When Can You Meaningfully Add Rates, Ratios and Fractions?

SIMON MOCHON

It is very common to see in the literature the remark that "rates can be added but ratios cannot". We will see that, in fact, both can be added as mathematical objects in several ways, but the question of meaningful answers is going to be situation-dependent. We can also find in the literature statements like "in order to interpret *a*/*b* as a ratio, *a* must be less than *b*" or "we cannot reduce ratios in general" (Borasi & Michaelsen [1985]). If this were true, what would then be the connection between fractions (with their equivalence property) and ratios? The first aim of this article is to clarify these contradictory ideas.

At the same time, we will also try to answer some questions about the normal way of adding fractions, like: Are there any assumptions behind this addition? Is it a meaningful operation within the context of a problem? Is it appropriate also to add rates and ratios in this manner?, and if not, what is the proper way?

A ratio is a relationship between two quantities. Implicitly, we assume this relationship to be of a quotient type (as opposed to a difference type). For example, if the price of 3 records is 13 dollars, we identify this with the constancy of the fraction 13/3. This association provides the ratio concept with an automatic but sometimes false proportionality inherited from the equivalence of fractions. In our example, we may translate that statement to others presumed to be equivalent, like 6 records will cost 26 dollars; but the real situation might be totally different: each record costing 4 dollars with one extra dollar for handling (mixed additive and multiplicative situation).

Even in cases of "true" proportionality, we have to be aware that this assumption is usually only a *mathematical approximation to the real situation*. For example, if a rowing crew covers 3 kilometers in 5 minutes, we know that, most likely, they won't be able to row 30 kilometers in 50 minutes.

The first point we are trying to make is that a single statement of a ratio, according to the situation, might or might not behave in a proportional manner and that this is in most cases a mathematical simplifying assumption. We will cite some other examples. We can say that in a population the ratio of children to adults is 3 to 2. If we are interested in the situation of how this ratio will change with time, we cannot assure that it will remain constant. The prices of things relate linearly (are proportional) to the quantity we purchase because we have designed our economy this way. In contrast physics, and most other areas of mathematical modeling, are filled with pseudo-linear relationships. For example, when we want to describe the lengthening of a bar with temperature, we assume that the change in length is proportional to the length of the bar and to the change in temperature. It is not difficult to convince oneself that this assumption will lead to the result that a sufficiently large negative change in temperature will make the bar disappear!!! In many applications we have to assume proportionality, or a constant rate, simply to make the analysis more manageable. (For a discussion on fractions, ratio and proportionality, see for example, Freudenthal [1983].)

Rates are not that different from ratios. We can say that a rate is the quotient form of a ratio transforming the relationship between two quantities into a single intensive quantity. Actually the rate is the quantity we get when we reduce a ratio to a unit ratio (one quantity relative to a single unit of the other quantity). On the other hand, any rate can be transformed into a ratio through a unit ratio. For example, the ratio 6 liters of gas in 75 kilometers can be transformed into the rate (75/6) 12.5 km./l. On the other hand, the mortality rate of 0.00003 can be expressed as 3 deaths in 100,000.

The main question we want to address here is the following: When can the conventional way of adding fractions,

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+cd}{bd}$$

or adding vectors,

$$(a,b)+(c,d)=(a+c,b+d),\ \text{i.e.}\ \frac{a}{b}\text{“}+\text{”}\frac{c}{d}=\frac{a+c}{b+d}$$

be used as a fair representation of a real situation involving either two fractions, two ratios (their quotient forms), or two rates? To begin the discussion, let's consider the following example.

Suppose that a basketball player has two series of 12 free shots. In his first series he fails 3 of them and in the second only 2. The fractions of shots missed in the two series are respectively 3/12 and 2/12. If we now look at his total performance we can see that he failed 5 shots out of the 24, with a fractional representation of 5/24. Why does the normal way of adding fractions (3/12 + 2/12 = 5/12) not apply in this case, and, instead, the quantities follow the relation:

$$\frac{3}{12}\text{“}+\text{”}\frac{2}{12}=\frac{5}{24}$$

which is the rule "add tops and bottoms"? (As shown above, this rule is equivalent to the addition of vectors.)

For the Learning of Mathematics **13**, 3 (November, 1993)
FLM Publishing Association, Vancouver, British Columbia, Canada

The answer to the last question (as we will see more clearly later) is that the whole (24 shots) to which the answer (5/24) refers is different from the wholes (12 shots) used for the addends (3/12 and 2/12). Note that in this example not only does the normal addition of fractions not give an appropriate answer, but also, if we simplify the original fractions (to 1/4 and 1/6), the "sum of tops and bottoms" (i.e. 2/10) won't give the right answer either.

The previous example suggests that, at least in applications, it would be worthwhile distinguishing ratios containing the total values of the quantities from reduced ratios. A reduced ratio loses information about the two quantities involved and retains only their relationship.

Ohlsson [1988] stresses that to understand the meanings of fractions we have to pay attention to both the mathematical theory and the real situations in which they are embedded. We fully agree with this statement. In fact, it is the physical situation that determines the appropriateness of the mathematical representation. Ohlsson proposes that an adequate mathematical theory of the concept of ratio is based on binary vectors (a,b) and gives the following examples to support this. If I get 8 miles per gallon during the first part of a journey but only 4 miles per gallon during the second, then the car's performance over the entire trip is not (8 + 4) = 12 miles per gallon. The correct analysis (according to Ohlsson) is (8,1) + (4,1) = (12,2) = (6,1) or 6 miles per gallon. Later in this paper it will be shown that this approach is also incorrect. Similarly, if one classroom has a ratio of 2 girls per 3 boys and another classroom has 4 girls per 3 boys, the combined class does not have 2/3 + 4/3 = 6/3 or 6 girls per 3 boys. In Ohlsson's paper it is claimed that the right description of this addition is (2,3) + (4,3) = (6,6) = (1,1), or one girl per boy. As we will see later, this way of adding ratios is only correct if the ratios involve the total values of the quantities, as in the example of the basketball player.

In another article by Lesh, Post & Behr [1988], these authors observed that because fractions and rates are themselves quantities, they can be added using the standard rules and that these rules do not make sense for ratios which are ordered pairs of quantities. They give as examples the following:

3 mi./hr. + 2 mi./hr. ≠ 5 mi./hr.

2 boys-to-3 pizzas "+" 3 boys-to-4 pizzas = ???

explaining that the seventeen-to-twelve answer (obtained by adding their fractional forms: 2/3 + 3/4) would not be a sensible answer.

It is true that rates and fractions can be added as numbers, but we should pay attention to the type of situations to which this addition can be applied. Referring to the first example, addition of speeds can be applied to relative speeds between two objects but cannot be applied to a single object with two different speeds in two parts of a trip. Looking at the second example, we could transform the two ratios into rates by dividing the two quantities. Therefore also the sum of these two rates would not give a sensible answer.

There seems to be a tendency to try to associate to each of these three types of quantities (ratios, rates, and fractions), according to their nature, a definite rule about the way that they should be added. This is probably the wrong approach. We have to keep in mind that these are really different mathematical expressions that can be transformed into each other. So if we are able to add fractions and rates, we can certainly also add ratios (using their equivalent fractional form, or in some other way, for example, by using vectors). On the other hand, if a rule does not make sense for ratios, it won't make sense for their equivalent rates either.

Really, the important issue at hand is: when do the different ways of addition give sensible answers within the real applications of these three quantities? The answers, obviously, will depend on the physical situations in ways we would like to describe. From the pedagogical point of view, this study strongly suggests that we should not detach these quantities and their operations from the real contexts they are embedded in—that is, we favor a realistic mathematics education approach. The following analysis gives the answer to the questions raised above, in a particular situation, from which we can draw more general conclusions.

In what follows next, we will frequently use the terms "reference whole" or "reference unit". We will specify now what we mean by the terms. In the case of fractions these terms refer to the whole that each fraction is related to, e.g. "the set of shots" in basketball example. In the cases of rates and ratios these terms seem more ambiguous, but they refer to the amount of the referential quantity (usually the second one). Thus, for example, in a rate of miles per gallon the "reference whole" would be "the total number of gallons". In a ratio of girls to boys, the "reference whole" will be "the number of boys". We need to clarify that these quantities are not *explicitly* contained within the fraction, the rate, or the ratio, since we know that these quantities only give information of a relative nature.

Rates, ratios, and fractions are statements of a relationship between quantities and do not give any information about the number or the amount of the quantities themselves (see Schwartz [1988]). Since this implies that these quantities cannot contain full information about a situation, we must expect, in general, to need to know the values of other variables to compute their sum meaningfully. This will be shown in the next sections

A mathematical analysis of rate

Consider the first example from Ohlsson about adding two rates representing miles per gallon on two parts of a trip. Let's call M_1 and M_2 the *total* number of miles in each part, G_1 and G_2 the *total* amount of gas used and r_1 and r_2 the two rates ($r_1 = M_1/G_1$ and $r_2 = M_2/G_2$). The rate representing the entire trip will then be:

$$r_t = \frac{M_1 + M_2}{G_1 + G_2} = \frac{G_1 r_1 + G_2 r_2}{G_1 + G_2} =$$

$$\left(\frac{G_1}{G_1 + G_2}\right) r_1 + \left(\frac{G_2}{G_1 + G_2}\right) r_2$$

As can be seen from the last formula (which does not really depend on the particular example we have chosen), the two partial rates do not simply add to give the total rate. Instead, we get a weighted sum with some related fractions as weights (the quotients in parentheses above).

If the rates are added as vectors in the way suggested by Ohlsson:

$$(r_1,1) + (r_2,1) = (r_1 + r_2,2) \equiv (\frac{r_1 + r_2}{2},1)$$

we would get the total rate to be the average of the two rates. Comparing this answer with the correct weighted sum, this vector addition is seen to be appropriate only for a very particular case (reference wholes the same: $G_1 = G_2$). Therefore, in this type of situation, rates generally cannot be added simply as quantities or as vectors (as Ohlsson suggested) to give meaningful results.

The formula above for the rate r_t can be applied in general to any situation in which we have two rates r_1 and r_2 in two parts of the situation, each one characterized by the reference wholes G_1 and G_2, respectively, when we want to obtain the rate on the combined whole $G_1 + G_2$. For example, a car moving at 80 mi./hr for 3 hours and then moving at 40 mi./hr. for 2 hours, will have, according to the general formula, an average speed given by: (3/5)80 + (2/5)40 = 64 mi./hr.

A mathematical analysis of ratio

If we now turn to ratios, almost the same analysis holds as the one described above for rates. Let's consider then the second example given here by Ohlsson about the two classrooms and let's assume that G_1, B_1, G_2, and B_2 are respectively the *total* number of girls and boys in the two classrooms. Then the total ratios of girls to boys can be expressed by $G_1 : B_1$ and $G_2 : B_2$ and the ratio of the combined class by $G_1 + G_2 : B_1 + B_2$. Usually, though, we have available only some reduced rates. Let's write these as $g_1 : b_1$ and $g_2 : b_2$. Since the reduced rates must describe exactly the same relationships between quantities as the total rates we must have the equalities:

$$f_1 \equiv \frac{g_1}{b_1} = \frac{G_1}{B_1} \text{ and } f_2 \equiv \frac{g_2}{b_2} = \frac{G_2}{B_2}$$

where f_1 and f_2 are defined to be the fractions representing the two ratios. Substituting these equalities into the ratio for the combined class we obtain:

$$B_1f_1 + B_2f_2 \ : \ B_1 + B_2$$

or, dividing by $B_1 + B_2$, we get the equivalent unit ratio describing the combined class:

$$(\frac{B_1}{B_1 + B_2})f_1 + (\frac{B_2}{B_1 + B_2})f_2$$

Again we can see clearly from the previous ratio that the addition of the fractional representations of two ratios ($f_1 + f_2$) would not give a correct relationship for the combined parts.

If we now add the reduced ratios as vectors to obtain $g_1 + g_2 \ : \ b_1 + b_2$, an identical type of reasoning to the one just applied to the total ratios will bring us to a similar expression for the unit ratio, except that the capital B's are changed to the lower case b's. This shows that the sum of the reduced ratios as vectors won't give, in most cases, an appropriate answer since the fractions $B_1/(B_1 + B_2)$ and $b_1/(b_1 + b_2)$ are only equal when the total ratios are reduced by the same factor. (This applies to the total ratios themselves.)

Note the similarity in the form of the unit ratio of the combined class just derived to the formula for the total rate given previously. This hints at the idea that operations with rates and ratios, when meaningful, should be symmetrical.

The above formula can also be applied in the general situation where two ratios (with fractional representations f_1 and f_2) describe two parts of a situation (with respective reference wholes B_1 and B_2) and we want to obtain the ratio for the combined situation. For example, if in two regions the voters in two political parties are in the ratios 2 to 5 and 3 to 2 respectively, and if the two reference wholes (for the second party) are 1000 and 2000, then the combined ratio for the two regions together will be (according to the general formula): (1000/3000)(2/5) + (2000/3000)(3/2) to 1, or, reducing this: 17 to 15.

Try now to solve the following problem using the above formula. A shirt with a marked value \$60 is sold at 50% off. A tie marked \$20 is 10% off. If we buy both, what is the equivalent total percentage reduction?

Addition of fractions

In order to extend the previous analysis to fractions, but at the same time to keep it readable, instead of a general approach that can always be followed, we will concentrate on one example and fix the values of some of the quantities involved. At the end we will give the general conclusions of the full treatment.

To fix ideas, we think of two cans of paint, one containing 1/8 of white paint (the rest black), the other one with 3/4 of the contents white (the rest black). If we mix the two cans, what fraction of white paint will the mixture contain? We can add the fractions to get 7/8 or add "tops and bottoms" to get 4/12. When, if ever, are these results correct?

Let's call A_1 and A_2 the amounts of paint in each can. Then the total amount of white paint will be:

$$\frac{1}{8}A_1 + \frac{3}{4}A_2$$

and the fraction of white paint, relative to the total amount of paint will be:

$$\frac{\frac{1}{8}A_1 + \frac{3}{4}A_2}{A_1 + A_2}$$

This is the correct result. Note that only in the special case when $A_1 = 8$ and $A_2 = 4$ does this fraction reduce to (1 + 3)/(8 + 4), which is what we would get applying the rule of adding "tops and bottoms". Therefore we see that this way of adding fractions is meaningful only if the denominators of the fractions represent the whole amount in each of the cans and if, additionally, we want to refer the result to the combined amount of paint.

On the other hand, we can relate the total amount of white paint to the amount of paint in one of the cans, let's say the first one. If we do this, we now obtain the fraction:

$$\frac{\frac{1}{8}A_1 + \frac{3}{4}A_2}{A_1}$$

The purpose of writing this is twofold. The first is to show that our original question (what fraction of white paint will the mixture contain?) was, strictly speaking, ambiguous. This fraction can be given (as we did above) taking the whole amount as the unit of reference, or the contents of one of the cans as a unit, or using some other unit. This might sound a little picky, but it is in fact a very important issue, as can be seen from our second remark:

If we take $A_1 = A_2$, the last fraction will reduce to 1/8 + 3/4. The conventional way of adding fractions finally appears. Thus, only when the amounts of paint in each can are equal, *and* the result is referred to this common unit, does the usual rule of adding fractions give a correct answer. (We will generalize this conclusion later on in the paper.)

In the above example the fractions involved express ratios and, as we saw, only in very particular situations does the addition of fractions correctly represent the solution to the problem. When, on the other hand, the fractions express quantities, the condition of a common unit of reference is satisfied automatically and, therefore, the addition of fractions is the proper representation of the problem.

It is important to be aware that adding fractions in the usual way, or as vectors, can give an incorrect mathematical representation of a real problem situation.

The two quotient formulas above, although not written with arbitrary fractions (to handle this we can replace the fractions 1/8 and 3/4 by general fractions f_1 and f_2), give the fraction that results when we combine two sets of sizes A_1 and A_2 each containing the fractions f_1 and f_2. The first formula gives the fraction relative to the combined whole and the second one relative to the first whole. For example, if in one year a person pays 1/4 of his $50,000 salary in taxes, and in the next year pays 3/5 of his $100,000 salary in taxes, then, he paid in taxes, according to the general formulas above, [1/4(50,000) + 3/5(100,000)]/150,000 = 29/60 of the total amount he earned, or [1/4(50,000) + 3/5(100,000)]/50,000 = 29/20, relative to his first salary.

Implications for the classroom and for research

Consider the four problems:

(a) I walked 1/2 km. yesterday and 1/4 km today, how far did I walk altogether?

(b) If a baseball player hits the ball once out of two times at bat (1/2) in one game and once out of four times (1/4) in a second game, what is his overall performance in the two games as a fraction of the total number of times at bat?

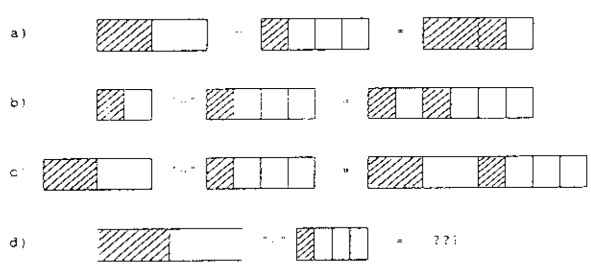
(c) 1/2 of cereal "Sweety" is sugar, 1/4 of cereal "Healthy" is sugar. If I put in a bowl equal amounts of these cereals, what fraction of the cereal in the bowl is sugar?

(d) In one classroom 1/2 of the children are boys and in another one 1/4 are boys. If we put the two groups together, what fraction of boys will we get?

These problems are completely different mathematically speaking, but students might see them as similar and solve them identically. The first is the only one that can be solved by the usual way of adding fractions (common units). In the second one, adding "tops and bottoms" will give us the right answer. The third and fourth need the more general formulas given before, although the fourth one, as stated, simply doesn't offer enough information.

Why, then, do we teach the "odd" conventional way to add fractions? Simply, but very importantly, because that is the way fractions should add *as numbers* and therefore this procedure is the basis for all operational higher mathematics. We cannot expect this algorithm however to fit blindly into all situations involving fractions.

We could represent these four problems graphically (using the same conventional part-whole interpretation, for comparison purposes) in the following manner:



It would be interesting to find out which type of representation the children have in mind when they perform the operation of adding fractions. The first is the only one compatible with the conventional algorithm. Note that the second and third (but not the first) reflect the idea of "combining" that we find in the addition of whole numbers. This suggests that children who have a part-whole representation of fractions (as given in these diagrams) and associate "sum" with combination will find the result of the usual algorithm for adding fractions incompatible with their mental representation and, for the same reason, will find the rule of adding "tops and bottoms" more logical (1/2 + 1/2 = 2/4 = 1/2). A graphical representation of addition within the measure subconstruct would not have this problem.

Our previous discussion, together with these diagrams, show the importance of the reference unit for fractions. We hardly stress in the classroom that, for example, "a half", "a fourth", etc., do not have an intrinsic meaning and that to be completely specified we need to refer them to a unit: "a half of...", "a fourth of ...", etc. (except in situations where the unit is implicitly accepted, as when 1/2, 1/4, ... are numbers). K. Hart (CSMS project, [1981]) found in children what appears to be a lack of reference to a unit in fractions when she posed problems of the type: "Mary spent 1/4 of her money, John spent 1/2 of his. Is it possible for Mary to have spent more than John?"

We too often assume that a fraction has a unique symbolic representation when, in fact, it has many, depending on the unit taken and the subconstruct of the fractions we are embedded in. For example, we often expect children to answer "one and a quarter" to the question: what fraction is represented in the following figure?

when, in reality, many interpretations are valid: "5" (one symbol as a unit), "5/8" (the whole figure as a unit), "5/3" or "3/5" (a ratio interpretation), "two and a half" (a column as a unit), etc. This again reflects what we said about the need to be explicit about the units.

Turning now to some of the questions we posed about ratios at the beginning. We certainly need to reduce ratios. Actually, that is the whole idea behind them. Nevertheless we have to be aware of the fact that some rules, like vector-type addition, can only be applied to total ratios and not to their reduced forms. As was suggested before in this article, this problem can be solved by making a distinction between these two cases.

Proportionality, as we mentioned earlier is a very delicate issue. On the one hand, we want students to realize that direct proportionality of two related quantities does not always hold and therefore must be checked using some of its properties, e.g doubling one quantity doubles the other, etc. But on the other hand, we expect them to disregard the cases where proportionality does not apply exactly (examples of this were given in the introduction). In these cases, we can only check that proportionality is "plausible" and assume it as a "reasonable" mathematical representation of the real situation we are looking at.

We are moving more and more in the direction of teaching as a way of guiding the students to rediscover mathematics (see for example, Gravemeijer *et al.* [1990] and Streefland [1991]). This kind of approach, though, forces the teacher to know more about the overall behavior of mathematical objects and their operations in different contexts, as we have explored in this paper

Conclusions

The single most important general conclusion of this paper is that the addition of fractions as numbers requires us to have the same unit of reference for the addends as well as for the result. This means that when we add two fractions in the context of a problem the result will be sensible only when the two fractions refer to some common unit and the result is needed also relative to this common unit. This sounds logical, but it is not trivial at all.

If we look back again at the four problems presented in the last section, we can identify only in the first this constancy of reference unit required for the addition of fractions to be appropriate. In the second example we actually have a 1/2 of 2 and a 1/4 of 4, not a common unit of reference. In the third there is a common unit for the two fractions but the result is required in relation to a different unit of reference: the total amount of cereal.

The requirement of common units will not be satisfied in applications related to fractions as ratios (for the reasons given below in the discussion of ratios), but will be fulfilled when fractions are used as measures (for example, in problem (a))

In the case of adding fractions as vectors ("tops and bottoms"), this procedure will almost always give an inappropriate result unless, by chance, the numerators and denominators of the fractions contain the total values of the quantities and, furthermore, the result is required relative to the combined whole (this is the case in problem (b) and the example of the basketball player).

In more general situations, the fractions cannot contain all the information and therefore the value of related fractions will be needed to calculate their combined sum. For these cases we need more general formulas (containing more relevant parameters of the problem) like the ones given in this paper.

Strongly connected with the previous issues is the importance, in general, of the reference unit for fractions, a point already stressed before.

If we now turn to rates and ratios, the work in this article suggests that the simple addition of these types of quantities, either as fractional quantities or as vectors, will not give an appropriate answer in the type of situations we are looking at, where we need to combine the given rates or ratios of the parts to obtain the corresponding description of the situation as a whole. This is really not very surprising. If we look at the general formulas deduced for rates and ratios we see they both have the combined wholes as reference ($G_1 + G_2$ and $B_1 + B_2$, respectively) in their denominators, which we already know is not a proper condition to be able to add them meaningfully as fractional quantities.

The analysis presented in this paper also demonstrates that only when the ratios contain the total values of the quantities will their vector addition give the correct result. If not, only the more general approach like the one given in previous sections will be appropriate.

The asymmetry implied in the adage that "rates can be added but ratios cannot" should not be taken very seriously since, although rates and ratios are different mathematical structures, they are equivalent in the sense that a ratio can always be transformed into a rate and vice versa. In fact this paper shows a strong symmetry between the formulas obtained for the "addition" of rates and ratios in combined situations.

References

Borasi, R. & Michaelsen, J. [1985] Discovering the Difference between Fractions and Ratios, *Focus on Learning Problems in Mathematics* 7, 53-63

Freudenthal, H. [1983]. *Didactical phenomenology of mathematical structures* (chapters 5 and 6). Reidel, Dordrecht

Gravemeijer, K., Van den Heuvel, M. Streefland, L. [1990]. *Contexts, free productions, tests, and geometry in realistic mathematics education*. State University of Utrecht, The Netherlands

Hart, K. [1981] Fractions. *Children's understanding of mathematics*, (pp. 66-81) John Murray, London

Lesh, R., Post, I. & Behr, M. [1988] Proportional Reasoning. In J. Hiebert & M. Behr (Eds.) *Number concepts and operations in the middle grades*, (pp. 93-118). Reston, VA: National Council of Teachers of Mathematics

Ohlsson, S. [1988] Mathematical Meaning and Applicational Meaning in the Semantics of Fractions and Related Concepts. In J. Hiebert & M. Behr (Eds.) *Number concepts and operations in the middle grades*, (pp. 53-92). Reston, VA: National Council of Teachers of Mathematics

Schwartz, J.L. [1989] Intensive Quantity and Referent Transforming Arithmetic Operations. In J. Hiebert & M. Behr (Eds.) *Number concepts and operations in the middle grades* (pp. 41-52). Reston, VA: National Council of Teachers of Mathematics

Streefland, L. [1991] *Fractions in realistic mathematics education: a paradigm of developmental research* Kluwer Academic Publisher, Dordrecht

---

Harmonious development is not a simple matter. Let us take, for instance, abstract thought, which at first sight seems to be wholly an advantage; concerning harmonious development, however, it has many disadvantages. Abstraction is the basis of verbalization. Words symbolize the meanings they describe and could not be created without the abstraction of the quality or the character of the thing represented. It is difficult to imagine any human culture whatsoever without words. Abstract thought and verbalization occupy the most important place in science and all social achievement. But at the same time abstraction and verbalization become a tyrant who deprives the individual of concrete reality; this in turn, causes severe disturbances in the harmony of most human activities. Frequently the degree of disturbance borders on mental and physical illness and causes premature senility. As verbal abstraction becomes more successful and efficient, man's thinking and imagination become further estranged from his feelings, senses and even movements [...] Thinking that is cut off from the rest of the man gradually becomes arid. Thought that proceeds mainly in words does not draw substance from the processes of the older evolutionary structures. Abstract thought that is not nourished from time to time from deeper sources within us becomes a fabric of words alone, empty of all genuine human content. Many books of art and science, literature and poetry have nothing to offer except a succession of words linked together by logical argument; they have no personal content.

Moshe Feldenkrais
