Famous Problems in Mathematics: an Outline of a Course

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This is a one-semester course at the 3rd year level offered in the department of mathematics at York University. (Some technical details about the course are given at the end.) The course has a significant historical component, but it is not a course in the history of mathematics. The historical perspective is, however, essential. One of the objectives of the course is to make students aware that mathematics *has* a history, and that it may be interesting, useful, and important to bring history to bear on the study of mathematics.

The course tries to legitimize in the eyes of students the point that it makes sense to *talk* about mathematics in addition to *doing* mathematics; that it makes sense to deal with ideas in mathematics in addition to dealing with "mathematical technology." In brief, the course attempts to make students more "mathematically civilized." (The term in quotes is the title of a "letter to the editor" written by Professor O. Shisha; it appeared in the *Notices of the Amer. Math. Soc.*, v. 30 (1983), p. 603)

THE THEMES

Before dealing with the so called "famous problems," let me list some ideas or themes which I try to pursue in the course, with brief indications of intent:

(a) The origin of concepts, results, and theories in mathematics

A relevant major theme of the course is that "concrete" problems often give rise to abstract concepts and theories. In fact, Dieudonné [26, Introduction] has argued that "the history of mathematics shows that a theory almost always originates in efforts to solve a specific problem." Our problems 1, 2, 3, 6, and 7 illustrate this point of view. For a discussion see [6, pp. 229-230], [33]*, [34]*, [67], [103], [110].

(b) The roles of intuition and logic in the creation of mathematics

Students often see only the logical side of the mathematical enterprise. But, in the view of Hadamard, "logic merely sanctions the conquests of the intuition." History often bears him out. On the other hand, there were times in the evolution of mathematics when logical rather than intuitive thinking was the creative force. (The creation of non-Euclidean geometry and set theory are prime examples.) For the working mathematician there is, of course, an ongoing interplay between intuition and logic. See, e.g., [5], [8], [16, v. 2]*, [22]*, [23], [53], [55], [94]*, [112], [113].

Changing standards of rigor in the evolution of mathematics

The concepts of "proof" and "rigor" in mathematics are not absolute but change with time. Moreover, the change is not necessarily from the less to the more rigorous—there are fluctuations in standards of rigor. An entire issue of the *Two Year College Mathematics Journal* (v. 12, no. 2, 1981) is devoted to the question of what a proof is. See also [22]*, [24]*, [36], [41], [42], [48]*, [64]*, [67]*, [73]*, [85]*, [114].

I think that what we have been witnessing recently (both pedagogically and professionally) is a reaction against the strict rigor and abstraction which have dominated mathematics for much of this century. Rigor is, of course, essential in mathematics, but "it ought," as Simmons [95] says, "to suit the occasion."

(d) The roles of the individual and the environment in the creation of mathematics

A sociological theory concerning the development of mathematics can be summarized succinctly and poetically by the following statement of J. Bolyai: "Mathematical discoveries, like springtime violets in the woods, have their season which no human can hasten or retard." As against this, note Cantor's dictum that "mathematics is entirely free in its development...... The essence of mathematics lies in its freedom." This "complementarity of freedom and necessity" (Weyl [107]) is explored. See [16, v. 1], [76], [94], [115]*, [116]*.

In classroom teaching, the human drama inherent in the creation of mathematics is often ignored. Even if there is a certain inevitability in mathematical creations, they are made by humans—humans with personalities, passions, and prejudices, which sometimes have a bearing on the mathematics they create. Cantor is a case in point. (For an analysis of the significance of Cantor's personality on the creation of his transfinite set-theory see [21]). The intent, then, is to pay attention to the creators as well as the creations of mathematics. See [7], [40]*, [78], [82]*.

(e) Mathematics and the physical world

The relationship between mathematics and the physical

* The books and articles which are most appropriate in a first encounter with a given theme or problem are indicated with an asterisk. This is not meant to discourage readers from consulting the other indicated sources. These often broaden, deepen, and reinforce in other ways the asterisked material.

world is a longstanding one. It has enriched both mathematics and our understanding of the physical world. Moreover, our view of this relationship has changed over time (especially in the 19th century). Witness the following words of Whitehead: "The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact." For an elaboration see [14], [16, v. 3], [47], [53], [57], [59]*, [60]*, [94]*, [101]*, [102]*, [108].

(f) The "relativity of mathematics"
By this I mean that mathematical truths are not absolute but depend on the context. For example, the statement "If $a + b = a + c$ then $b = c$" is true in the domain of, say, real or complex numbers but false in the domain of transfinite numbers. Again, the equation $x^2 + 1 = 0$ has no solutions in the domain of real numbers, two solutions in the domain of complex numbers, and infinitely many solutions in the domain of quaternions. See e.g. [57].

(g) Mathematics—discovery or invention?
This question arises more or less naturally in connection with various mathematical developments in the 19th century which are dealt with in the course. Moreover, one need not opt for one characterization or the other. Davis and Hersh [22] suggest that the typical working mathematician is a Platonist on weekdays and a formalist on weekends (thus viewing mathematics at one time as a discovery and at another as an invention). See also [5]*, [63], [94]*, [101].

The above themes are, of course, of major importance in the history and philosophy of mathematics, and one can not treat them exhaustively in a one-semester course. They are, however, central to the course. Moreover, they are not dealt with one by one (as listed above), but rather are discussed in the course of dealing with the problems. So much for the underlying themes. Now to the "famous problems."

THE PROBLEMS

The content of the course is flexible and one can choose the problems more or less as one pleases. Here are some of my choices. They are dictated by personal taste, by the level of the course, by the fact that the subject matter of the problems is usually not dealt with in the standard courses, and, most importantly, by the relevance of the problems to the themes which I am trying to expound. (Note that the "problems" are interpreted quite broadly.)

I will describe nine problems—some in detail, others very sketchily. The problems are independent of each other (although some reinforce one another) and can be dealt with in any order.

1. Diophantine equations

These are equations in two or more variables in which the solutions sought are integers. Diophantine equations are fundamental in number theory; their study has inspired the development of important concepts (see below). I begin this topic with the equations $x^2 + y^2 = z^2$ and $x^2 + 2 = y^3$. The first equation goes back to Diophantus (ca 250 A.D., and in one form or another to the Babylonians ca 1600 B.C.), whose work apparently inspired Fermat's conjecture about $x^n + y^n = z^n$ (see below). The second equation is a special case of another famous diophantine equations, namely $x^2 + k = y^3$ (studied by Fermat and others), for which no solution is yet known for general k. I proceed to solve these two equations "formally," and analogously, as follows:

(a) On factoring the left hand side of $x^2 + y^2 = z^2$, we get $(x + yi)\cdot(x - yi) = z^2$. This is now an equation in so called "Gaussian integers," that is, "numbers" of the form $a + bi$, where a and b are ordinary integers. Now, the set Z of (ordinary) integers has the property that if $ab = c^2$ ($a, b, c \in Z$) and a and b are relatively prime, then $a = u^2$ and $b = v^2$ for some $u, v \in Z$. (i.e., if ab is a square with a and b relatively prime, so are a and b.) Proceeding in the same manner in the set G of Gaussian integers, it follows from $(x + yi)(x - yi) = z^2$ (assuming that $x + yi$ and $x - yi$ are relatively prime Gaussian integers) that each of $x + yi$ and $x - yi$ is a square in G. That is, $x + yi = (a + bi)^2$ for some $a, b, \in Z$ (and also $x - yi = (a - bi)^2$, which we do not need). Then, $x + yi = (a^2 - b^2) + 2abi$ and hence $x = a^2 - b^2$, $y = 2ab$. From this one finds, using $z^2 = x^2 + y^2$, that $z = a^2 + b^2$. Conversely, one easily verifies that $x = a^2 - b^2$, $y = 2ab$, $z = a^2 + b^2$ is a solution for any $a, b \in Z$. Thus we have found all solutions of $x^2 + y^2 = z^2$, namely $x = a^2 - b^2$, $y = 2ab$, $z = a^2 + b^2$, where a and b are arbitrary integers.

(b) The equation $x^2 + 2 = y^3$ yields, on factoring, $(x + \sqrt{2}i)\cdot(x - \sqrt{2}i) = y^3$, which can be considered as an equation in the domain $D = \{a + b\sqrt{2}i: a, b \in Z\}$. Since the product of two elements of D is a cube, it follows (assuming that $x + \sqrt{2}i$ and $x - \sqrt{2}i$ are relatively prime and that the result in Z carries over to D) that each of $x + \sqrt{2}i$ and $x - \sqrt{2}i$ is a cube. In particular, $x + \sqrt{2}i = (a + b\sqrt{2}i)^3$ for some $a, b, \in Z$. Thus $x + \sqrt{2}i = (a^3 - 6ab^2) + (3a^2b - 2b^3)\sqrt{2}i$ and, equating real and imaginary parts, $x = a^3 - 6ab^2$ and $1 = 3a^2b - 2b^3 = b(3a^2 - 2b^2)$. Since $a, b \in Z$, we must have $b = \pm 1$ and $3a^2 - 2b^2 = \pm 1$ and, subsituting $b = \pm 1$ into the last equation, we get $3a^2 - 2 = \pm 1$. Hence $3a^2 = 3$ ($3a^2 = 1$ is impossible, and therefore we must choose the "+" sign in ± 1; thus $b = +1$). From $3a^3 = 3$ we get $a = \pm 1$ and with $b = 1$ this yields $x = a^3 - 6ab^2 = \pm 1 \mp 6 = \pm 5$. Since $x^2 + 2 = y^3$, we get $y^3 = 27$ and $y = 3$. One notes that $x = 5$, $y = 3$ and $x = -5$, $y = 3$ are indeed solutions, and hence they are the only solutions of $x^2 + 2 = y^3$.

ANALYSIS OF THE ABOVE "SOLUTIONS"
Once the two equations in (a) and (b) are "solved," we go back to examine carefully and to justify various steps in the method of solution. The basic questions that must be answered are: What are the essential properties of the (ordinary) integers that are to be carried over to the other two types of "integers" (i.e., G and D), and how can this be done? To answer these questions one introduces the concept of a *unique factorization domain* and develops enough "machinery" relevant to such domains to close the logical gaps in the "formal" solutions. For further details see [3]*, [10], [38], [84]*, [100].

Although the above procedure is probably the reverse of what is done in standard courses (in which one would first define a unique factorization domain and then, perhaps, give an application to the solution of diophantine equations), I have found it to be a good way of motivating the introduction of the concept of a unique factorization domain. In fact, it is my experience that students are much more receptive to "digesting" abstract concepts when their introduction is motivated by concrete problems.

If an instructor wants to spend more time on this topic, the following interesting diophantine equations pursue a similar theme:

(c) $n = x^2 + y^2$; (d) $n = x^2 + y^2 + z^2 + w^2$; (e) $x^3 + y^3 = z^3$ and, more generally, $x^n + y^n = z^n$.

(c) This is Fermat's problem of determining which integers can be represented as sums of two squares. Since $n = x^2 + y^2 = (x + yi)(x - yi)$, one applies some of the results developed above for Gaussian integers (in connection with the equation $x^2 + y^2 = z^2$) to resolve this problem rather quickly. For details see [3]*, [10], [38]*, [50, pp. 112-113].

(d) This is Lagrange's theorem, namely that every integer can be represented as a sum of four squares. Here $n = x^2 + y^2 + z^2 + w^2 = (x + yi + zj + wk)(x - yi - zj - wk)$, where $x \pm yi \pm zj \pm wk$ are conjugate quaternions, and the problem can be dealt with in a manner similar to (c) above. (One requires, of course, some knowledge of quaternions—see problem 4). For details see [10, pp 127-133], [50, pp 329-335]. An alternate, short and interesting algebraic proof of Lagrange's theorem, conceptually related to (c) above, is given in [102].

(e) To show that $x^3 + y^3 = z^3$ has no nontrivial integer solutions one factors $x^3 + y^3$ into $(x + y)(x + wy)(x + w^2y)$, where $w = 1/2(-1 + i\sqrt{3})$ (a primitive cube root of 1), and uses the fact that $B = \{a + bw: a, b \in Z\}$ is a unique factorization domain. For details see [3], [27], [45]*, [49].

If one considers Fermat's equation $x^p + y^p = z^p$ for any prime p, one can "prove" in a similar manner to the case $p = 3$ that the equation has no nontrivial integral solutions (!) One must, of course, assume that the domain of cyclotomic integers $D_p = \{a_0 + a_1w + a_2w^2 + \dots + a_{p-2}w^{p-2}: a_i \in Z,$ w a primitive p-th root of $1\}$ is a unique factorization domain. (See [3, p. 103] or [12, pp. 160-163] for details of such a "proof.") It is precisely this assumption which Lamé made in 1843 when he announced that he had proved Fermat's Conjecture. He was, of course, unaware that D_p is not a unique factorization domain for every p. See [3], [30], [31].

It is such equations as the above (i.e., (a) to (e), but especially (e)) which have given rise to a new branch of mathematics, namely *algebraic number theory* and, in particular, to such concepts as *unique factorization domain, ring, field, ideal.* They provide a very good illustration of our "theme (a)," namely that concrete problems often give rise to abstract concepts and theories. See [3]*, [10], [25], [30], [31], [45]*, [86], [92]*.

2. Distribution of primes among the integers

The study of prime numbers has fascinated and challenged some of the greatest mathematicians of all time, from Greek antiquity to the present. The purpose of this "problem" is to give students a sense of that fascination and challenge. It is also to show that important questions about natural numbers cannot be settled by restricting one's attention only to the natural numbers. The basic problem is that the natural numbers (or even the integers) have too little structure. Thus, in Problem 1 one enlarged the domain of integers to that of algebraic numbers so as to be able to employ algebraic methods, and in this problem one extends the domain of integers to that of real or complex numbers in order to enable one to use the tools of analysis.

At about the mid-eighteenth century Euler stated that "mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate." Some of the facts which attest to the irregularity, the "mystery," in the distribution of primes are:

(i) Numerical evidence suggests that there are infinitely many primes that are as close together as possible — the so called "twin primes" (e.g., 11, 13; 107, 109; 10006427, 10006429). At the same time,

(ii) there are arbitrarily large sequences of consecutive composite integers (e.g., $10^6! + 2$, $10^6! + 3$, ..., $10^6! + 10^6$ is a sequence of 999,999 consecutive composite integers, where $n!$ denotes n factorial.) Yet

(iii) the next prime after any given prime cannot be too far removed from it — there exists a prime between n and $2n$ for any integer n. (This is the so called Bertrand Postulate.)

Euler's apparent pessimism did not prove to be entirely justified. For, although we find no regularity in the distribution of the primes when considered individually, Gauss found regularity in their distribution when considered collectively. Thus, Gauss tried to describe not "how" but "how often" the primes occur in the integers. We are referring, of course, to the Prime Number Theorem which Gauss (and independently Legendre) conjectured, but was unable to prove, namely that $\pi(x)$ is asymptotic to $x/\log x$, where $\pi(x)$ is the number of primes $\leq x$ (i.e., $\lim_{x\to\infty} \pi(x)/(x/\log x) = 1$). Davis & Hersh [22, p. 210] said of this theorem that "it is one of the finest examples of the extraction of order from chaos in the whole of mathematics."

Attempts to prove the Prime Number Theorem stimulated the development of the branch of analysis called complex function theory, and this, in turn, led Hadamard and de la Vallée Poussin (independently) to a proof of the theorem in 1896. The starting point for the use of analysis in number theory, which eventually led to a new branch of number theory, namely *analytic number theory,* was Euler's own work.

In 1737 Euler showed that $\sum_{n=1}^{\infty}(1/n^s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$, where s is any real number > 1. (Euler was a master of formal manipulation of series. Inspired by Leibniz' result that $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \dots$, he proved in 1736 that $\pi^2/6 = 1 + 1/2^2 + 1/3^2 + \dots$, and soon afterwards that

$1 + 1/2^{2n} + 1/3^{2n} + \ldots = \pi^{2n} \cdot q$, q a rational number, n any positive integer. This apparently led him to the series $\sum_{n=1}^{\infty} (1/n^2)$ and to the discovery of the above identity.) Euler noted that his identity gives a new proof that there are infinitely many primes (by assuming the contrary and taking limits of both sides as $s \to 1$), and that it can be used to show that the series $\Sigma\, 1/p$ diverges (take the log of both sides). Moreover, an elementary argument, based on a similar idea, proves that there are infinitely many primes in the two arithmetic progressions $\{4n + 1\}$ and $\{4n + 3\}$ (Cf. Dirichlet's theorem on primes in an arithmetic progression — (b) below.)

In 1859 Riemann attempted to prove the Prime Number Theorem by introducing the so calle *zeta function* $\zeta(s) = \Sigma(1/n^s)$, where now s was a complex variable with real part > 1, and noted that Euler's identity extends to this complex domain. This led him to the very famous, still unsolved, *Riemann Hypothesis* concerning the roots of $\zeta(s)$. We discuss the known results about the Riemann Hypothesis, and the relationship of the Hypothesis to the Prime Number Theorem.

Another aspect of the problem of the distribution of primes has to do with prime-producing formulas. Since, as the above evidence suggests, it is very unlikely that a formula can be found which will produce *all the primes and only the primes,* what about formulas which will produce a subset of the primes (and possibly also composites)? We deal with some such formulas. For example:

(a) $f(n) = 2n^2 + 29$ is prime for $n \leqq 28$.

$f(n) = n^2 - 79n + 1601$ is prime for $n \leqq 79$.

$f(n) = n^2 + n + 41$ is prime for $n \leqq 39$.

The last formula is an instance of the formula $f_q(n) = n^2 + n + q$, which takes on primes for all $n \leqq q - 2$ if an only if $q = 3, 5, 11, 17, 41$ (a result due to Euler). These are precisely the values of q for which the domain A_d of "integers" is a unique factorization domain (see Problem 1). (A_d equals $\{a + b\sqrt{d}: a, b \in Z\}$ if $d \equiv 1 \pmod 4$, and $\{(a + b\sqrt{d})/2: a, b, \in Z$, a & b both even or both odd$\}$ if $d \equiv 3 \pmod 4$, where $d = 1 - 4q$ is the discriminant of $f_q(n)$.) See [37] for details.

The above formulas are a good illustration of the failure of "scientific induction" in mathematics.

(b) As we mentioned above, Euler showed that $f(n) = 4n + 1$ and $f(n) = 4n + 3$ yield infinitely many primes as n ranges over the positive integers. In 1837 Dirichlet effected a grand generalization of this result by showing (using fairly deep analytic tools) that $f(n) = an + b$ yields an infinite number of primes for any fixed relatively prime postive integers a and b.

It is not known whether $f(n) = 2^{2^n} + 1$ and $f(n) = 2^n - 1$ produce infinitely many primes as n ranges over the positive integers. The former are the famous *Fermat numbers,* the latter the equally famous *Mersenne numbers.* (The Fermat numbers are connected with the question of the construction of regular polygons with straightedge and compass, the Mersenne numbers with the question of the determination of even perfect numbers.)

(c) W.H. Mills showed in 1947 that there exists a real number a such that $[a^{3^n}]$ is a prime for every integer n, where for any real number x, $[x]$ denotes the greatest integer $\leqq x$. It was subsequently shown that there are infinitely many such a's (in fact, their cardinality is that of the continuum), although not a single specific value for a is known.

It is known that no polynomial (with integer coefficients) will produce only primes. It was a remarkable achievement when Matijasevich produced (in 1970) a polynomial which assumes all the primes and only primes for its *positive* values. The polynomial, of degree 37 in 24 variables, was the result of deep insights into Hilbert's Tenth Problem. See [52] for details.

Despite the many interesting and powerful results obtained on the distribution of primes since Euler's statement, it is quite fitting to conclude with the following quote from H. Weyl who, more than 200 years after Euler, echoed the former's sentiments: "It is most surprising to find that the distribution of primes among all natural numbers is of such a highly irregular and almost mysterious character."

For references on various aspects of Problem 2 see [4]*, [22], [28]*, [29], [39], [45]*, [46], [49], [56], [62], [80], [87], [88], [117]*.

REMARK ON PROBLEMS 1 AND 2

In addition to providing illustrations of some of the themes mentioned above (e.g., (a), (b), (e)), the study of number theory, as exemplified in the first two problems, sheds light on the following points:

(i) "Simplicity" in mathematics is complex (there is an abundance of "simple" questions to which there are, as yet, no answers).

(ii) To study problems in a given system (in this case, the integers) it is often very helpful to enlarge the system (a recurrent theme in mathematics)

(iii) The computer can be a useful tool in the study of various branches of mathematics

Moreover, number theory, I find, is a good topic with which to begin a course such as outlined here. It is of intrinsic interest to students, and it lends itself, perhaps more than many other topics, to student participation; students are willing, even eager, to respond to questions and make conjectures. This puts them at ease and sets the proper tone for the remainder of the course — a relaxed atmosphere and an ongoing dialogue between instructor and students.

3. Polynomial equations

The Babylonians knew how to solve quadratic equations (essentially by the method of completing the square) about four thousand years ago. Little progress was made in the theory of algebraic solution of equations for the next 3 500 years, until the 16th century Italian school of algebra made a fundamental breakthrough by giving an algebraic solution of the cubic equation (and soon thereafter the quartic equation). We focus on this breakthrough, which is intimately related to the discovery (invention?) of complex numbers. Students think that it was, in fact, the quadratic equation (e.g., $x^2 + 1 = 0$) which led to the introduction of complex numbers. This is not the case. It was the cubic which gave rise to complex numbers. The "why" and "how" of this interesting "story" are explored. The subsequent evolution of the complex numbers is briefly dealt with. The *complex numbers* are an interesting case study of the genesis, evolution, and acceptance of a "mathematical system." See [15], [19], [20]*, [61], [68], [70], [99]*.

Some indication is given of the theory of polynomial equations beyond the quartic; in particular, how the study of permutations of the roots of a polynomial equation aids in the study of solutions of the equation — an important source of the rise of the *group concept*. See [2, v. 1], [3], [27], [58]*, [61], [71]*, [101], [103].

This problem illustrates themes (a), (d), (e), and (g).

4. Are there numbers beyond the complex numbers?

The answer depends on what we mean by "numbers." We explore the historical evolution of the various *number systems* (indicating gains and losses at each stage of the evolutionary process), and then introduce the *quaternions* and the *octonions* (Cayley numbers), indicating how these, in turn, led to the study of *non-commutative algebra*. For details see [1], [13], [32], [61], [66]*, [74], [99]*, [104].

This problem illustrates themes (a), (b), (d), (e), (f), and (g).

5. Why is (-1)(-1) = 1?

This is an instance of the general problem of the (logical) justification of the laws of operation with negative numbers. It became a pressing problem (for both pedagogical and professional reasons) at Cambridge University around 1830. (In fact, the very existence of negative numbers came into question.) G. Peacock and others set themselves the task of resolving this problem by codifying the laws of operation with numbers. This was perhaps the earliest instance of *axiomatics in algebra*. The seeds of "abstract algebra" that emerge here are:

(i) The manipulation of symbols for their own sake (so called symbolical algebra); interpretation comes later.

(ii) Some freedom to choose the laws obeyed by the symbols.

We discuss some of these issues, focussing on the following:

(a) Reasons why the problem of the negative numbers became a burning issue at the time.
(b) Some proposed solutions, especially Peacock's, embodied in his "principle of the permanence of equivalent forms."
(c) Reactions to the symbolical approach to algebra.
(d) Implications of the symbolical approach for subsequent developments in algebra (e.g., the works of De Morgan, Hamilton, Boole, Cayley).

Pointing out some of the limitations in Peacock's development, we next take a more modern, Hilbertian approach to the problem of negative numbers. Just as Hilbert "defined" (characterized) the real numbers axiomatically as a complete ordered field, so we characterize the integers as an ordered integral domain in which the positive elements are well ordered. Once this is done we can, of course, prove such "laws" as $(-1)(-1) = 1$ (and more generally $(-a)(-b) = ab$), $a \cdot 0 = 0$, and others.

The following are some issues which we discuss in this context:

(a) How can we prove a law such as $(-1)(-1) = 1$? This question leads to the concept of axioms. (We cannot prove everything.)
(b) What axioms do we set down in order to give a complete description of (to define, to characterize) the integers? This question enables us to introduce the concepts of ring, integral domain, ordered structure.
(c) How do we know when we have enough axioms? This questions permits us to introduce the concept of completeness of a set of axioms (to be elaborated in Problem 9).
(d) What does it mean to characterize the integers? This questions sets the stage for the introduction of the concept of isomorphism. (We have characterized the integers by means of a set of axioms when any two systems satisfying these axioms are isomorphic. Thus, for example, the axioms for an ordered integral domain do not characterize the integers since the rationals are also an ordered integral domain, and the integers and rationals are not isomorphic, as can readily be shown.)
(e) Having characterized the integers, do we now have perhaps too many axioms? (In fact the commutativity of addition can be derived from the other axioms for an integral domain.) Here we come face to face with the concept of independence of a set of axioms (see Problem 9).
(f) Are we at liberty to pick and choose axioms as we please? This leads us to the concept of consistency of a set of axioms (again, to be elaborated in Problem 9) and, more broadly, to the question of "freedom of choice" in mathematics.

For details on the symbolical approach see [13]*, [89]*, [90], [91], [97]; as for the Hilbertian approach see [9], [75].

This problem illustrates themes (a), (b), (c), and (g).

REMARK ON PROBLEMS 3, 4, AND 5

Problems 3, 4, and 5 come from algebra and are an indication of the transition from "classical" algebra (the study of polynomial equations and laws of operation with "numbers") to "modern" algebra (the study of axiomatic systems). In fact, I often begin teaching a course in abstact algebra with Problem 5. Moreover, a "nonstandard" course in abstract algebra, in which "concrete" problems motivate the introduction of abstract concepts, can be structured around Problems 5, 1, 3, and 4.

6. Euclid's parallel postulate

This problem gave rise to the creation of *non-Euclidean geometry*, the re-evaluation of the *foundations of Eucliean geometry*, and the study of *axiomatics*. It is an excellent topic for raising many interesting issues (e.g., what is mathematics?) and, in particular, addressing *all* the themes

(a) to (g) mentioned above. For details see [5]*, [11], [15]*, [23], [33], [34], [35], [44]*, [54], [59], [63], [69], [77], [79], [111].

7. Uniqueness of representation of a function in a Fourier series

The study of *Fourier series* had a great impact on subsequent developments in mathematics. The problem of unique representation was addressed by Cantor and this led him to the creation of *set theory* and the clarification of the concept of *the (actual) infinite*. (For the origin of Cantor's set theory in the study of Fourier series see [21], [43]).

In this problem we are not concerned so much with Cantor's technical achievements in set theory as with his conceptual breakthrough in coming to grips with the actual infinite, and the consequences of this for mathematics beyond. On the technical side, we study some *cardinal arithmetic,* and *algebraic and transcendental numbers*. (Recall Cantor's proof that there is a continuum of transcendental numbers.) For details see [8], [15], [17]*, [18], [20], [34], [54], [63], [72]*, [79]*, [83], [93]*, [99], [103], [105]*, [111], [118].

This is an excellent topic for illustrating themes (a), (b), (d), (f), and (g).

8. Paradoxes in set theory

Various approaches to resolving Russell's paradox concerning the set $N = \{x: x \notin x\}$ led to different *axiomatizations of set theory* in the early 20th century. (E.g., Russell's theory of types forbids asking if $N \in N$; the Zermelo-Fraenkel theory forbids the formation of N; the von-Neumann-Gödel-Bernays theory classifies N as a class but not as a set.) among other causes, these axiomatizations led to various *philosophies of mathematics* (logicism, formalism, intuitionism). For details see [5], [8], [15]*, [16, vols. 1 & 2], [22]*, [34], [43]*, [60], [63]*, [79]*, [93]*, [111].

The problem helps illustrate themes (a), (b), (c), (d), and (f).

9. Consistency, completeness, independence

Here we study the *continuum hypothesis* and, especially, *Gödel's theorem* — one of the greatest mathematical achievements in this century — and their impact on mathematics and beyond. For details see [18], [22], [23], [34], [51], [60], [63], [65]*, [79], [81]*, [93]*, [98]*, [111].

These matters illustrate themes (b), (c), (f), and (g).

REMARK ON PROBLEMS 6, 7, 8, AND 9

In addition to illustrating the various themes as indicated, these problems relate to questions in the philosophy of mathematics, and especially to the fundamental question about the nature of mathematics.

GENERAL REMARKS ON THE COURSE

(i) The technical aspects of the course (which constitute about 1/3 to 1/2 of the course) are not very demanding. Many students, however, find the intellectual aspects challenging. To deal with ideas in mathematics, to be asked to read independently in the mathematical literature, to write "mini-essays," are tasks which mathematics students are not — but should become — accustomed to.

(ii) No textbook is used. However, *many* references are given and students are expected to go to the library and *read* some of them! (See the list of *References* below.)

(iii) The prerequisites for the course are any two mathematical courses. Students with only this minimum prerequisite are asked to take concurrently at least one or two more mathematics courses. (One is looking for the elusive quality of "mathematical maturity" rather than for specific technical proficiency.)

(iv) In a one-semester course one can deal adequately with only some (at most six) of the above nine problems.

OTHER PROBLEMS

Here are a few more problems (technically somewhat more demanding) which may be considered in such a course.

(a) The Königsberg Bridge Problem; the Euler-Descartes Theorem for polyhedra; the Four-Colour Theorem; (motivated the development of graph theory, topology).
(b) Measurement — length, area, volume; (motivated the development of the integral).
(c) "Exotic" functions; space-filling curves; (motivated the rigorization and arithmetization of analysis).
(d) Isoperimetric problems; other maxima and minima problems; (motivated the creation of the calculus of variations).
(e) Aspects of Fourier series; (led to a re-evaluation of a number of fundamental concepts of analysis such as function, integral, convergence).

REFERENCES

The books and articles listed below represent various levels of sophistication. Some are suitable mainly for the instructor; many can be read by students. The following "difficulty code" may be helpful: S = intended mainly for students. T = intended mainly for teachers. B = can be read with profit by both students and teachers.

The code is, of course, only a rough guide. In particular, it is not meant to discourage students from consulting teachers' sources. At the same time, teachers will find much that is of interest in students' sources. There are also very brief annotations of some of the less familiar *books*.

T 1. A.A. Albert (ed.), *Studies in modern algebra*. Math. Assoc. of America, 1963. A good survey of major themes of abstract algebra in the first half of the twentieth century.

B 2. A.D. Aleksandrov et al., *Mathematics: its content, methods, and meaning*. 3 vols. M.I.T. Press, 1963. A rich compendium.

S 3. R.B.J.T. Allenby, *Rings, fields and groups*. Edward Arnold Publ, 1983. A welcome addition to books on abstract algebra.

B 4. T.M. Apostol, *Introduction to analytic number theory*. Springer-Verlag, 1976. Comprehensive and well-written text.

B 5. S.F. Barker, *Philosophy of mathematics*. Prentice-Hall, 1964. The focus is on geometry (mainly non-Euclidean) and on number concepts.

T 6. E.T. Bell, *The development of mathematics*. McGraw Hill, 1945 (2nd ed.). Lively but provocative.

S 7. E.T. Bell, *Men of mathematics*. Simon & Schuster, 1965. Colorful but overstated.

T 8. P. Benacerraf & Putnam, *Philosophy of mathematics: selected readings*. Prentice-Hall, 1964 (reprinted by Cambridge Univ. Press, 1983). A good selection of classics on set theory and logic.

S 9. G. Birkhoff & S. MacLane, *A survey of modern algebra.* 4th ed. Macmillan, 1977 (1st ed., 1941). A classic; still well-worth reading.

S 10. E.D. Bolker, *Elementary number theory: an algebraic approach.* W.A. Benjamin, 1970.

I 11. R. Bonola, *Non-Euclidean geometry.* Dover, 1955 (orig. 1911). An historical outline.

I 12. Z.I. Borevich & I.R. Shafarevich, *Number theory.* Academic Press, 1966. An excellent book on advanced topics.

S 13. C.B. Boyer, *A history of mathematics.* Wiley, 1968.

I 14. F.E. Browder, Does Pure Mathematics Have a Relation to the Sciences? *American Scientist* 64 (Sept./Oct. 1976), 542-549.

S 15. D.M. Burton, *The history of mathematics: an introduction.* Allyn & Bacon, 1985. A welcome addition to books on the history of mathematics. At the level of Boyer's book (item 13 in this bibliography), but contains more technical mathematics. Useful references at the end of each chapter.

B 16. D.M. Campbell & J.C. Higgins (eds.), *Mathematics: people, problems, results.* 3 vols. Wadsworth, 1984. A nice, varied selection of readings.

S 17. R. Courant & H. Robbins, *What is mathematics?* Oxford University Press, 1941. A classic.

I 18. J.N. Crossley et al., *What is mathematical logic?* Oxford University Press, 1972. Five brief essays and an historical survey, by experts in the field.

I 19. J.N. Crossley, *The emergence of number.* Upside Down A Book Company (Victoria, Australia), 1980. Historical survey of the natural, real, and complex numbers.

S 20. T. Dantzig, *Number: the language of science.* Macmillan, 1930 (Free Press, 4th ed., 1967). Still well-worth reading.

I 21. J.W. Dauben, *Georg Cantor: his mathematics and philosophy of the infinite.* Harvard University Press, 1979. Scholarly biography.

B 22. P.J. Davis & R. Hersh, *The mathematical experience.* Birkhäuser, 1981.

I 23. H. De Long, *A profile of mathematical logic.* Addison-Wesley, 1970.

I 24. R.A. De Millo et al., Social Processes and Proofs of Theorems and Programs, *The Mathematical Intelligencer* 3 (1980), 31-40.

I 25. L.E. Dickson, Fermat's Last Theorem and the Origin and Nature of the Theory of Algebraic Numbers. *Ann. Math.* 18 (1916-17), 161-187.

I 26. J. Dieudonné, *A panorama of pure mathematics.* Academic Press, 1982. A technical survey of themes from 20-th century mathematics as seen by Bourbaki.

S 27. H. Dörrie, *100 great problems of elementary mathematics: their history and solution.* Dover, 1965. A very nice collection. Excellent source for students' projects.

S 28. U. Dudley, Formulas for Primes. *Math. Mag.* 56 (1983), 17-22.

I 29. H.M. Edwards, *Riemann's zeta function.* Academic Press, 1974.

I 30. H.M. Edwards, *Fermat's last theorem: a genetic introduction to algebraic number theory.* Springer-Verlag, 1977. A delightful book for the expert in the field.

S 31. H.M. Edwards, Fermat's Last Theorem, *Scientific Amer.* 239 (Oct. 1978), 104-122.

S 32. H. Eves, *An introduction to the history of mathematics.* 5th ed. Saunders College Publ., 1983 (orig. 1953).

S 33. H. Eves, *Great moments in mathematics (before 1650).* Math. Assoc. of Amer., 1980.

S 34. H. Eves, *Great moments in mathematics (after 1650).* Math. Assoc. of Amer., 1981.

S 35. H. Eves & C. Newsom, *An introduction to the foundations and fundamental concepts of mathematics.* Holt, Rinehart & Winston, 1958.

I 36. S. Feferman, What Does Logic Have to Tell Us About Mathematical Proofs? *Math. Intelligencer* 2 (1979), 20-24.

I 37. D. Fendel, Prime Producing Polynomials and Principal Ideal Domains. *Math. Mag.* 58 (1985), 204-210.

S 38. H. Flanders, A Tale of Two Squares — and Two Rings. *Math. Mag.* 58 (1985), 3-11.

S 39. M. Gardner, Patterns in Primes are a Clue to the Strong Law of Small Numbers. *Scientific Amer.* 243 (Dec. 1980), 18-28.

B 40. C.C. Gillispie (ed.), *Dictionary of scientific biography.* 16 vols. Scribner's, 1970-1980. Scholarly but very readable. An excellent source.

B 41. J. Grabiner, Is Mathematical Truth Time-Dependent? *Amer. Math. Monthly* 81 (1974), 354-365.

I 42. J. Grabiner, Changing Attitudes toward Mathematical Rigor: Lagrange and Analysis in the Eighteenth and Nineteenth Centuries. In: *Epistemological and social problems of the sciences in the early 19th century,* ed. H. Jahnke & M. Otte. D. Reidel, 1981, pp. 311-330.

B 43. I. Grattan-Guinness (ed.), *From calculus to set theory, 1630-1910: an introductory history.* Duckworth, 1980. Six essays, written by well-known contemporary historians of mathematics.

S 44. M.J. Greenberg, *Euclidean and non-Euclidean geometries: development and history.* 2nd ed. W.H. Freeman & Co., 1980 (orig. 1972). Very suitable as a text.

S 45. E. Grosswald, *Topics from the theory of numbers.* Macmillan, 1966. A nice selection from algebraic and analytic number theory. Recently reprinted.

S 46. R.K. Guy, Conway's Prime Producing Machine. *Math. Mag.* 56 (1983), 26-33.

I 47. R.W. Hamming, The Unreasonable Effectiveness of Mathematics. *Amer. Math. Monthly* 87 (1980), 81-90.

B 48. G. Hanna, *Rigorous proof in mathematics education.* Ont. Institute for Studies in Educ. Press, 1983.

B 49. G.H. Hardy & E.M. Wright, *An introduction to the theory of numbers.* 4th ed. Oxford University Press, 1960 (orig. 1938). A classic.

S 50. I.N. Herstein, *Topics in algebra.* Blaisdell Publ. Co., 1964.

I 51. D.R. Hofstadter, Analogies and Metaphors to Explain Gödel's Theorem. *Two Yr. Coll. Math. Jour.* 13 (1982), 98-114.

I 52. J.P. Jones et al., Diophantine Representation of the Set of Primes. *Amer. Math. Monthly* 83 (1976), 449-464.

I 53. M. Kac & S.M. Ulam, *Mathematics and logic: retrospect and prospects.* The New Amer. Library, 1969.

S 54. E. Kasner & J.R. Newman, *Mathematics and the imagination.* Simon & Schuster, 1967 (orig. 1940). Engaging, even after forty-six years.

I 55. J.G. Kemeny, Rigor vs. Intuition in Mathematics. *Math. Teacher* 54 (1961), 66-74.

S 56. I. Kleiner, The Distribution of Prime Numbers Among the Integers. *Ontario Math. Gaz.* 20 (1982), 20-26.

S 57. I. Kleiner & S. Avital, The Relativity of Mathematics. *Math. Teacher* 77 (1984), 554-558 & 562.

I 58. I. Kleiner, The Evolution of Group Theory: A Brief Survey. *Math. Mag.* 59 (1986).

S 59. M. Kline, *Mathematics in western culture.* Oxford Univ. Press, 1953.

S 60. M. Kline (ed.), *Mathematics in the modern world.* W.H. Freeman & Co., 1968.

B 61. M. Kline, *Mathematical thought from ancient to modern times.* Oxford Univ. Press, 1972. Half the book (ca 600 pages) is devoted to the history of mathematics of the 19th and early 20th centuries.

S 62. M. Kline (ed.), *Mathematics: An introduction to its spirit and use.* W.H. Freeman & Co., 1979.

B 63. M. Kline, *Mathematics: the loss of certainty.* Oxford Univ. Press, 1980.

B 64. G. Kolata, Mathematical Proofs: The Genesis of Reasonable Doubt. *Science* 192 (June 1976), 989-990.

B 65. G. Kolata, Does Gödel's Theorem Matter for Mathematics? *Science* 218 (Nov. 1982), 779-780.

B 66. A.G. Kurosh, *Lectures on general algebra.* Chelsea Publ. Co., 1963. Should be better known.

B 67. I. Lakatos, *Proofs and refutations: the logic of mathematical discover.* Cambridge Univ. Press, 1976. An important, recent perspective on the philosophy of mathematics.

S 68. Leapfrogs, *Imaginary logarithms*. Leapfrogs, 1978 Excellent elementary historical account
S 69. Leapfrogs, *Alternative geometries* Leapfrogs, 1980
S 70. Leapfrogs, *Complex Numbers* Leapfrogs, 1980
S 71 L.R. Lieber, *Galois and the theory of groups*. The Galois Institute of Mathematics and Art, 1932
S 72. L.R. Lieber, *Infinity.* Rinehart & Co , 1953. Charming.
T 73. Yu. I. Manin, How Convincing is a Proof? *Math Intelligence* 2 (1979), 17-18
S 74. K O. May, The Impossibility of a Division Algebra of Vectors in Three Dimensional Space. *Amer. Math. Monthly* 73 (1966), 289-291
S 75 N.H McCoy, *Introduction to modern algebra* Allyn & Bacon, 1968 Good elementary text
B 76. R.K Merton, Singletons and Multiples in Scientific Discovery: A Chapter in the Sociology of Science *Proc. Amer Philos Soc.* 105 (1961), 470-486
S 77. H. Meschkowski, *Noneuclidean geometry*. Academic Press, 1964
S 78. H. Meschkowski, *Ways of thought of great mathematicians*. Holden-Day, 1964
S 79 H. Meschkowski, *Evolution of mathematical thought* Holden-Day, 1965
B 80. H.L. Montgomery, Zeta Zeros on the Critical Line, *Amer Math. Monthly* 86 (1979), 43-45
B 81. E. Nagel & J.R. Newman, *Gödel's proof.* New York Univ. Press, 1958. Both philosophical and technical aspects are considered. Very well written.
B 82 J.R Newman (ed), *The world of mathematics*, 4 vols. Simon & Schuster, 1956
S 83 I. Niven, *Numbers: rational and irrational* Random House, 1961
S 84. C.S. Ogilvy & J T. Anderson, *Excursions in number theory*. Oxford Univ Press, 1966. Delightful.
I 85 J. Pierpont, Mathematical Rigor, Past and Present *Bull Amer. Math Soc* 34 (1928), 23-53
B 86. H Pollard & H G. Diamond, *The theory of algebraic numbers* Math. Assoc. of Amer., 1975
I 87 C. Pomerance, Recent Developments in Primality Testing. *Math Intelligencer* 3 (1981), 97-105
S 88 C. Pomerance, The Search for Primes *Scientific Amer.* 247 (Dec. 1982), 136-147
I 89. H Pycior, George Peacock and the British Origins of Symbolical Algebra. *Historia Math.* 8 (1981), 23-45
B 90 H. Pycior, Historical Roots of Confusion Among Beginning Algebra Students: A Newly Discovered Manuscript *Math Mag.* 55 (1982), 150-156
I 91 H. Pycior, Augustus De Morgan's Algebraic Work: The Three Stages *Isis* 74 (1983), 211-226
S 92. F. Richman, *Number theory: an introduction to algebra* Brooks/Cole Publ Co , 1971 A good introduction to ideas of abstract algebra via concrete problems in number theory.
B 93. R. Rucker, *Infinity and the mind: the science and philosophy of the infinite* Birkhäuser, 1982 Engrossing. Written in an informal style.
B 94. W L. Schaaf (ed), *Our mathematical heritage*. Macmillan Co , 1963 A selection of essays on "what mathematics is about."
S 95 G F. Simmons, *Differential equations, with applications and historical notes*. McGraw-Hill, 1972. Excellent text.
B 96. C. Small, A Simple Proof of the Four-Square Theorem. *Amer. Math Monthly* 89 (1982), 59-61
B 97 G.C. Smith, De Morgan and the Laws of Algebra. *Centaurus* 25 (1981), 50-70
S 98. R.M. Smullyan, *What is the name of this book? the riddle of Dracula and other logical puzzles*. Prentice-Hall, 1978 Enchanting.
S 99. E. Sonheimer & A. Rogerson, *Numbers and infinity: a historical account of mathematical concepts* Cambridge Univ Press, 1981. Very readable.
S 100. H. Stark, *An introduction to number theory* Markham Publ. Co., 1970 A nice account, written by a master of the subject
B 101 L A Steen (ed.), *Mathematics today: twelve informal essays*. Springer-Verlag, 1978
I 102. I Stewart, The Science of Significant Form *Math Intelligencer* 3 (1981), 50-58
S 103 H. Tietze, *Famous problems of mathematics*. Graylock Press, 1965 An historical account. Little technical mathematics.
B 104 B L. Van der Waerden, Hamilton's Discovery of Quaternions. *Math Mag* 49 (1976), 227-234
S 105. N. Ya. Vilenkin, *Stories about sets* Academic Press, 1968. Written in an informal and engaging style. A great delight
I 106 A. Weil, *Number theory: an approach through history* Birhäuser, 1984. Scholarly account.
I 107. H. Weyl, A Half-Century of Mathematics. *Amer. Math Monthly* 58 (1951), 523-553
I 108 E.P. Wigner, The Unreasonable Effectiveness of Mathematics in the Natural Sciences *Comm. Pure & Appl Math.* 13 (1960), 1-14
I 109 R.L. Wilder, The Nature of Mathematical Proof. *Amer. Math. Monthly* 51 (1944), 309-323
I 110 R L Wilder, The Origin and Growth of Mathematical Concepts. *Bull. Amer. Math Soc* 59 (1953), 423-448
S 111. R.L. Wilder, *Introduction to the foundations of mathematics* Wiley, 1965
I 112 R.L. Wilder, The Role of the Axiomatic Method. *Amer. Math. Monthly* 74 (1967), 115-127
B 113 R.L Wilder, The Role of Intuition *Science* 156 (1967), 605-610
I 114. R.L. Wilder, Relativity of Standards of Mathematical Rigor. *Dict. of the Hist. of Ideas* 3 (1968), 170-177
S 115 R L Wilder, *Evolution of mathematical concepts: an elementary study.* Wiley, 1968
B 116. R.L Wilder, *Mathematics as a cultural system* Pergamon Press, 1981 On the same theme as his earlier book (item 115 above), but dealing with more advanced mathematics.
B 117 D Zagier, The First 50 Million Prime Numbers *Math Intelligencer* 0 (1977), 7-19
S 118. L. Zippin, *Uses of infinity*. Random House, 1962. One of over twenty elementary booklets in an excellent series (New Mathematics Library) sponsored by the Mathematical Association of America. (Our item 83 is another in this series)

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