The Student's Construction of Quantification

ED DUBINSKY, FLOR ELTERMAN, CATHY GONG

The ability to work with existential and universal quantification of logical propositions is one of the most important and useful tools for accessing a vast array of mathematical ideas Quantification is, on the other hand, one of the least often acquired and most rarely understood concepts at all levels, from secondary school on up — even, in many cases, into graduate school.

It is not hard to provide a litany of topics that are very difficult for students and appear to depend on quantification. At the very elementary level the difference between an algebraic equation which may have a solution and an identity which is "always" true is the difference between an existential and a universal quantification. The same is true of the distinction between proof and counterexample How many of us have found students puzzled by our rejection of their "proof" which consists of checking a small number of examples? Is it possible that the difficulty here is simply a lack of understanding of what it means for something to be *true for every value of x?*

Cornu [4] has suggested that lack of understanding of quantification is an often insurmountable barrier for students in developing a sophisticated understanding of limits and continuity Ralston [13] has expressed the opinion that quantification is too difficult for students in the first two years of university If both of these educators are correct, then this could help explain the lack of success our students have with understanding calculus [5] and a host of other topics.

This list includes linear independence, compactness, and inverses in groups, to mention only a few These concepts illustrate the very practical necessity for students to be able to express a quantified proposition in formal language, to negate the statement, and to reason about both the original statement and its negation. It therefore seems that finding something out about understanding quantification, how it is learned, and what we as teachers can do to help might contribute to the goal of improving all students' understanding of advanced mathematical ideas

We believe that to understand a mathematical concept an individual must construct something in her or his own mind; that it is possible, through research, to determine, in detail, ways in which this can take place; and that one can develop instructional treatments designed explicitly to stimulate the specific constructions suggested by the research.

It is the purpose of this paper to propose a constructivist analysis of learning the concept of quantification In a subsequent paper [8], we intend to report on an instructional treatment based on this analysis which makes heavy use of orchestrated computer experiences to induce students to make the constructions described here

Our analysis is based on and contributes to a general theory of mathematical knowledge and its acquisition that we are attempting to develop from the ideas of Jean Peaget as expressed for example, in [3]. (For earlier but somewhat more detailed descriptions of our theory, see [6], [7], [9].) According to this theory, what the individual constructs are *schemas* A schema is a more or less coherent collection of mental objects and mental processes for transforming objects. When faced with a new situation or, what we may refer to in mathematics as a *problem,* an individual is said to be *disequilibrated* and may attempt to *reequilibrate* by *solving the problem* The process of *equilibration* results in the construction or reconstruction of schemas and it is this construction process (which we refer to generically as reflective abstraction) for various mathematical concepts that our theory attempts to describe. There are a number of ways in which the constructions can occur. We will describe them in the next section and examples will be provided in the context of our application of the general theory to the particular concept, quantification.

It is important to note, however, that this general theory is not the sole source of our analysis Obviously, as mathematicians, we have an understanding of quantifications and we may have some awareness of the nature of this understanding. There is no way to avoid the effect, explicit or implicit, that our own ideas of quantification will have on the analysis This is useful because any description of quantification must not only be "mathematically correct" but must also embody all of the subtleties and ramifications of this complex concept and reflect its varying role in the full spectrum of mathematical endeavours.

On the other hand, there is more than one way to describe a mathematical idea As is the case with the development of number in children [11], not all of these correspond to how an individual might construct the concept. This can only be determined empirically by observing students in the process of trying to understand the concept. Their obstacles and successes can give important clues to the nature of the ongoing construction process which, of course, can only be observed indirectly

The observations considered in this paper are taken from an informal study of a Discrete Mathematics class taught by the first author in collaboration with the other two authors at the University of California, Berkeley, in Spring, 1986 Quantification was a major topic in this course and the instructional treatment used computer experiences with the programming language SETL [14]. The design of the experiences was based on our emerging notion of how this concept could be constructed by students.

After the topic was covered, each student was given an

For the Learning of Mathematics 8, 2 (June 1988)
FLM Publishing Association, Montreal, Quebec Canada

in-depth interview during which time he or she was asked to perform certain tasks (i.e. solve problems) connected with quantification and to explain their reasoning. Excerpts from these interviews will be given in Section 3. The full set of protocols are available from the first author.

In earlier papers we have discussed how our approach, combining a constructivist analysis with a particular way of using computer experiences, can be used to help students construct the concept of mathematical induction ([6],[7]) and other topics in discrete mathematics ([1],[2]). The present work is an attempt to extend these investigations to another important mathematics topic, quantification.

Generalities on schemas and their construction

A schema consists, in the first instance, of mental *objects* and *processes* for transforming objects. These two are not so different as might appear. According to Piagetian theory, the way that an object is known is through *actions* which the knower performs on it. This is as true of mental objects and mental actions as it is of physical objects and their transformations. Thus a part of learning involves the acquisition, through reaction to experience or even spontaneously, of new actions with which to know objects. This is a form of construction, of course, but a more profound development occurs when a person makes an internal construction corresponding to one or more of these actions. This is a form of reflective abstraction called *interiorisation* and the result is what we are calling a *process* in a schema. It is much more than a copy of the external action. It depends on the objects and processes which already exist for the learner and will often involve other schemas which the learner possesses. It is the intelligent response to an external stimulus and although it "represents" the external actions it also depends on the knowledge of the learner at the time that the interiorisation takes place. Piaget has considered that this construction is more than analogous to the biological notion of phenotype in evolution theory [12]. An important class of interiorisations is generated in the construction of mental processes corresponding to the actions of functions given by an explicit algebraic formula or implicit in the analysis of a problem. The ability to do this is a very useful tool in mathematical modelling, understanding composition functions, and many other topics.

Thus, processes come from objects in the sense that they are interiorised actions on objects. On the other hand, objects are contructed by *encapsulating* processes which is another form of reflective abstraction. Encapsulation in this sense is a Piagetian notion which echos a very modern trend in mathematics. It occurs, according to Piagetian theory, very early in the development of intelligence with the emergence of the concept of *permanent object*. This concept consists in the encapsulation of the process of performing transformations in space which do not destroy the physical object. In mathematics, one of many examples of encapsulation occurs in the construction of the *dual* of a *vector space*. The dual is another vector space whose vectors are obtained by encapsulating the linear function (each of which is a process) from the original vector space to the underlying field.

It is also possible to construct new processes from two or more existing processes by linking them together. This can be done by simple composition or, as in the case of mathematical induction (see [9]), a more complicated matching of "inputs" and "outputs". We refer to this kind of reflective abstraction as *coordination*.

Finally, there is *generalisation*. It can occur that a problem points to an existing object and a process which is already present for the learner and the situation suggests that this process may be applied to this object. If this has not been done before, then, for the subject, both the process and the object are transformed into slightly more powerful versions and this is also a kind of construction.

The schema for mathematical functions is an important example which illustrates this formulation. A student's function schema at some point may include the transformation of numbers through the application of algebraic formulas. The objects are numbers and the processes are the algebraic manipulations. At the same time the student may understand regular geometric shapes such as equilateral triangles, squares, etc., and be familiar with actions on them such as rotations and reflections. When these actions are interiorized to become internal processes, the student can think of them independently of any physical presence, and can compose or reverse them to imagine additional processes. These coordinations lead to the construction of a collection of processes for transforming geometrical objects. If the student comes to think about such transformations as functions whose domains and ranges are geometrical objects positioned in space — or as permutations of the vertices which is another kind of function, then the function schema has been generalized. Moreover, the transformation processes can themselves be encapsulated to become objects and the coordinations taken as functions or operations on transformations. This is not only a further generalisation of the function schema but leads to the mathematical concept of transformation groups.

The quantification schema

In the quantification schema, the objects are propositions and one kind of processes are interiorisations of logical operations on sets of propositions. Processes are encapsulated to obtain more complex propositions and in this way, the schema is generalised.

Another source of processes is the interiorisation of an internal process corresponding to a logical statement presented in natural language, or derived from a situation. The examples which we will consider in this paper are the following two statements.

Statement 1 For every tire in the library, there is a car in the parking lost such that if the tire fits the car, then the tire is red.

Statement 2 Amongst all the fish flying around the gymnasium, there is one for which there is, in every Computer Science class, a Physics major who knows how much the fish weighs.

We will describe the internal processes corresponding to such statements.

Finally, we will consider two kinds of actions on the objects: negation of a proposition and reasoning about a proposition.

The action of negation will be a major concern in our analysis of quantification. It is extremely important in mathematics, for example in making proofs by contradiction or looking for counter-examples. At a conceptual level, when complex logical statements are used to define a concept, it may be said that in order to understand what something is, it is essential to understand what it is not. Throughout the development of quantification the same three methods of negation are used.

- The most mechanical method of negating a proposition is to express it in formal language and apply rules such as DeMorgan's law from memory. We will call this *negation by rules.*
- A method of negation that is transitional towards using an understanding of the statement is what we will call *negation by recursion.* The statement is "parsed" so that it appears as a single operation between two propositions (which themselves may be composed of other statements). The idea is that this "top-level" form is negated, either by rules or by meaning, and then the individual parts are negated. For example, to negate an expression such as

  $$(A \Rightarrow (B \wedge C)) \Rightarrow (C \wedge (A \vee B)) \qquad (1)$$

  the first step is to see it as

  $$P \Rightarrow Q$$

  where $P = (A \Rightarrow (B \wedge C))$ and $Q = (C \wedge (A \vee B))$. This is negated to obtain $P \wedge Q$, after which Q is negated and the expression for that and for P are substituted back.
- The most powerful, although most difficult, method of negation is to *negate the meaning,* that is, the student has a mental representation of a set of situations that correspond to the statement being true and can then take the complementary set of situations which corresponds to its falsity. This configuration is then expressed formally, if desired. For example, if a statement depends on a variable, then its negation is a statement which depends on the same variable and has the property that for every value of the variable, the truth value of the negation is the opposite of the truth value of the original statement.

Thus, in its most advanced form, negation amounts to reasoning about a proposition, which is the second kind of action that we will consider. We will also consider reasoning in the form of making particular assumptions about a statement and then trying to determine what effect this has on its truth or falsity. For example, in Statement 1 above, what is known if it is assumed that there is at least one red car in the parking lot?

Our description of the construction of the quantification schema is divided into three stages. First, the preliminary stage which concerns the set of schemas which the student should possess in order to begin to understand and work with quantification; second, the single-level quantification in which an existential or universal quantifier is applied to a set of propositions; and finally, the full schema in which two or more quantifications are coordinated to form the complex logical statements of advanced mathematics.

Preliminary stage

At the very beginning, objects are restricted to simple declarations, statements of fact. The only internal processes that are present consist of checking (a possibly imagined) reality to determine the truth or falsity of the statement. Negation amounts to checking a different reality.

As the student progresses through this stage, the propositions become more complicated in two ways: linking two or more propositions by means of the standard logical operations (and, or, implies, etc.) and the introduction of variables whose values may be unknown or changing. Both of these give rise to internal processes. For the former, there is the act of coordinating several propositions with the logical connectors and, for the latter, the student must imagine running through all values of the variable in its domain set to see what is the declarative statement in each case. These processes are used for example, in thinking about the truth or falsity of the entire statement. The following phrase from Statement 1 illustrates both kinds of proposition.

if the tire fits the car, then the car is red

The variables are *tire, car;* the two simple declarations are, *the tire fits the car, the car is red* and the linking operation is *implication.*

The protocols of responses to the question of determining the truth or falsity of Statement 1 show that students struggle with these complexities. Here is one in which the student is explicit about the set (pile) of tires and of (all of the) cars. He is clearly iterating through these sets and checking the truth value in every case.

CHI: You have a pile of tires right here in the library and the cars are right across in the parking lot. You have to figure out whether its true or false. OK, so I'd take all those tires down to the parking lot and . . . for every tire that's there I would test to see whether . . .

I: "For every tire?" How would you select a tire?

CHI: I would just randomly pick a tire and see if it fits on, say the first car I saw. And if it fits . . . er . OK . . .

I: So you would pick a tire. What would you do for that tire?

CHI: I would put it on one of the cars. And if it fits and it wasn't red . . if the tire fits and it wasn't red, then this whole statement is false, because that one tire didn't work, and it has to work for all the tires.

I: That tire didn't work if you just find one car for which its false?

CHI: Oh! Oh! Oh! That's right . OK . . I forgot about that. Let's see . . .

I: You started right. Let's see, you're taking a tire . .

CHI: Oh, and I have to test it on all the cars. And if there is one of them that fits either it doesn't fit all the cars, or if it does fit a car, then that car is red. For every one of those tires, if that's true, then the whole thing is true. And if one tire fails to meet that, then its false

At the end, he has difficulty with the quantification, but this is the business of a later section

In general, students appear to have considerable difficulty with negation. When the statement is complicated, the student will often focus on negating one part or one value of the variable and not consider the entire statement. Only after these statements are dealt with (cognitively, not explicitly) as Boolean-valued functions of one or more variables can the student understand the negation of the statement as a second statement which is another Boolean-valued function with the same variables and having the property that for any set of values of the variables, the truth value of this new statement (the negation) is exactly opposite to the truth value of the original statement, given the same set of values for the variables. We will see examples of this in Section 3 3

Single-level quantification

A single-level quantification is a proposition of one of the following two forms

$$\forall x \in S, P(x) \qquad \exists x \in S \ni P(x)$$

which are read, respectively, *for all x in S, P(x) is true* and *there exists an x in S such that P(x) is true*. Referring back to our examples of Statements 1 and 2, the following propositions of this kind appear.

> there is a car such that if the tire fits the car, then the car is red
>
> in every computer science class, the rest of the statement is true.

We may consider the single level quantification in a more mathematical context. For example, if $b_1, \ldots, b_k$ is a list of integers, then $b_1 > 5, \ldots, b_k > 5$ is a list of propositions and one can use it to construct the following single proposition,

$$(b_1 > 5) \land (b_2 > 5) \land \ldots \land (b_k > 5)$$

Our concern is that the student should see this expression, not as a piece of formal syntax, but as a "signifier" of a situation in which there is a bunch of numbers (k of them, actually) and each one has a certain relation (positional?) to the number 5. There are variations of this picture, some of them, corresponding to the situation in which our statement is true, that is, all of the numbers are greater than 5 and some of them in which it is not, that is, some of the numbers are ≤ 5. We would expect the student to be able to translate back and forth between the formal and descriptional specifications of the proposition

In order to construct the quantification schema at this level, we suggest that the student must coordinate the two developments described in the previous section in connection with simple declarations. A list of propositions as in (2) must first be seen as a Boolean-valued function on a set $\{1, \ldots, n\}$ of positive integers. For more general propositions, this domain must be generalized to an arbitrary set. The action of connecting (via conjunction or disjunction) all of the propositions in the list must be carried over to the more general situation of a proposition which depends on a variable and then must be applied to connecting all of the propositions that arise from evaluating the Boolean-valued function at all points in its domain.

The overall result is an interiorized process of iterating through a set of propositions depending on one or more variables and applying a quantification to form a single proposition whose value is the truth or falsity of either all of them (universal quantification) or at least one of them (existential quantification).

We asked students in this study to explain how they (or the computer) would determine the truth or falsity of Statements 1 and 2. If the interiorised process has been constructed, then it is possible that it will be revealed in the form of an explanation that describes an iteration through a set, and evaluating a proposition at each point. The additional feature that distinguishes this stage from the end of the preliminary stage is that the iteration is controlled by the quantification (Notice how CHI above (page 6) seems to be confused about quantifying his iteration through all of the cars.) Contrast this with the following student. She keeps the tire fixed and seems confused about the implication, but she does iterate over the set of cars and controls her iteration with an existential quantifier

I: Well, okay, say I just gave you some. There is a bunch of tires in this room. What would you have to do to decide if this is a true statement or not?

NGU: Outside of taking it outside and testing it, I guess that . . .

I: That's fine, what would you do? How would you test them?

NGU: Take it outside and look for a car that, I guess look for any car that's red, and then all the cars that are red, and then test the tire to see if it fits. Okay, and if, if one of them fits, the statement is true. And if none of them fits, then the statement is false

According to our analysis, the next step in the development of single-level quantification is to encapsulate the interiorised process into an object. Once this is done (or, more accurately, as part of doing this) actions can be applied to the object. One action is to reflect on it and reason about it, for example, to realize that in Statement 1, if there is one red car in the parking lot, then the statement must be true. Another action that can be applied is negation. We can consider negation of a single-level quantification in the context of the three methods of negation described earlier.

- The rule for negating a single-level quantification is to replace the universal (alt existential) quantification by an existential (alt universal) quantification and to replace the Boolean function by the same function but with its truth values reversed

- Negation by recursion is very similar to the rule It consists of making the change of the quantification and then negating the rest of the statement. This ephemeral distinction becomes sharper and useful in higher level quantifications
- Negation of the meaning consists in realizing that a universal (existential) quantification asserts that every (at least) one of a collection of propositions has the value true, so its negation is the assertion that at least (every) one of them is false.

Before passing on to higher level quantifications it is important to note that the encapsulation of single-level quantifications is critical for working with several quantifications. The single-level quantifications must be objects so that an operation (composition) can be applied to them to obtain a new object

Two, three, many quantifications
It is at this point that quantification goes beyond the syllogisms of Aristotle ("All men are mortal," "Socrates is a man," etc) and moves towards the complex ideas of advanced mathematics. (For every –, there is a ⅛ such that for every x,.) Considering Statement 1 as a whole, we have already seen examples (NGU page 8) where the student's explanation (which could be an indication of her internal process) has little connection with the actual meaning of the statement Here is an example with Statement 2.

WON: Well, you look into every Computer Science class and you try, if you can, to find a student who knows how much a fish weighs in the gymnasium. Any fish in the gymnasium Only a student has to know how much that fish, any fish, weighs

I: Okay, any fish For any fish?

WON: Yeah, any fish

I: How many students have to know?

WON: Only one student

I: Just one student?

WON: Yeah, well it doesn't, okay, at least, one; I could say at least one, not . . .

Interiorisation
So, how is a student to go about reconstructing the quantification schema at this higher level? Our suggestion is that he or she begins by taking two single-level quantifications which have been encapsulated and thinks in terms of combining those objects to make a new object. This is done by returning to the two internal processes corresponding to these two propositions and linking them together by substituting the entire second proposition for the proposition-valued function that the first process is quantifying. Then the resulting process is interiorised and finally encapsulated to obtain the new object. Thus, the schema for a single-level quantification is extended by coordinating two instances of it to reconstruct it at a higher level

A major cognitive skill (or act of intelligence) that we feel is required here is the ability to move back and forth between an internal process and its encapsulation as an object This specific mental activity occurs in working with a number of major concepts in mathematics, in particular, the concept of function.

Thus, for example, if a student encounters Statement 1 (either explicitly presented or arising in some context) then we suggest that understanding this statement includes the construction of an internal process as follows First the students must decompose the statement into two propositions such as

> For every tire in the library, it is the case that . there is a car in the parking lot such that if the tire fits the car, then the car is red.

or, to use mathematical symbols to express these two encapsulated processes,

$$\forall t \in L \; A(t)$$
$$\exists c \in P \ni F(t,c) \Rightarrow R(c)$$

where $\forall$ stands for *for all*, $\exists$ stand for *there exists*, $\in$ means *is an element of*, $\ni$ means *such that*, L is the set of tires in the library, P is the set of cars in the parking lot, $A(t)$ is the truth or falsity of the statement in the second line, $F(t,c)$ is the truth value of the statement that tire t fits car c, and $R(c)$ asserts that car c is red

It is important that the student mentally converts these objects into two dynamic processes The first corresponds to iterating t through L, checking the value of $A(t)$ and controlling the iteration with a universal quantifier The second requires that a particular t is given and then c is iterated through P, checking the value of the implication $F(t,c) \Rightarrow R(c)$ and controlling this iteration with an existential quanitifier It is these two processes which must be linked together to form a new internal process that will be the student's understanding of Statement 1 The linking consists of replacing $A(t)$ in the first expression by the entire second expression

If the student has made this construction and is then asked to explain how to determine whether Statement 1 is true or false, the response might contain some form of a description of the constructed process. The protocols show a very definite progression on this point. It begins with students like the following who is clearly making two iterations but without any sign of quantifiers and, perhaps as a result, the coordination of the two iterations is very weak.

AOK: Well, you would iterate through every tire in the library and you would look at every car in the parking lot and see if the tire fit the car and if the car is red. Run through all the set of tires in the library and cars in the parking lot and see if this is true or not

There is a gradual improvement in the protocols culminating with succinct mathematically mature formulations as in the following.

ELK: Well, let's see. One way to do it would be to look at every tire in the library and then go out and find at least one car in the parking lot so that if *this* tire fits the car and check to see if the car is red

The interiorised process corresponding to a two-level quantification must be encapsulated in order to proceed to three and higher-level quantifications. If there really are, say, three variables present, then the effect of grouping two of them to form a two-level quantification and then encapsulating the process of making this quantification is to obtain a proposition-valued function of a single variable This can be quantified to obtain the process for a three-level quantification The entire activity can then be repeated indefinitely for higher level quantifications

Here is an example in which the encapsulation of the two-level quantification is made and used explicitly.

VLA: Okay, I would look at a set of fish among the set of all available fishes and I would have to iterate over that and of course the condition is that as soon as I find the first one for which the rest of the long expression holds, I stop right then and there

I: Can you explain to me what would be the rest of the whole expression? How would you check that?

VLA: Yeah, that was just the first step

I: Good

VLA: I got . . . I'm picking a fish and then I have to start iterating over a set of available classes Here I'll have to go through everyone of them for that fish And then I would have to go through a set of students in the class Here we're dealing with an exists so that as soon as we find the first one that matches the rest of the conditions, its fine And then I would run that function on the student.

I: What would you ask about the student?

VLA: I would ask if the student knows the weight of the fish.

Notice how VLA keeps returning to the particular fish. This suggests that he may be thinking about the *rest of the long expression* as a proposition-valued function of fish. Notice that he is not extremely strong in his understanding and may be weakening when he tries to *match the rest of the conditions*. But he seems to recover by the end

Success in understanding the meaning of Statement 2 is particularly impressive because of the "traps" which are contained in the wording. The opening phrase, *Amongst all the fish* ... could mislead the student into thinking that the iteration over fish is controlled by a *universal* quantifier wheras the meaning of the entire statement forces it to be *existential*. Also, the insertion of the phrase *in every Computer Science class* in the middle of the quantification over students is confusing and could suggest that the order of iteration should be *exists fish, exists student, for all classes*. The purpose of these traps is to prevent students from interpreting the statement by following strategies of simple linguistic translation

This raises another point that might warrant a brief digression Sinclair in [15] and Freedman and Stedman in [10] argue that understanding such statements should not be used as an indicator of children's ability to reason logically because of the ambiguities of language and the fact that words used in quantifications mean different things linguistically than mathematically This is certainly the case, but the point here is that it is the business of mathematical analysis to resolve these ambiguities, at least in situations that arise in a mathematical context Statement 2 is given in phraseology that is intentionally confusing The issue for the student is not whether he or she guesses the correct linguistic meaning The question is whether he or she is able to construct a mental action and interiorise a process corresponding to the statement. If more than one action is actually specified by the statement, then ambiguity and uncertainty remain. In this case, however, as in most mathematical contexts, there is only one possible action that really corresponds to the statement and that is what resolves the ambiguity. To see this and make the construction is, in our opinion, a valid test of logical reasoning.

Our interpretation of understanding three-level quantifications is supported by another, overall observation we can report In every case in which a student, in explaining how the truth value of Statement 2 was determined, grouped two of the three iterations, the *Computer Science class* iterations was grouped with the *Physics major* iteration. No other grouping occurred This included all of the students who were successful in giving a complete explanation as well as those who were able to construct a two-level quantification but had difficulty incorporating the third. This strengthens our suggestion that the way this statement is interiorised is to first construct the process,

> for all Computer Science classes, there exists a Physics major who knows how much the fish weighs,

next interiorize this process and then encapsulate it to obtain a proposition valued function of the single variable fish and finally to apply an existential quantifier to this function

Negation

The above analysis of what it means to understand higher level quantifications is very helpful when the student attempts to negate the statement In order to observe this action and see how what the student does corresponds to the three ways of negating — by rules, by recursion, and by meaning — described above, we asked the students to negate Statement 1, in their head or using pencil and paper, and then we asked them to explain how they did it

Here is a protocol of a student who seems to be using rules His negation is correct as far as the quantification goes, but his negation of the implication is not correct

BOW: There is a tire in the library, such that, for all the cars in the parking lot, if the tire, if the tire doesn't fit the car, then the car is red.

I: If the tire doesn't fit the car, the car is red. Okay, can you explain to me what went through your mind as you tried to negate the sentence? And if it help to draw a picture, go ahead and do so.

BOW: Okay, the first thing was that I remember that the negation of the universal quantifiers, I guess the for, the for all and there is, is the opposite Because to me, there is, for all in the universe set, and the other thing to remember, that p implies q, the negation of that is q and not p. So I tried to negate the first part and then kind of the second. And that's basically from applied formulas

This protocol shows two things wrong with relying on one's memory for rules: they can be remembered incorrectly (the correct negation of "p implies q" is "p and not q") and, in the context of other complications (such as quantification) the remembered rule can be forgotten in the midst of applying it (the actual application here is "not p implies q" which is both incorrect and different from the remembered "q and not p") Notice also, in the explanation, the coordination between quantifications is lost

An important observation to report is that several students got the quantification part correct in their negation but negated the implication incorrectly — and every one of these seemed, in their explanation, to be using rules None of them said anything that sounded like "negation by recursion" On the other hand, every student who did explain her or his procedure by dealing first with negating the implication and then thinking about "the rest of the statement" (which is what we mean by recursion) got the negation correct Here is one example

AME: There is a tire in the library such that for every car in the parking lot, the tire fits the car and the car isn't red Right?

I: Very good ... When you were trying to negate the statement, what went through your head?

AME: Well, okay. First of all, I looked at it and I said, right, these are familiar constructs — "for every tire" in the library means, for all tires in the group of them in the library. And then, "there is a car" means that there exists a car in some other set and then "if the tire fits the car then the car is red" is an implication. So I sort of broke it down into parts

I: Would you say that you used an explicit procedure to negate this statement?

AME: Oh yes.

I: Can you describe it?

AME: Well, like when I first did it, I recognised these things so I sort of went through it mechanically So I knew that if you wanted to negate something, you know "for all *x*, *P*," then you say "exists *x* such that not *P*". And so on for the other two

Notice how she is very explicit about iterating the variables through sets, breaking the statement down, but then putting it together by coordinating the quantification If we infer that by *P* she means the rest of the statement, then this is a precise description of the method of negation by recursion.

All students who tried to negate the higher level quantifications directly from the meaning became totally confused Without a great deal of experience this is probably much too difficult.

Reasoning about propositions

A number of questions were asked to determine if the students were able to think about the meaning of the statements. They were asked, assuming that Statement 1 is true, to compare the possible numbers of tires with numbers of cars. Most said (or agreed when prompted) that all three relations were possible. A similar question was asked about Statement 2 with similar results Another type of question was to give the student additional information about the situation (no tire fits a car, there are no Physics majors taking Computer Science, etc.) and to ask them if this permitted any definite conclusions about the truth or falsity of the statement This was completely different from anything they had experienced in class and proved to be very hard.

Conclusions

We conclude this paper with a summary of the construction of the quantification schema according to our analysis We may call this description a *genetic decomposition of quantification* (cf. [6,7,9])

The construction begins with simple declarations that are made more complicated in two ways First, two or more declarations are coordinated by linking them with the standard logical connectors Second, variables are introduced and the learner interiorises the process of iterating this variable through its domain, checking the truth or falsity of the proposition-valued function for each value of the variable.

The transition from this preliminary stage to the stage of single-level quantifications is achieved by coordinating these two extensions. That is, the propositions that are obtained for the various values of the variable are all connected by a conjunction or they are all connected by a disjunction, in any case resulting in a single proposition. These connectors are replaced by a universal or existential quantification. Thus the learner interiorises a process of iterating a variable through its domain to obtain a set of propositions and applying a quantifier to obtain a single proposition.

The transition to two-level quantifications requires the encapsulation of this last process to obtain a single proposition as the result of quantifying a proposition-valued function Now when analysing a statement which is a two-level quantification involving two variables, the subject begins by parsing it into two quantifications. There is an inner quantification whose proposition-valued function also depends on an additional variable. There is also an outer quantification over this same additional variable. The idea is to coordinate these two objects to obtain a third which will be a two-level quantification. This proceeds by fixing the additional variable and applying the single-level quantification schema to the inner quantification and encapsulating the result to obtain a proposition. But this proposition depends on the value of the additional variable and so there is a proposition-valued function to which the other quantification may be applied, exercising the single-level schema once again. Altogether then, the learner has constructed and interiorised a process which is a nested iteration over two variables This process is again encapsulated to a single proposition so it will be possible to proceed to higher-level quantification

Given a statement which is a three-level quantification, the subject groups the two inner quantifications and applies the two-level schema to again obtain a proposition which depends on the outermost variable. This proposition-valued function is then quantified as before to obtain a single quantification.

The entire procedure can now be repeated indefinitely and with an awareness of this possibility, the learner has constructed a schema which can handle quantifications which are nested to any level.

References

[1] Baxter, N. & Dubinsky, E A new approach to teaching abstract mathematical concepts *The Journal of Mathematical Behaviour* in press

[2] Baxter, N., Dubinsky, E & Levin, G *Learning discrete mathematics with ISETL* New York: Springer, in press

[3] Beth, E.W. & J. Piaget, *Mathematical epistemology and psychology* (trans. W. Mays) Dordrecht: Reidel, 1966 (Original published 1965)

[4] Cornu, B. Apprentissage de la notion de limite: conceptions et obstacles Thèse de doctorat de troisième cycle. L Université Scientifique et Medicale de Grenoble. Grenoble, 1983

[5] Douglas, R G (ed.) *Toward a lean and lively calculus* Report of the Conference/Workshop to develop Curriculum and Teaching Methods MAA Notes, No 6 Math Assoc. Amer 1986

[6] Dubinsky, E. Teaching mathematical induction I, *The Journal of Mathematical Behaviour* 5, 305-317 (1986)

[7] Dubinsky, E. Teaching mathematical induction II. *The Journal of Mathematical Behaviour* in press

[8] Dubinsky, E On learning quantification. *The Journal of Mathematical Behaviour* in press

[9] Dubinsky, E & P Lewin, Reflective abstraction and mathematics education: the genetic decomposition of induction and compactness *The Journal of Mathematical Behaviour,* 5, 55-92 (1986)

[10] Freeman, N H & Stedman, J A How Children deal with natural language quantification In Kurcz, I , Shugar, G W & Danks J.H (eds.) *Knowledge and language* Amsterdam: North Holland, 1986

[11] Piaget, J *The child's conception of number* New York: Norton (Original published in 1941)

[12] Piaget, J. *Biology and knowledge* (trans B Walsh) Chicago: University of Chicago Press, 1971 (Original published 1967)

[13] Ralston, A private communication

[14] Schwartz, J.T., Dewar, R B.K , Dubinsky, E. & Schonberg E , *Programming with sets* New York: Springer, 1986

[15] Sinclair, H Constructivism and the psychology of mathematics. *Proceedings of the Eleventh International Conference of Psychology of Mathematics Education* Montreal, July, 1987, pp. 28-41

Corrections to the article:

Gert Shubring, On the Methodology of Analysing Historical Textbooks: Lacroix as Textbook Author *For the Learning of Mathematics* 7, 3 (November 1987)

page 41, right half, last para , line 6:
. . Lacroix was one of the few French . .

page 42, right half, first para , lines 8 and 9:
. . . at the same time: the Ecole Normale of the year III, the

page 42, last entry of the COURS COMPLET list:
Traité du Calcul différentiel et du Calcul intégral

page 43, right half, 3rd para , line 15:
. "Doctrine of forms or mathematics"

page 44, right half, last line should read:
the number of editions neither reliably expresses the success

page 47, right half, 2nd para., line 8
could not win a distinction

page 49, note 5, 2nd para line 10 the reference is:
Schubring 1981 a
This reference has to be supplemented on page 50
Schubring, Gert: (1981a) The Conception of Pure Mathematics as an Instrument in the Professionalization of Mathematics In: Mehrtens, H / Bos H / Schneider, I. (eds.): *Social history of nineteenth century mathematics* . Birkhäuser: Basel 1981, 111-134

page 49, note 20. A second paragraph has to be inserted:
The most decisive change was the replacement in 1803 of the "livres élémentaires" by "livres classiques": No longer was it the progress of science which gave the orientation for school knowledge, but rather the values of tradition and stability (Schubring 1984, 371-2). Consequently, school mathematics as an *open* system became replaced by a *closed* system of knowledge

page 50, entry Schubring 1981, line 3, read:
. . Bukarest 1981. In: Reihe Diskussi —

page 51, last entry:
the author's name is *Chevallard, Yves* and should therefore appear in the second place in list B