Tacit Models and Mathematical Reasoning

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The present paper deals with a number of examples of tacit, concrete models which influence the students' mathematical concepts and operations Since these types of models remain active even during the formal operational period, a new perspective of the Piagetian stage theory should be considered and special metacognitive procedures should be adopted to overcome the conflicts.

In his famous book "Knowing and Being" Michael Polanyi writes: . . "I shall speak of contributions made to scientific thought by acts of personal judgement which cannot be replaced by the operations of explicit reasoning. I shall try to show that such tacit operations play a decisive part not only in the discovery, but in the very holding of scientific knowledge" [Polanyi, 1969]

The concept of "tacit knowledge" should be considered as a fundamental one for scientific reasoning. In our opinion, cognitive psychology has not dedicated enough attention to it. One may assume that the Piagetian influence and the information processing approach have played a certain role in hindering the progress of systematic research in that direction In the present paper we intend to focus on one of the main aspects of tacit cognition, namely tacit models, but before continuing a remark is necessary In Polanyi's conception, the process of sense-giving (conferring a unitary meaning on a certain conglomerate of data) is based on an act of integration which is *basically* tacit. "No explicit procedure can produce this integration," writes Polanyi [1969, p. 191]. In our opinion, these tacit operations are not, as a matter of fact, inaccessible to an explicit analysis. They *are* accountable —in principle—by resorting to adequate means. This is an hypothesis of fundamental practical importance: If the tacit process of integration leads to incorrect solutions, *the remedial activity has the option of identifying and analysing these, initially hidden, mechanisms and submitting them to the individual's control* "Tacit" does not mean, in our view, mysterious, irrational, genuinely unaccountable It means only a way of increasing the productivity of the intellectual process This is, in fact, the basic task of a mental model acting as a substitute for a complex, abstract, difficultly accessible original A mental model makes an essential contribution in the integration process, in conferring a unitary and directly accessible sense on an ensemble of data. The process is considerably simplified and any conflict is avoided if the model imposes, tacitly, its constraints on the reasoning process.

Mathematical concepts and operations are basically abstract, formal constructs. Their meaning and their coherence are not guaranteed by empirical evidence but rather by axiomatic constraints.

The main psychological problem is that we are not *naturally* equipped to manipulate concepts and operations, the consistency of which is not supported by some empirical evidence. To think by manipulating pure symbols which obey only formal constraints is practically impossible Consquently, we produce models which confer some behavioral, practical, unifying, meaning to these symbols. Moreover, as we have said, these models tend to replace, tacitly, the original in our reasoning process. Very often, the model is suggested by the initial, empirical reality from which the mathematical concept has been abstracted.

What happens is that we continue to resort, tacitly, to the primitive sources of the abstract mathematical concepts, long after these sources should have lost their impact on the reasoning endeavour (as a consequence of mathematical education).

Although there is presently a growing interest concerning the role of models in the reasoning process (especially in science and science education), little has been said with regard to the role of mental models in mathematical reasoning

Many of the difficulties students are facing in science and mathematics education are due to the influence of tacit intuitive models acting uncontrolled in the reasoning process.

Given two systems, A and B, B may be considered a model of A, if, on the basis of a certain isomorphism between A and B, a description or a solution produced in terms of A may be reflected, consistently, in terms of B and vice versa

Models may be intuitive or abstract; external or mental; tacit or explicit; analogical or paradigmatic; primitive or elaborate

In the present paper we consider mainly mental, intuitive, tacit, primitive models More explicitly, we refer to representations of certain mathematical, abstract notions which develop themselves at an initial stage of the learning process and which continue to influence, tacitly, the interpretations and the solving decisions of the learner The term "tacit" means, simply, that the individual is not aware of that influence or, at least, of the extent of it.

Our thesis is that a main task of the psychologist interested in mathematics education is to identify such models and to suggest the means by which the student may become able to control their influence. In what follows we will consider a number of examples.

The concept of set

Linchevski and Vinner [1988] have analyzed a number of misconceptions held by elementary school teachers concerning the mathematical concept of set. They have identified the following misconceptions: (1) The subjects consider that the elements of a set must possess a certain explicit common property. (2) A set must be composed of more than one element. The ideas of an empty set or of a singleton are rejected. (3) Repeating elements are considered as distinct elements. (4) An element of a set cannot be an element of another set. (5) To these we may add a fifth common misconception, i.e. that two sets are equal if they contain the same number of elements

A very simple interpretation may account for all these misconceptions. If the model one has in mind when considering the concept of set is that of a *collection of objects*, all these misconceptions are predictable. An empty collection, or a collection containing only one object, are obviously nonsense We never constitute classes of objects absolutely unrelated conceptually (your name, a pair of old shoes, and the imaginary number i!). In every practical situation two identical elements, but existing separately (for example, two dimes), are counted separately The same object cannot be, at the same time, in two different containers Two collections of objects are considered equal if they contain the same number of elements

We do not affirm that the student identifies, explicitly, consciously, the mathematical concept of set with the notion of a collection of concrete objects. What we affirm is that, while considering the mathematical concept of set, what he has in mind—*implicitly* but *effectively*—is the idea of a collection of objects with *all* its connotations. There is no conflict here The intuitive model manipulates *from behind the scenes*, the meaning, the use, the properties of the formally established concept. The intuitive model seems to be stronger than the formal concept The student simply forgets the formal properties and tends to keep in mind those imposed by the model. And the explanation seems to be very simple: The properties imposed by the concrete model constitute a coherent structure, while the formal properties appear, at least at first glance, rather as an arbitrary collection. The set of formal properties may be justified as a coherent one only in the realm of a clear, coherent mathematical conception

In our opinion, the influence of such tacit, elementary, intuitive models on the course of mathematical reasoning, is much more important than is usually acknowledged. Our hypothesis is that this influence is not limited to the pre-formal stages of intellectual development Our claim is that even after the individual becomes capable of formal reasoning, elementary intuitive models continue to influence his ways of reasoning. The relationships between the concrete and the formal in the reasoning process are much more complex than Piaget supposed. The idea of a tacit influence of intuitive, primitive models on a formal reasoning process does not seem to have attracted Piaget's attention In fact, our information processing machine is not controlled only by logical structures but, at the same time, by a world of intuitive models acting tacitly and imposing their own constraints.

The equals sign

Let us consider a second example In a study published in 1977, Ginsburg pointed out that elementary school children interpret the signs + and = in terms of actions to be performed [Ginsburg, 1977]. As a consequence they would not accept a sentence like $\square = 3 + 4$ as such and would claim that it is written backwards

Ginsburg also found that children reject a sentence like $3 = 3$ as being meaningless because it contradicts the idea that the equals sign expresses a process by which, combining some ingredients, one produces a certain result Behr, Erlwanger and Nichols [1976] mention the same type of attitude in children in grades 1 to 6. The sentence $3 = 3$ is interpreted like $6 - 3 = 3$ and $7 - 4 = 3$ Similarly, a sentence like $4 + 5 = 3 + 6$ gave rise to the following commentary: "After "=" should be your answer. It's the end and not another problem." [Cf Kieran, 1981, p. 319]. The conclusion is that students tend to interpret the equals signs not in equivalence terms, but rather in the light of an input-output model from which the properties of reflexivity and symmetry are absent. In such a model one has, on the left side, the initial ingredients, while on the right side one represents their product If one intends to make a cake, one mixes, let's say, flour, sugar, milk, eggs, etc (*several* ingredients) and one obtains finally the cake, *one* product.

Lesley Booth [1988], referring to the students' attempts to simplify expressions such as $2a + 5b$, writes: "In arithmetic, symbols such as + and = are typically interpreted in terms of actions to be performed, so that + means to actually perform the operation and = means to write down the answer " [Booth, 1988, p. 14] Consequently, one may find students who write: $2a + 5b = 7ab$ or $a + b = ab$, but who will never write $ab = a + b$ or $7ab = 2a + 5b$. Similar results have been found in twelve-to-fourteen-year-old students [Kieran, 1981] and even in seventeen-year-old students [Wagner, 1977].

These are not accidental mistakes. They can be explained if one grasps the genuine but hidden meaning the students confer on the equals sign: a sign indicating the transformation of initial (multiple) ingredients into a final (unique) product.

It is only an illusion that, with time, as the students proceed to higher classes, they learn to confer on the equals sign the equivalence meaning rather than the operational meaning Asking high school or college students to find the decimal equivalent of $1/3$, they willingly write $1/3 = 0.3333\ldots$ On the other hand, they would hardly accept that $0.3333\ldots$ *equals* $1/3$ They will claim that 0.3333 *tends* to $1/3$. Consequently, the equals sign does not represent, for those subjects, a symmetric relation Intuitively, the relation $1/3 = 0.3333\ldots$ is accepted, while the relation $0.3333 = 1/3$ is not accepted because one can never have on the left side *all* the needed ingredients (an infinity) to produce the result. The asymmetry is an effect of the asymmetrical, tacit, input-output model

The operation of subtraction

Let us consider a third example drawn from the recent literature, namely the operation of subtraction. One knows, today, that students make various systematic mistakes in performing subtraction and many such "bugs" have been identified. I do not intend to enter into details. I want only to specify that at least a number of these bugs might be predicted from the primitive role of subtraction.

If you have in a container a number A of objects (for instance marbles) and you want to take out a number of them, B, (the primitive model of the operation of subtraction) you can do it only if $B < A$. If $B > A$ the student will tend to reverse the operations $B - A$. For instance [Resnik, 1983, pp. 73]:

$$\begin{array}{r} 326 \\ -\underline{117} \\ 211 \end{array}$$

Another possibility, derived from the primitive model, is just to write down) when $B > A$: You take out from the container as much as you can and the container remains empty. For instance:

$$\begin{array}{r} 542 \\ -\underline{389} \\ 200 \end{array}$$ [Resnik, ibid]

If the student has learnt the patent of "borrowing," several situations may occur. The most typical difficulty appears when the student has to "borrow" from 0. If $B > A$ you borrow from the next container, but if this container is empty, then you may write 0, or you may borrow from the bottom, or you may skip over the empty container and try a third one.

Borrow from bottom instead of zero:	$\begin{array}{r} 702 \\ -\underline{368} \\ 454 \end{array}$
Borrow across zero:	$\begin{array}{r} 602 \\ -\underline{327} \\ 225 \end{array}$

[For misconceptions in subtraction see also Maurer, 1987, and Resnik, 1982]

The division of a segment and of a wire

In the course of a research now in progress, the following questions were asked [Fischbein, Oster, Stavy and Tirosh, 1988]:

(1) "Let us consider a segment AB. Let us divide the segment into two equal parts. Let us divide each equal half again into two equal parts. Let us continue to divide the segments obtained in the same way. Will the process of dividing come to an end?"

(2) "Let us consider a length of copper wire. Let us divide the wire into two equal parts. . . . ." And the question continues in exactly the same way as in question (1)

In fact, the two problems are fundamentally different. In the first problem, one considers a geometrical segment, not a material one, and the process of division may go on, in principle, indefinitely.

In the second problem one considers a material wire, namely copper. In this case, one supposes that the process of division will come to an end, the final moment of division being represented by the atoms of copper.

Certainly, the two problems may also be interpreted differently. The segment AB may be considered as a drawing, as a fine ink line. The atoms of copper may not be considered as the final step of a division process. Atoms are composed of more elementary particles (electrons, protons, and others). Mentally, one may entertain the possibility that these elementary particles are also divisible.

The predicted correct reactions to the two problems are: In the case of the segment AB the process of division is endless, and in the case of the copper wire the division processing stops when reaching the atomic level (after which the quotient elements lose their identity).

Our hypothesis was that the students will chose *one* solution which will then become a model for both problems. This research is now in progress and therefore we do not possess, yet, statistical results. But from the reactions recorded so far, it seems that our hypothesis has been confirmed. Let us quote some examples:

> *Guy* (Grade 12) (The segment) "The process is infinite because one can always divide into two parts and this because the sequence of numbers is infinite." (The wire) "Technically, the process is limited but conceptually, something always remains and therefore the process is infinite."
>
> *Oded* (Grade 11) (The segment) "The process is endless because the geometrical segment is constituted from points." (The wire) "The process is endless, and this is exactly as with the segment."
>
> *Sara* (Grade 11) (The segment) "The process will come to an end when one reaches all the points and it will be impossible to continue to divide." (The wire) "The process will come to an end, and this is exactly as with the segment—the same principle. It will happen when we reach the smallest part, the atom."
>
> *Gali* (Grade 8) The segment "It will become extremely small and at the end we will not be able to divide any more and therefore the process will come to an end." (The wire) "Yes, the process will come to an end because we will not be able to continue to divide."

These examples give us an idea of how a tacit model works. Though the concepts of a geometrical segment and that of a certain material substance—copper—have been encountered by the students in two absolutely different contexts, the external similarity and the apparent identity of the process (division) tend to produce a *unique mental model* adaptable to both situations. For some subjects it is the formal concept which dominates. For others, it is the material version which captures the imagination. But most of the subjects seem to be consistent in their choice, feeling no difficulty in using the same model for two essentially different situations. The model dominates the interpretation not as a conscious, controlled device but rather "from behind the scenes," imposing its structural, self-consistent system of constraints on the data considered.

What is particularly interesting in the present example is that the abstract version (the infinitely divisible segment) may play the role of the tacit model and not the concrete reality as one usually supposes. As a matter of fact, the geometrical segment represents what we have called a *figural concept,* an entity which appears subjectively as an abstract, pure, ideal entity—like every concept—and, at the same time, to be intuitively representable and manipulable as if it were a real object. The segment can be divided because it is *subjectively* real (a concept cannot be divided) and it can be *infinitely* divided because it has, nevertheless, an ideal nature. This double nature of geometrical concepts explains their fundamental role in modeling, mathematically, real situations. The copper wire ceases to be a conglomerate of atoms—the end of the process of division—and becomes an infinitely divisible *segment.*

The mental model adopted tacitly, spontaneously becomes a genuine substitute for the original, eliminates those properties which would be inconvenient for it and exposes its own properties to mental analysis.

Multiplication and division

A final example often discussed in recent years refers to the operations of multiplication and division. It has been assumed that the primitive model for multiplication is repeated addition and that, for division, one may consider two models: partitive division and quotitive division (measurement) [Fischbein *et al.* 1985].

Let us focus on the operation of multiplication. The repeated addition model (putting together two or more disjoint, equivalent collections of objects) imposes a number of constraints. First, one has to distinguish the *operand* (the magnitude of the collections) and the *operator* (the number of equivalent collections). The operand can be any positive number, but the operator must be a whole number. One may certainly say "3 times 0.65", but "0.65 times 3" has no intuitive meaning. A second constraint of the repeated addition model is the property that multiplication "makes bigger."

It has been found that even high school pupils and college students face difficulties when asked to solve simple multiplication problems which contradict the above constraints.

Let us consider the following two problems:

> "From 1 quintal of wheat you get 0.75 quintals of flour. How much flour do you get from 15 quintals of wheat?"
>
> "1 kilo of detergent is used in making 15 kilos of soap. How much soap can be made from 0.75 kilos of detergent?"

These problems have been addressed to Italian pupils enrolled in Grades 5, 7 and 9. They were asked only to choose the appropriate solving operation, not to perform the calculation. For the first problem the percentages of correct answers were, respectively, 79% (Grade 5), 74% (Grade 7) and 76% (Grade 9). For the second problem the percentages of correct answers, for the same grades, were respectively: 27%, 18% and 35%. Both problems are solved by the same multiplication: 15×0.75, but in the first case the operator is the whole number while in the second case the operator is the decimal [Fischbein *et al,* 1985, p. 9, 10]. The most suprising finding is that the effect appears not only in young students but also in older ones who should certainly have accumulated a large amount of experience of manipulating decimal numbers. These students had no idea that their difficulty was produced by the repeated addition model influencing, tacitly, from "behind the scenes," their solving decisions [see also Bell *et al.* 1987; Harel, Post and Behr, 1988; Mangan, 1986; Verschaffel, De Corte and Van Coilie, 1988; Tirosh, Graeber and Glover, 1986]. Verschaffel, De Corte and Van Coilie [1988] have shown that in area problems the multiplier-as-a-decimal effect does not appear. This is an additional argument for the validity of the theory. In a problem in which the student has to calculate the area of a rectangle, the presence of decimal numbers (even for both dimensions of the rectangle) does not influence the correct answer. In this case, the student has simply to use a formula and the distinction between operator and operand does not intervene.

In the above example, a learned model, used initially for didactical reasons in an explicit way, becomes, later on, in older students, a tacit model, the presence and influence of which is completely ignored.

The characteristics of implicit mental models

Let us try to summarize the common characteristics of intuitive, elementary, implicit mental models.

(a) A fundamental characteristic of a mental model is that it is a *structural entity.* A model, like a theory , is not a simple isolated rule but, rather, a global, unitary, meaningful interpretation of a phenomenon or a concept. A model implies, usually, *a cluster* of rules, of constraints. It happens, very often, that a person presents various misconceptions with regard to a certain phenomenon. These misconceptions may seem to be totally unrelated and yet, after an appropriate analysis, we may find that all of them are imposed by the same model!

(b) A second characteristic of an implicit model is its *concrete, practical, behavioural* nature, even if the model is an abstract construct, as is the case with the line segment mentioned above. Multiplication means putting together several sets of the same size, a set is a collection of objects, the equals sign represents an input-output process, etc.

(c) A third characteristic of this type of model is its *simplicity,* its elementary, even, I would say, its trivial character. These implicit substitutes acquire their privileged role in the process of reasoning just because they are simple, economical, directly representable in terms of action.

(d) Despite the fact that they are so simple, *they are usually able to impose a number of constraints*. The operation of division is represented, primarily, by the action of dividing a collection of objects into a number of subcollections. This is a simple concrete, elementary operation, but it imposes a number of constraints: The divisor must be a whole number, smaller than the dividend, and the quotient must be smaller than the dividend also. A problem in which the numbers do not comply with these rules has no direct intuitive solution [Fischbein *et al*, 1985]

(e) Fifth, a mental model, like every type of real model, is an *autonomous* entity with its own rules and not an entity the behaviour of which depends on some external constraints [see, for the autonomy of models, Williams, Hollan and Stevens, 983]. It is important to emphasize this aspect in order to understand how a model, having its own rules and parameters, may impose them on the original situation, even going beyond what is scientifically acceptable.

(f) The sixth and fundamental characteristic of this type of model is its *robustness*, its capacity to survive long after it no longer corresponds to the formal knowledge acquired by the individual. As has been shown, high school and college students make the same types of mistakes and manifest the same misconceptions as younger subjects do, which may be explained by their use of the same implicit, primitive models

How can this robustness be explained?

First of all, it is related to our way of thinking. The role of formal, conceptual structures is essentially to control and not to invent. We invent, we understand, by resorting, basically, to concrete representations which mediate between abstract meanings and the course of some concrete activity. This is certainly a trivial affirmation. What is not trivial is that, very often, the concrete substitutes not only inspire and stimulate the reasoning process but, in fact, *control its course*

The robustness of certain mental models can thus be explained by their fundamental importance for the reasoning process. They inspire and, very often, even control it.

Secondly, acting usually in a tacit manner, their concrete, primitive substitutes remain very much uncontrolled. The individual, being unaware of these influences, does not try to intervene and change or replace them.

Thirdly, one may assume that these primitive models owe their robustness to their instrinsic qualities: simplicity, concreteness, immediacy. One may assume that these properties correspond to basic tendencies of our reasoning processes in general.

All these remarks raise not only fundamental theoretical problems (as mentioned above), but also complex didactical ones

It seems that the reasoning process continues to be controlled, to a great extent, by concrete representations "from behind the scenes," even during what has been termed the formal operational period. The process of liberating the reasoning activity from primitive constraints is not a spontaneous one. The process has to be initiated and achieved systematically by appropriate educational means

One should devote much more experimentation to developing techniques which would enable students to become aware of the influence of their own tacit, intuitive constraints, and, on the other hand, one should help students to build efficient conceptual control systems which would control the impact of these models. This is an extremely difficult and complex problem, by and large relating to metacognition. I would recommend the following steps:

— The first thing to do is to analyse the systematic errors appearing in the students' solutions to a certain problem.

— The errors may point to constraints arising from a certain tacit model. Through discussion in the classroom, the students can be helped to become aware of the respective model and of the potential conflict generated by the constraints of the model and those formally established by mathematical definitions and theorems. The students themselves may deduce, from the presumed tacit model, the constraints it imposes, and compare them with those derived from definitions and theorems.

— Once the students become aware of the conflict, they should be asked to solve problems, related to the same domain, but presented in various contexts and involving new aspects. (With regard to class discussion as a metacognitive technique, see also Schoenfeld [1987], p. 201-209).

For instance, let us suppose that the students have been asked to compare the sets of points of two segments of different lengths. Some of the students claim that the two sets contain the same number of points (both sets are infinite), while others claim that the longer segment contains more points because it is longer

The teacher then explains that two infinite sets cannot be compared by the usual means because we are not naturally equipped for dealing with infinite sets. Our mental schemes are adopted to finite sets or objects. When claiming that the longer segment contains more points, we are, in fact, influenced by the pictorial model of a point (a small ink spot). This model acts tacitly and it is practically impossible to get rid of it. But as a matter of fact a point is a pure concept. It represents a position, but it has no dimensions. When we are dealing with statements about points, we should resort to purely formal rules. On the other hand, declaring that "both sets are infinite" is simply avoiding the problem because, again, the relations "more," "less," "equal" do not have the same meaning in the domain of infinite sets as they have in the domain of finite sets—to which we are naturally adapted. Consequently other means must be used, purely formal ones, in order to deal with infinity.

Next, the students may be asked to compare the set of natural numbers with that of positive even numbers. If a student says: "Let's be careful, we again have to do with infinity, the answer is not trivial", he has certainly learned something from the previous discussion

Finally, if the students are asked to compare the set of rational numbers with the set of irrational numbers, they will learn that two infinite sets may *not* be equivalent. Although the domain remains the same—the domain of infinity—the context and the specific aspects may change [see Tirosh, 1985].

If such discussions take place often enough in the classroom, one may hope that the following didactical achievements will be recorded:

— The students will become more circumspect with their primary solutions and their primitive interpretations. They will learn to analyse carefully their generalizations and conclusions in the light of the formal constraints. They should not be taught that intuitive models are always wrong, but that their applications of them in certain circumstances may not be adequate. Such distinctions require a number of intellectual skills which only practice will develop.

— Secondly, the students may accumulate useful information concerning specific intuitive models—already identified—which may interfere with the reasoning process and distort it. We assume that it would be of great importance for the students to learn to identify the practical situations which have inspired their intuitive models. As a consequence, the students will learn, *through particular examples,* about the tendency of these primitive models to survive even *after* they have acquired the corresponding correct, formal definitions and theorems.

— Thirdly, one may hope that, as an effect of such metacognitive practices, the students will develop general habits and adequate skills in order to analyse the concepts and the operations they are dealing with when trying to identify an eventual intuitive, tacit, model

Gavelek and Raphael write, referring to the problem of transfer: " . the concept of metacognition addresses one of the perennial problems of instruction—that of transfer or generalization of what has been learned. It is reasoned that to the extent that individuals know what and how they know, such higher-order knowledge should be utilizable across different settings [Gavelek and Raphael, 1985, vol. 2, p. 129]

In other words, if individuals know what they know and how they know, for instance, with regard to a certain mathematical concept, they will get generalizable information which may be used in learning other mathematical concepts. But the same may be said about metacognitive skills themselves. One may assume that students will get basic, general skills for analyzing the sources of their intuitive obstacles and, first of all, the impact of tacit models on their mathematical reasoning. This is certainly a matter of psychology. But is it not, also, an extremely important and very powerful component of genuine mathematical activity?

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