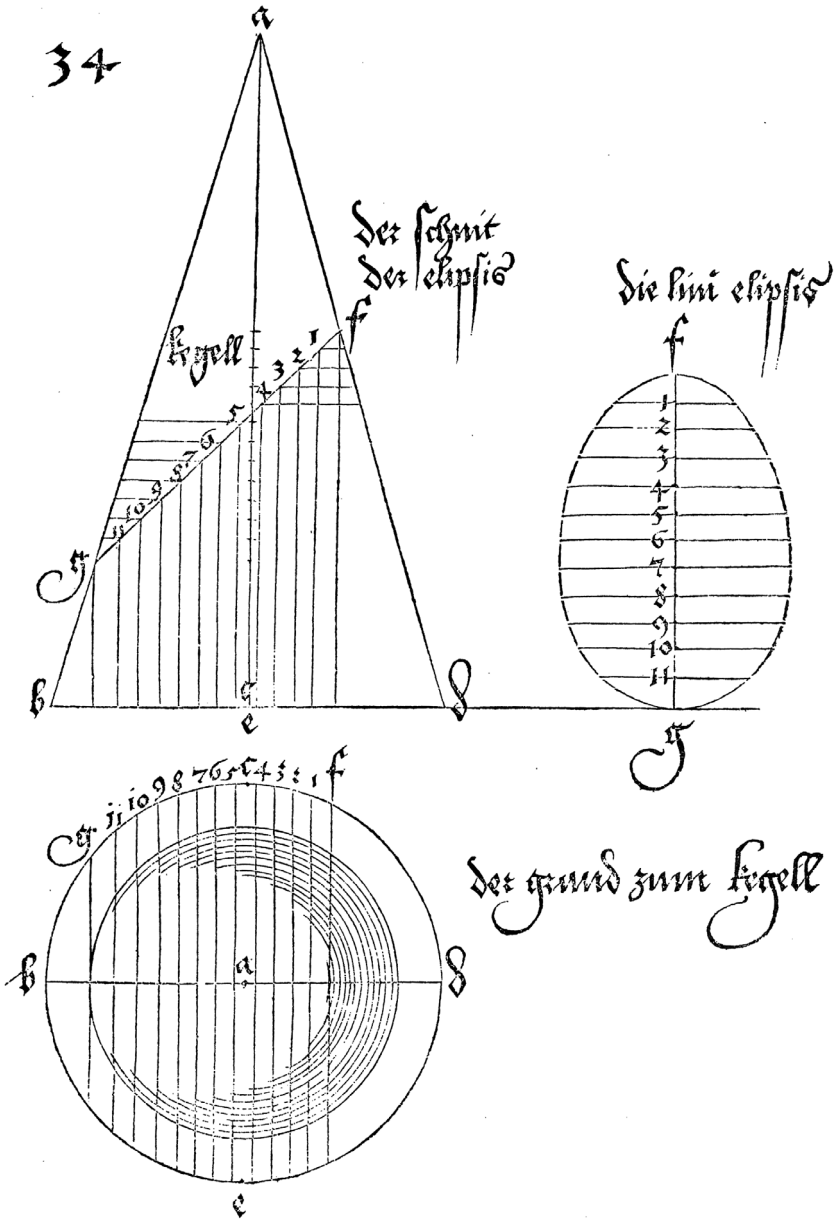

Albrecht Dürer's diagram showing how to construct an ellipse from an elevation and plan of a cone, from his 1525 *Unterweisung der Messung mit dem Zirkel und Richtscheit [Instructions for Measuring with Compass and Ruler]*. In principle, his method should work, but in this case the ellipse is noticeably egg shaped, and it is interesting that Dürer does not find this odd.