FUNCTIONS OF SYMBOLIZING ACTIVITY: A DISCUSSION

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Researchers who study mathematical learning have engaged in considerable debate about the role of social interaction in learning (Cobb, Jaworski, & Presmeg, 1996; Confrey, 1995; Davis, Sumara, & Kieren, 1996; Ernest, 1995; Lerman, 1996, 2000; Sfard, 2007; Steffe & Thompson, 2000a; van Oers 1996). In these debates, researchers using radical constructivism as a background theory have been criticized for not sufficiently attending to student-teacher communication (or student-student communication) (Davis, Sumara, & Kieren, 1996; Ernest, 1995; Lerman, 1996; Thompson, 2000; von Glasersfeld, 2000). Rather than rendering radical constructivism inadequate for studying student-teacher communication, these types of critiques can spur researchers who use this background theory to clarify and expand their positions (*cf.* Simon, 2009). It is in this spirit that I present a framework for studying the *functions* of students' and teachers' symbolizing activity.

The purpose of the article is to use the functions of students' and teachers' symbolizing activity to characterize student-teacher communication and to investigate how the functions contribute to acts of learning for both teacher and student. I lay the groundwork for this purpose by first describing key theoretical ideas – a scheme, symbolizing activity, communication, and learning. Then I suggest five ways that students' and teachers' symbolizing activity can function. Finally, I use these functions to analyze a sample piece of data from a three-year constructivist teaching experiment. The sample analysis illustrates how the functions of symbolizing activity help to characterize student-teacher communication, and how such communication contributes to a student's and a teacher's learning.

Schemes, symbolizing activity, communication, and learning

A *scheme* is a "package" of mental operations that are organized toward accomplishing a goal (Piaget, 1970). A scheme consists of three parts – an assimilatory mechanism, an activity, and a result (von Glasersfeld, 1995, 2001). When a person encounters an experiential situation, he or she uses current schemes to interpret it. Such interpretations may trigger records of prior operating, and the person may come to assimilate the situation as one that involves a particular type of activity. The activity of a scheme involves mental operations such as ordering, pairing, and uniting (joining), and a person uses these operations to produce a result.

When a person produces a material component during the functioning of his or her schemes (*i.e.,* a person creates some form of perceptually available material), then a person is engaged in what I call *symbolizing activity* (*cf.* Ernest, 2006; Kaput, Blanton, & Moreno, 2008). Symbolizing activity includes verbalizing (*e.g.,* a student saying the word "four" or providing a more extended verbal explanation), gesturing (*e.g.,* a student using his fingers to measure the length of a segment), and notating (*e.g.,* a student writing "13 × 5" or drawing a segment in a computer microworld). [2]

Although there are differences among these three forms of symbolizing activity, one reason to study them together is that when two or more people communicate, the different forms of symbolizing activity often function in concert with one another (Nemirovsky & Monk, 2000; Radford, Bardini, & Sabena, 2007; Kaput, Blanton, & Moreno, 2008). For example, when a student responds to a teacher's question, the student might gesture, respond verbally, and produce some form of notation. Similarly, a teacher's response to a student might involve more than a single form of symbolizing activity. For this reason, I situate my study of the functions of symbolizing activity within a model for communication – and view communication as an overarching function of symbolizing activity.

I consider *communication* to involve people in reciprocally assimilating what each person perceives to be other peoples' symbolizing activity (Thompson, 1999, 2000). Figure 1 illustrates the process of communication between two people. In Figure 1 person A might begin communicating with person B by generating some type of symbolizing activity (*e.g.,* a question, a verbal remark, a written problem). For communication to proceed, person B would need to assimilate person A's symbolizing activity, using available schemes. If person B perceives that the assimilated situation requires some type of response, then he would determine what an appropriate response would be, and generate his own symbolizing activity (a response to a question, a verbal remark, *etc.*). Then a similar process for person A might take place: Person A assimilates person B's symbolizing activity, determines an appropriate response, and generates further symbolizing activity. [3]

To illustrate this model of communication along with the ideas of scheme and symbolizing activity, I present a brief data excerpt between a teacher-researcher (me) and a student.

Excerpt 1: Carlos responds to the Outfits Problem [4]

T: You have four shirts and three pairs of pants. How many outfits could you make?
….
C: [Carlos draws a diagram of a shirt and writes "×4" and a pair of pants and writes "×3" (fig. 2)].

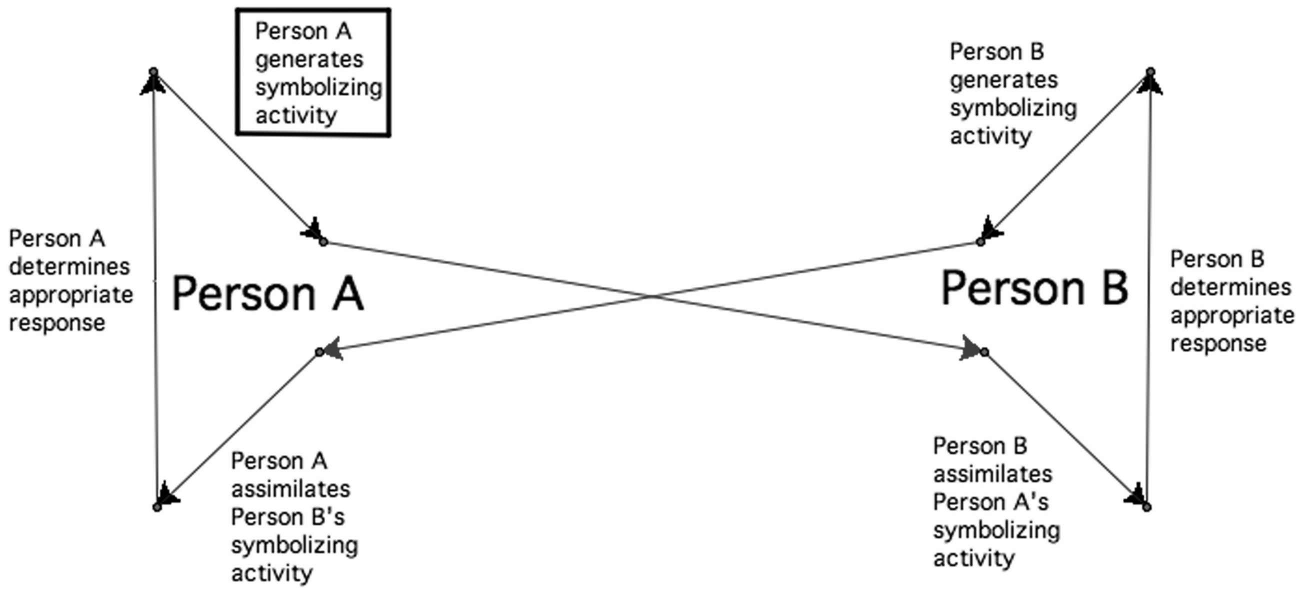

Figure 1. The boxed text marks the beginning of communication between two people; the arrows can be followed from there (cf. Thompson, 1999, p. 2)

….
C: [Carlos writes next to the shirt he has drawn "B, R, G, W" and next to the pair of pants "B, W, R." (fig. 3a)] Let me see. There would be the blue shirt and the blue pants and then there would be the blue shirt and the white pants, and then the blue shirt and the red pants [Carlos writes "BB," "WB," and "BR" in a column]. And then you would do that for each one. And then … [quietly] and then it could be [Carlos finishes writing out four columns (fig. 3b)]. So you count all these up and you get … there's three here, three here, three here, and three here. You add these two up you get six. You add these two up you get six. You add these two and you get twelve. [Carlos writes fig. 4.]
T: Okay

In this data excerpt, I (person A) began communicating with Carlos (person B) by presenting him with a question (line 1) – a form of symbolizing activity. Carlos's creation of Figure 2 was part of his establishment of an experiential situation, and so was part of how he assimilated the teacher's symbolizing activity. Figure 2 suggests that Carlos assimilated my question using two composite units – the shirts and pairs of pants. He then solved the problem by ordering the shirts – a first shirt, a second shirt, a third shirt, and a fourth shirt – as well as ordering the pairs of pants – a first pair of pants, a second pair of pants, and a third pair of pants. His symbolizing activity for this process was assigning a color to each pair of pants and a color to each shirt. He then paired [5] the first shirt with the first pair of pants, the first shirt with the second pair of pants, and the first shirt with the third pair of pants, and repeated this process for each of

Figure 2. Carlos's initial notation

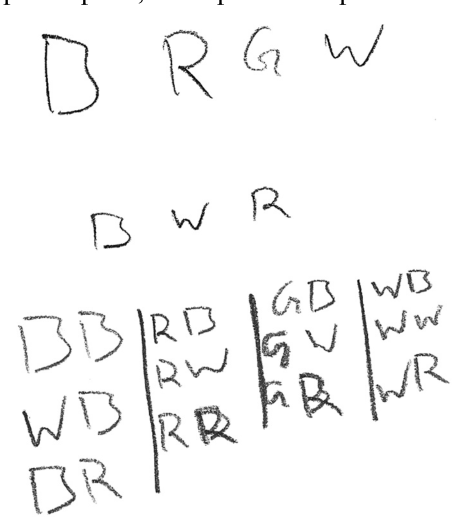

Figure 3. Carlos's notation for the shirts and pants (top) and his notation for the outfits (bottom)

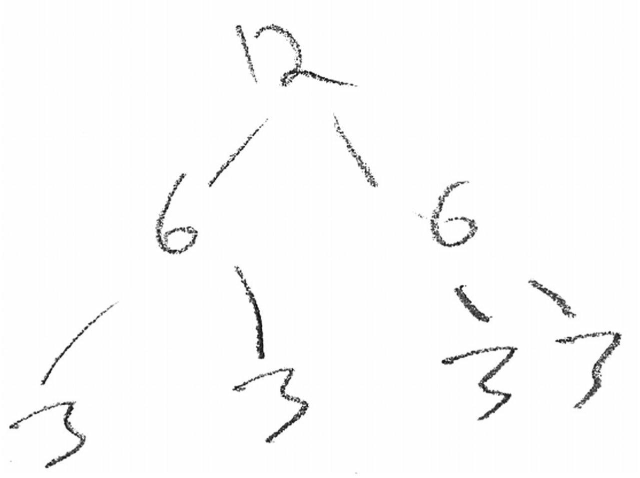

Figure 4. Carlos determines the total number of outfits.

the other shirts, symbolizing these actions with Figure 3. Finally, he united the outfits into four units of three, two units of six, and one unit of twelve, symbolizing these actions with his verbal statement and the numeral "12". My response, "okay", was intended to indicate to Carlos that I was able to assimilate his symbolizing activity to a way of operating that was sensible to me.

Although brief, this data excerpt situates schemes and symbolizing activity in the context of the model for communication that I presented above. In doing so, it illustrates the three parts of Carlos's scheme: He assimilated my question using two composite units – the shirts and pairs of pants; the activity of his scheme was ordering the shirts and pants, pairing shirts with pants to make outfits, and uniting the outfits together; and the result of his scheme was the total number of outfits he could make. The data excerpt also highlights the relationship between Carlos's symbolizing activity and mine. My symbolizing activity initially was the source for Carlos to establish an experiential situation. Then Carlos's symbolizing activity occurred during all three parts of his scheme, and it was sufficient for me to attribute a way of operating to him that was within the realm of my expectation for how a student might solve the problem, which led to my response, "Okay."

The above data excerpt, however, does not provide insight into significant learning on either Carlos's part or mine. That is, in the data excerpt, I found out that Carlos could solve the problem in a way that I could interpret, and so I did not need to modify my ways of operating (in the moment) to interpret what he had done. Similarly, Carlos's way of operating seemed solidly established, and so his response did not seem to involve learning on his part.

Here I use the term *learning* to mean a modification or adaptation of any of the three parts of a scheme or a reorganization of schemes in relation to each other (*cf.* Piaget, 1970; von Glasersfeld, 1995). Learning that occurs in the context of communication involves a person making adaptations to his ways of operating in relation to other peoples' symbolizing activity (*cf.* Thompson, 1999; 2000). In the case of student-teacher communication, a teacher's learning might involve the establishment of a new scheme that he or she attributes to a student, while a student's learning might involve a novel scheme that he or she creates in the process of problem solving activity (*cf.* Steffe & Thompson, 2000b).

Functions of symbolizing activity

To make more specific how students' and teachers' symbolizing activity functions, it is useful to specify the role and goals of the people engaged in communicating. For this reason, I situate my discussion of the functions of symbolizing activity within a discussion of the role and goals of teachers and students who are engaged in a constructivist teaching experiment. Specifically, I highlight one way that a teachers' symbolizing can function and four ways that a students' symbolizing activity can function. These five functions are not intended to be a comprehensive list of how symbolizing activity can function – rather, they grew out of the sample data that I present later. [6]

Teachers

Generally, one of the central goals of a constructivist teaching experiment is for the teacher to learn ways of operating from his or her students (Confrey & LaChance, 2000; Steffe & Thompson, 2000b). In order to achieve this goal, a teacher tries to act in harmony with students' current ways of operating by de-centering from his or her own ways of operating in order to see his or her students' ways of operating. In the process of harmonizing with a students' current ways of operating, a teacher poses problems, asks clarifying questions, and chooses particular tools (*e.g.,* a computer microworld or paper and pencil) aimed at *making students' ways of operating "visible"* (*cf.* Kaput, Blanton, & Moreno, 2008).

I use the term "visible" to mean that a student's symbolizing activity can open the opportunity for a teacher to make a model of the student's ways of operating. I do not intend the term to mean that a person's symbolizing activity necessarily makes his or her ways of operating obvious or clear to other people. Rather, engaging in symbolizing activity is a pre-condition for seeing another person's ways of operating. For example, in Data excerpt 1 Carlos's assimilation of my question and his subsequent response made his way of operating in the Outfits Problem visible to me because I could assimilate this way of operating to my expectation of how a student might solve this problem. The process of making students' current ways of operating visible constitutes one way that a teacher's symbolizing activity functions for him or her in interaction with students.

Students

The role of a student in a constructivist teaching experiment is to be open to working on problems that the teacher poses, to communicate with the teacher and other students about his or her reasoning about a problem, and to listen to and interpret the solutions of other students involved in the experiment. In this process, one way that a students' symbolizing activity functions is that it may be a necessary part of *enacting a way of operating.* I consider it to be a necessary part of enacting a way of operating when a student is unable to carry out his or her scheme without the use of some kind of perceptually created material; *i.e.,* the student is unable to carry out a scheme in visualized imagination. Determining whether and when a student is constrained to carrying out his or her scheme with the aid of some type of perceptual material requires more than a single observation. Nonetheless, a hypothetical response to the Password Problem is useful in illustrating this idea.

> *The Password Problem:* Your school requires that you have a two-letter password to log in to your computer account. How many possible passwords could you create?

In working on the Password Problem, a student might write in rows "A—26 letters, B—26 letters, …, Z—26 letters." Then he might count the number of rows while pointing to each row and write "26 × 26." A researcher might judge that the student's symbolizing activity was necessary for the enactment of his way of operating, in part because the student did not curtail his way of operating prior to concluding that the result was "26 × 26." Instead, the student wrote down all of the possible letters that could go with A, all of the possible letters that could go with B, and continued to do so all the way to Z, and then counted the number of rows. So, a researcher (or teacher) might judge that the student's symbolizing activity functioned as a necessary component of enacting his scheme.

A second function of a student's symbolizing activity is that it can involve *monitoring* and *self-regulation.* That is, students can use their symbolizing activity as a way to keep track of, or *monitor,* their ways of operating (*cf.* Thompson, 2000), which can in turn engender acts of self-regulation. By an act of self-regulation, I mean that a student re-initiates her activity in response to disequilibrium that she experiences while she is interacting with others (*cf.* Steffe's *auto-regulation,* 1991, 1994; Pirie & Kieren, 1989). Acts of self-regulation and monitoring are intertwined in that when a student engages in an act of self-regulation, it often indicates that she is monitoring her ways of operating. In turn, once a student has engaged in an act of self-regulation, it is common for her to find a new way to monitor her scheme, which may include a change in her symbolizing activity (*e.g.,* a student may use a new type of notation that better helps to monitor her operating).

Once again, a hypothetical response to the Password Problem can illustrate the ideas of monitoring and self-regulation. A student might respond to the Password Problem by writing the notation "AA, AB, AC, AD, AE" to symbolize the first five possible passwords. In doing so, the student's notation enables him to monitor which passwords he has created. After the student writes "AE", a teacher might ask the student whether he can use some other type of notation. In response, a student might begin using new notation by writing "A—26 other letters, B—26 other letters, C—26 other letters, *etc.,*" to symbolize the possible passwords. In re-initiating his activity, by using new notation, the student engaged in an act of self-regulation, which was occasioned by his original use of notation to monitor his solution, and his interaction with the teacher.

A third function of students' symbolizing activity is that it can serve as a way for a student to *explain* her solution. In such a situation, a student may carry out part or all of the activity of his or her scheme in visualized imagination without initially producing any gestures, verbal expressions, or written symbols. For instance, in the example above, a student might sit in silence while imagining putting the letter A with 26 letters, the letter B with 26 letters, *etc.* Subsequently, the student might write "26 × 26" to record the result of her way of operating. If asked why such a problem involves multiplication, the student might elaborate on her response with an explanation such as, "you could put the letter A with the letters A thru Z to make 26 passwords, then you could put the letter B with the letters A thru Z to make 26 more passwords, and you would do that twenty six times." In such a situation, the primary function of the student's symbolizing activity is to explain her solution.

A fourth function of students' symbolizing activity is that it can *point to the results of prior operating.* That is, as a student becomes proficient with solving particular problems, his symbolizing activity may point to the results of prior operating without him needing to carry out the activity of his scheme in full. In such a situation, the student no longer has to carry out activity either with perceptual material or in visualized imagination (*cf.* von Glasersfeld, 1991, 1995). For example, a student might respond immediately to the Password Problem by stating that the answer would be "twenty-six times twenty-six" or by writing down the corresponding notation. If asked why such a problem involves multiplication, a student might elaborate on his response with an explanation similar to the one in the previous paragraph. Because the student was able to provide an explanation, the student's verbalization "twenty-six times twenty-six" pointed to a way of operating that he could regenerate. However, this way of operating did not need to be carried out in order for the student to experience the situation as involving multiplication. The student simply assimilated an experiential situation to the results of prior operating.

Table 1 provides a summary of the functions of symbolizing activity that I have discussed above.

An illustrative example: functions of symbolizing activity

Now that I have discussed some functions of teachers' and students' symbolizing activity, I provide a data sample from a constructivist teaching experiment. The teaching experiment was conducted with eight middle school students during the students' sixth, seventh, and eighth grades. The

Functions of a Teacher's Symbolizing Activity	Functions of a Student's Symbolizing Activity
Making students' ways of operating visible	Enacting a way of operating
Opening the possibility for students to engage in acts of learning	Engaging in acts of monitoring and self-regulation
	Explaining a way of operating
	Pointing to the results of prior operating

Table 1. Functions of symbolizing activity

transcript presented here is taken from work with two eighth graders, Michael and Deborah, and illustrates the functions I have identified above. My analysis focuses on the functions of Michael's and my symbolizing activity in the interaction, and how these functions contributed to both of our learning.

Excerpt 2: Michael's solution to the Outfits Problem

T: Let's suppose in your closet you have three shirts and four pairs of pants. How many different outfits could you make?

M [Michael and Deborah respond immediately and at almost the same time]: Three.
D: Twelve.
T [responding to Deborah]: Twelve? [Looking at Michael] Let's see, can you show how you are thinking about that?
M [under his breath]: Wait.
T [Hands each student a piece of paper and pencil]: Here go ahead.
….
M [Begins to draw a picture of one pair of pants and one shirt and writes 4 next to the pants and 3 next to the shirts (fig. 5a).]: You could take one and make one outfit. [Michael draws a tally mark to the right of the shirt and one to the right of the pants (fig. 5b).] Take another one, [Michael draws a second tally mark to the right of the shirt and one to the right of the pants], and take another one [Michael draws a third tally mark to the right of the shirt and one to the right of the pants]. And then you have one pants [left over]. Wait, do we do every possibility?
T: Every possibility you could possibly get.
M [Michael draws three tally marks followed by four tally marks, making two rows of tally marks (fig. 6).]: You could do that, that, that [Michael connects the first tally mark in the second row with each tally mark in the first row.] You could do a lot of possibilities of this one. [Michael is silent while he draws a line from the second tally mark in the second row to each tally mark in the first row. He repeats this process for the final two tally marks in the second row.] There is three for every pants, so basically twelve.

Both Michael's and Deborah's initial response to the Outfits problem, "three" (line __) and "twelve" (line __) respectively, were immediate. I was surprised by Michael's response, and was unable to assimilate it to a way of operating. So I posed a question, "Let's see, can you show how you are thinking about that?" (lines __–__) Because Michael's and Deborah's initial responses were immediate, it is likely that both students used the number words *to point to* previous ways of operating, without carrying out the activity that would produce this result. In turn, my question was intended to *make Michael's way of operating more visible* because I was unsure why he had responded as he did.

Prior to making his way of operating more visible, Michael said "wait" (line __), indicating that my question and Deborah's response created a state of disequilibrium for him. At that moment, however, he proceeded with his solution that led to the result three. During his solution, he used written (figs. 5a & 5b) and verbal symbols (lines __–__) in order to *explain* to me why he concluded the result should be three. His explanation indicated that he paired the first pair of pants with the first shirt, the second pair of pants with the second shirt, and the third pair of pants with the third shirt, leaving one pair of pants left over.

At this point, I was able to assimilate Michael's symbolizing activity to a way of operating that made sense to me – so his symbolizing activity allowed me to see why he initially concluded the result should be three. For Michael, producing a left over pair of pants created a perturbation, and he asked the question, "Wait, do we do every possibility?" (line __–__). Michael's question suggests he *monitored* how he had paired shirts with pants to make outfits during his explanation (lines __–__). It seems likely that he monitored his initial solution because Deborah had responded differently than he had. By monitoring his activity he became aware that he should be able to use the left over pair of pants to make an outfit, just as he had used the first three pairs of pants to make outfits. Michael's monitoring, and my confirmation that he should do "every possibility [he] could possibly get" (line _), occasioned an act of *self-regulation* – he reinitiated his activity in response to his interaction with Deborah and me. In doing so, he produced new tally marks, drew lines from each of the tally marks in the first row to the first tally mark in the second row (fig. 6), and once again verbalized part of his solution (lines __–__).

After Michael had drawn lines from each of the tally marks in the first row to the first tally mark in the second row, (lines

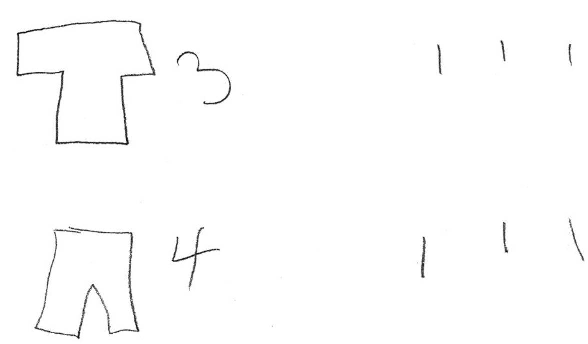

Figure 5. Michael's initial drawing and tally marks

Figure 6. Michael's tally marks

__-__), he did not yet seem to know the total number of outfits he would create, because he stated, "you could do a lot of possibilities of this one" (line __). In particular, he did not yet seem to know for sure why Deborah had responded, "twelve" (line __). So, this way of operating seemed novel to Michael. His new way of operating involved him in pairing the first shirt with all of the pairs of pants, the second shirt with all of the pairs of pants, *etc.* Because this way of operating seemed novel to him, his symbolizing activity seemed to be necessary for him to respond to the problem. He needed it to *enact* his new way of operating.

When Michael finished Figure 6 he concluded, "There is three for every pants, so basically twelve" (line __). One interesting difference between this verbal expression and Michael's initial response of "three" is that he had to carry out the activity of pairing the first shirt with the first pair of pants, the first shirt with the second pair of pants, *etc.* prior to providing it. This difference helps to highlight the distinction between when a student uses a verbal expression *to point to* the results of prior operating without having to carry out activity in a situation, and when a student may be constrained to using his symbolizing activity to *enact* a way of operating.

It is also noteworthy that Michael produced his solution that resulted in three outfits first, and only afterwards produced a second solution that resulted in twelve outfits. It may seem plausible that Michael carried out his initial solution only because I asked him to show me how he got that solution. However, over the course of the next three teaching episodes, Michael often initially solved similar problems as he initially solved the Outfits Problem, and only later incorporated his new way of operating. So, a more viable interpretation of why Michael carried out his initial solution is that he could not assimilate Deborah's verbal expression "twelve" using a way of operating that he had already established. Rather, Deborah's response of "twelve", as well as her comments in other situations, were an impetus for Michael to monitor his way of operating. Yet he still had to produce his already established way of operating (or the beginning of it) before he could make an interpretation of Deborah's responses.

What learning took place?

In this example, my learning of Michael's way of operating was occasioned by Michael elaborating on his initial response of three: His elaboration made his way of operating visible to me. This learning entailed my differentiation between two schemes that I attributed to Michael – a one-to-one correspondence scheme, and a lexicographic pairing scheme. The one-to-one correspondence scheme entailed his putting shirts and pants in one-to-one correspondence to make outfits, and led to the result of three outfits. The lexicographic pairing scheme involved his ordering the shirts and pants to differentiate between a first shirt, a second shirt, *etc.,* and then pairing shirts with pants to make twelve outfits. This distinction has been useful in my subsequent interactions with students around similar problems. Michael's learning involved an adaptation to all three parts of his one-to-one correspondence scheme (*i.e.,* the assimilatory part, the activity, and the result), which emerged over the course of the next three teaching episodes. The original site of this adaptation (*i.e.,* his solution of the Outfits Problem) was occasioned by my question and Deborah's different response. My question occasioned an explanation of his initial solution, and Deborah's different response occasioned him monitoring his initial solution, and occasioned his subsequent act of self-regulation.

Attending to the functions of symbolizing activity

I have proposed a framework for studying the functions of teachers' and students' symbolizing activity, built out of scheme theory and a conception of communication based on reciprocal assimilation of symbolizing activity. I have demonstrated how this framework helps to characterize student-teacher communication in mathematical interaction, and how it enables an analysis of how the functions of symbolizing activity contribute to students' and teachers' learning. However, the framework I have proposed raises a significant question that is worthy of consideration in further theoretical discussions and research studies: How can the functions of symbolizing activity be delimited? This question arises because the definition I have given of symbolizing activity is broad, and the number of ways that such activity might function in student-teacher communication is wide and varied. One approach for addressing this question is to tightly link the study of the functions of symbolizing activity to situations where a researcher deems learning has occurred for teacher and student. Tightly linking functions of symbolizing activity to moments of learning may help to delimit the functions under consideration. It also makes a central goal of developing the framework providing accounts of learning that are situated in the context of characterizing student-teacher communication. Therefore, this article, and the development of the framework, responds to von Glasersfeld's (2000) call for researchers who use a radical constructivist perspective to highlight the role of student-teacher communication in their research on learning.

Notes

[1] Portions of the research reported in this manuscript were part of my doctoral dissertation, *Students' construction of algebraic symbol systems,* completed in 2007 at the University of Georgia, Athens under the direction of Leslie P. Steffe. All research reported in the manuscript was conducted at the University of Georgia.

[2] I use the word *notation* broadly for any type of graphic record. Other researchers have differentiated among different kinds of graphic records, but I do not go to this level of detail here (*e.g.,* Kaput, Blanton, & Moreno, 2008, p. 29; Lehrer, Schauble, Carpenter, & Penner, 2000).

[3] Thompson (1999, 2000) notes that this definition of communication is compatible with symbolic interactionists who take as problematic the issue of how two physically disconnected systems can communicate with one another (*e.g.,* see, Bauersfeld, 1980; Cobb, Jaworski, Presmeg, 1996).

[4] In the data excerpts, T stands for the teacher-researcher (me), C stands for Carlos, M stands for Michael, and D stands for Deborah. Comments enclosed in brackets describe students' nonverbal action or interaction from the teacher-researcher's perspective, and four ellipses indicate omitted dialogue.

[5] *Pairing* is used to imply that a student has established an outfit as a new type of unit that has been created from putting a shirt with a pair of pants.

[6] I structure what follows by identifying that certain functions of symbolizing activity pertain to teachers and other functions pertain to students. However, it is not my intent to identify these functions exclusively with a student or teacher (*i.e.,* a teacher might use their symbolizing activity in a

way that I identify as a function of students' symbolizing activity). I simply structure my discussion this way because it fits with the sample data analysis I provide. For other examples of how symbolizing activity functions for teachers and students see the dissertation mentioned in note [1].

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