Communications

Standing on the shoulders of giants: a response to Foster

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Colin Foster, in issue **42**(3), has presented a very interesting but quite provoking point of view to which I would like to respond in this short communication. I will not be very original in this contribution; my aim is only that of recalling some important considerations that I learnt from reading other researchers' work. Hence, I will mainly draw on the work of others, 'giants' of mathematics education, on whose shoulders we all stand.

A very hazardous pursuit

In his article, Foster presents a representation of multiplication based on the Cartesian plane. As he relates, his research group is building a curriculum (and related materials) seeking *coherence* in representations. Such coherence is achieved by "prioritising a single representation of number—the number line" (p. 21). The author recognises that such seeking of coherence could be considered detrimental but supports his position by stating that, "It is possible that multiple representations […] could result in an overall less powerful picture for the student than might be obtained with one carefully-chosen representation" (p. 25).

The representation presented by Foster appears to be a good *model* for multiplication in the sense of Fischbein (1987). Indeed, it is self-consistent and, at the same time, is easily relatable to other models of multiplication. As Foster argues, it has the potential for what Fischbein calls *heuristic efficacy* and it may be intuitive because it draws on the number line. Hence, I am not claiming in any way that the chosen representation is not as good as others. However, I take a semiotic point of view to argue that Foster's assumption, that multiple representations may result in some sort of confusion, is inadmissible. The very core of my argument resides in what I consider a mathematical object to be. In any theory of semiosis, the representation (or sign, representamen, *zeichen, etc.*) is distinguished from what is represented (or object, reference, *gegenstand, etc.*). In decades of research about mathematics teaching and learning, Raymond Duval (*e.g.,* 2006) has wondered how students can "distinguish the represented object from the semiotic representation used if they cannot get access to the mathematical object apart from the semiotic representations" (p. 107). According to him, the *dissociation* between the representation and the object happens only when non-congruent registers are confronted. It is not just a matter of choosing the most suitable representation (whatever this may mean). If they are not confronted with several representations, the students will never distinguish the object from its representation; learning about the object (multiplication in this case) will never happen. Based on Duval's work and my research, I believe not only that "presenting different models to pupils is relevant, but also that the operation of putting in relation the models has the potential to be particularly productive" (Maffia & Mariotti, 2018, p. 35).

As Duval (2006) states:

> Too often, investigations focus on what the right representations are or what the most accessible register would be in order to make students truly understand and use some particular mathematical knowledge. With such concern of this type teaching goes no further than a surface level. (p. 128)

Of course, interpreting this quotation requires specifying what a 'surface' or a 'deep' level is, and this depends on what we consider as successful teaching/learning. In the conclusion of his article, Foster contrasts "trying to find the quickest, easiest way to address each narrow skill" with "the long-term investment of building the most powerfully useful and coherent models" (p. 26), and he takes the second position. However, what "powerfully useful" may mean must be addressed. In this section I argued that privileging one representation is hazardous (using Duval's words). In the next section, I will state that the concept of 'usefulness' may change considerably when we consider not only individual learning but frame the teaching/learning of mathematics as a cultural endeavour—which means considering ethnomathematics.

Mathematics as culture

Addressing the usefulness of teaching/learning requires wondering what we teach mathematics for. According to the answer we provide, we may distinguish mathematical education from mathematical training (Bishop, 1988). As noted by Ubiratan D'Ambrosio (1994), "we are under pressure from educational authorities, community leaders, parents, and students themselves to get 'better results' to improve our marks, to be better in our marksmanship" (p. 444). I would classify the classroom activity realised for these purposes as mathematical training. While I am strongly convinced that Foster does not want to emphasise these activities, I see the risk of diverting in that direction when looking for the most useful unique representation for multiplication (or any other mathematical content). This is because, by forgetting other representations that have been shared by different populations during history, we are not considering that mathematics is a cultural activity.

Taking the point of view of ethnomathematics means recognising "that every cultural group generates its own ways of explaining, understanding, and coping with reality, transmits and organizes these ways into techniques […] and diffuses them through the group; improving and transmitting them from generation to generation" (D'Ambrosio, 1994, p. 449). The cultural history of a community (comprehending each and all individuals) allows us to understand how techniques have been generated by our ancestors and then inherited through generations. Many of the representations that appeared in the course of the history of mathematics have then been abandoned for others that were more useful—at that moment, for someone, for some purpose. This is the case for geometry as mean of proving arithmetical theorems (*e.g.,* largely used by Euclid) which has been replaced by algebraic notations. Rectangles were the main means for representing multiplication in the Hellenistic period, in al-Khwarizmi's work, and they were still used by Italian mathematicians in medieval times. In Radford's (2008) words "artifacts are bearers of the historical cognitive activity deposited in them by previous generations […], in using them in the course of our activities the subjective domain and the cultural-objective one become imbricated into each other" (p. 451), and representations are symbolic artefacts (Rabardel, 1995) that have shaped the mathematical activity of our ancestors and we—as members of our own culture—should know that.

The rectangular representation that is chiefly criticised by Foster—'algebra tiles'—is a cultural product and being able to interpret it allows understanding the mathematical activity of the past. In my research, I described how Italian fifth graders who knew how to use and interpret rectangular representations of multiplication were able to make sense of Tartaglia's original words and infer from them the definition of prime numbers (Maffia, 2019). This was possible because they could notice that some numbers can be only represented as rectangles having a side that is long as the number itself. While the same (pure) numbers were used to represent the length of the side and the area of the rectangle, it was the different roles of the two measures that allowed children to understand what a prime number was according to a mathematician of another time. At least (but not only) for this reason, I would describe the rectangular model as a *powerfully useful* representation.

> We certainly know that different symbolisations have been developed in different cultures and it is very likely that there are differences in values also […] How unique these values are, or how separable a technology is from its values must also remain open questions. (Bishop, 1988, p. 187).

To conclude, I would like to stress again how valuable the representation proposed by Foster is, but, taking a semiotic and ethnomathematical perspective—based on the arguments presented through the giant voices of Duval, D'Ambrosio, Radford, Bishop, and my smaller humble one—I would strongly suggest reconsidering the pursuit of a unique, powerful, useful representation of multiplication, to be prioritised above all others.

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Fraction as narrative: reflection on the conversations of Hewitt, Pimm and Sfard

AEHEE AHN

How is mathematics different in different languages? The conversations of Hewitt and Pimm, in issue **41**(2), and Sfard, in issue **42**(1), inspired me to re-think the relations between fractional language and fraction conceptions. Hewitt and Pimm talked about a classroom situation in which a Chinese teacher (called Ms Dai) accepted Figure 1b as a match of the fraction $3/5$ but rejected Figure 1a because it is not equally partitioned. This short communication proposes the idea of a fraction as a narrative, as one explanation for the behaviour of the Chinese teacher.

Sfard discussed the teacher's reaction with reference to the Chinese language structure used for fractions. She suggested that the teacher's rejection was caused by Chinese fraction language. Whereas English fractions name the product of operations, fractions in Chinese imply actions such as partitioning or dividing as well as the outcome. If Ms Dai's reactions are related to the Chinese language structure, I wondered what it would be like if I (a Korean native) interpreted her reaction through a Korean lens. Like Sfard, I tried to interpret Ms Dai's logic based on Korean language structure by building on Sfard's argument that fractions are actions in Chinese.

The order of reading and writing fractions in Korean is the same as in Chinese. The Korean words for fractions are based on Chinese words. The meanings and the usage of each Korean fraction word are very similar to those of Chinese words (see Table 1).

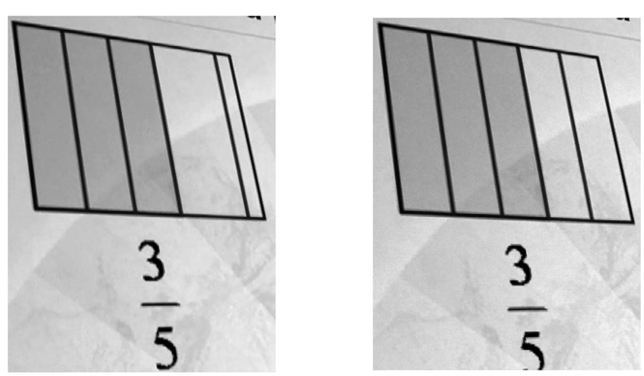

Figure 1. Drawings used in Ms Dai's lesson.

Table 1. Chinese and Korean language for fractions.

Fraction language structure	Chinese	Denominator – 分 (parts) – 之 (of) – Numerator
	Korean	Denominator – 분 (parts) – 의 (of) – Numerator

The Korean word 분 (read as 'bun') comes from the Chinese word 分 (fēn) and is used as a noun and a verb with the meaning of parts, pieces, partition, and divide, like 分. The word does not originally imply the meaning of 'equal', but in fractions it is commonly used to include the meaning of equal partitions. The next word 의 (read as 'ui') is from the Chinese character 之 (zhī), and instead of being used alone, it is used with other words as a postposition to represent the relationship between the whole and the part, and has a similar meaning to the English word 'of' [1]. Pimm translated the Chinese fraction name as 'denominator (equal) parts of (which) numerator'.

The evidence of the fraction names show that the Korean and Chinese languages basically describe fractions in terms of whole-part relations: not from part to whole but from whole to part. In Korean, I believe whole-part thinking not only refers to the order of considering a whole first and then parts later. It also reflects the action process of constructing fractions as narratives.

Fractions in Korean tell series of actions narratively. As an example, I am reminded of teaching fractions with visual representations to primary school students in Korea. Firstly, while saying (reading, writing, or thinking) a denominator (number), I drew a rectangle on a blackboard. While saying 분 or drawing a fraction horizontal line, I partitioned the rectangle into several equal parts. The two actions are closely interrelated and often performed consecutively. Next, while saying 의, I stressed whole-part relations between a numerator and the whole rectangle figure I drew. Lastly, while saying a numerator (number), I shaded some parts in the partitioned rectangle. In a class of words, the operative description flows from a denominator to a numerator. A denominator refers to a whole implying unitising; 분 means equipartitioning; 의 shows whole-part relations; and a numerator implies shading or choosing parts. Every word of Korean fraction names is linked to actions from unitising to partitioning to shading, along with the focus shift from a whole to parts.

In terms of narrative, equal partitioning related to a denominator sets up the context of unit fractions and is embedded as orientation in fractional actions. In other words, the essential first step of the narrative is equal partitioning, and unit fractions involving equal partitioning assume the initiation of the narrative. Equal partitioning provides the set-up context, then the narrative has been completed by associating with a numerator in company with shading or selecting actions. Labov (2006) symbolised the most reportable event as e_0 and used e_{-n} for past events, such as e_{-1}, e_{-2}, *etc.,* depending on the order of the series of events. According to the Labov's symbolisation of narrative, I narratively describe fractions as follows:

e_{-2}	Equipartitioning a whole
e_{-1}	Shading or choosing some of equal parts
e_0	A fraction

A fraction e_0 is a completed event and the most reportable event. The recursive series of fractional actions precedes e_0, and equipartitioning as an initial event is the orientation of a fraction.

Then, how does Korean fraction language compare to English fraction language? I examined the difference by taking the example of $^3/_5$ and tackling it from my view of fractions as narrative. The English $^3/_5$, commonly read as three-fifths, makes me think of a rectangle with three coloured areas out of a partitioned whole as an outcome or an object, similar to the comment made by Hewitt and Pimm. English fractions mean how much of an area is or how many are shaded out of a whole, which is synthetic. Whereas, in Korean $^3/_5$ translates as five-parts-of-three and makes me think of two steps: unitising and partitioning a whole into five identical areas and then finding three parts out of the partitioned whole. As an analytical process, a whole is considered first and then parts can exist. The denominator 5 provides the assumed context of equal partitioning and presupposes the unit fraction $^1/_5$. Then, the fraction $^3/_5$ is completely constructed with the numerator 3. I summarise the difference visually in Figure 2.

If I interpreted Ms Dai's rejection from my perspective, the fraction representation that is not equally partitioned (Figure 1a) is false. The fraction $^3/_5$ already assumes an equal partition of a whole, but Figure 1a went wrong from the assumed setting. Even though the outcome of shaded parts matches the region of $^3/_5$, this is unlikely to be considered true since the first action, non-equipartitioning, is contrary to the starting situation of the fraction narrative.

Then, I wondered about Korean teachers' reactions. Since the Chinese and Korean language of fractions are similar, would Korean teachers' reactions be similar to Ms Dai's? In order to see how Korean teachers react to the visual representation that is not equally partitioned, I had a talk with three Korean primary school teachers individually. They all have around 15 years of teaching experience in Korea. While showing Figure 1, I asked their thinking about the images. The question was 'what do you think of the drawings for $^3/_5$?'. To tell you the result first, all the three teachers rejected Figure 1a, as Ms Dai did. I undertook to understand their conceptions or logic, and I paid attention to where their focus on fractions had been. One teacher (called Ms Kim) said,

> This shaded part can be five 분 [parts] 의 [of] three [$^3/_5$] of the figure but fractions have to be equally partitioned. Some out of some equal parts. At first I thought both are correct but soon I think thinking of only three

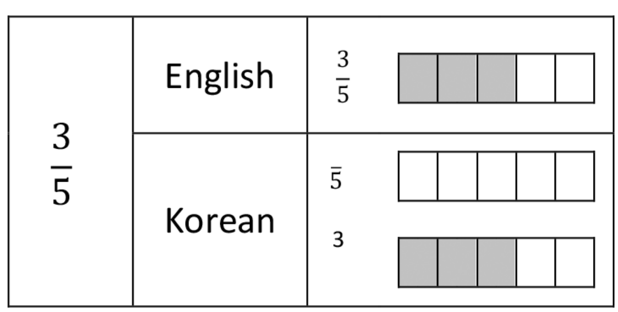

Figure 2. Fraction $^3/_5$ in English and in Korean.

Figure 3. Fraction rectangle missing one line.

> shaded parts in the rectangle focuses on the numerator three and that overlooks the precondition of fraction concepts equal partitioning.

Thinking of fractions as narrative, Ms Kim considered fractions with actions from equipartitioning to shading. She stressed equipartitioning in fraction concepts, mentioning it three times (equally partitioned, equal parts, and equal partitioning) in our short conversation. She linked partitioning actions to a denominator and regarded a denominator as a fundamental feature for fractions compared to a numerator. In her logic, partitioning as a first step was wrong, thus the representation Figure 1a is false. The other teachers answered similarly to her. This shows equal partitioning is embedded as an assumed context in Korean fractions and the fractional actions begin from it.

Through the next step, I was able to see more clearly that the teachers' focus is on action process more than lines or the results of drawing fraction representations. As a next step, I showed Figure 3 to the Korean teachers.

Figure 3 is an insufficient drawing that misses one line and therefore does not clearly show the representation of the fraction $^{3}/_{5}$. Before showing the drawing, I guessed the Korean teachers' answer that this is also false. They said, however, this is a tentative state, not true or false, since they cannot determine whether or not equipartitioning is done properly yet. This might be what Hewitt and Pimm said in the subtitle of their conversation: 'true, false, or somewhere in-between'. It might be somewhere in between true and false. If I symbolise $^{3}/_{5}$ as e_0 as a completed event, the lines and shadings play a role in showing the series of events preceding the e_0. The $^{3}/_{5}$ of Figure 3, however, does not clearly provide the initial process e_{-2} due to the one line missing. Missing one line makes it difficult to trace what happened recursively. If one line is added to have equal five congruent parts, this figure can be true, and if one line is drawn to have non-congruent parts, this figure can be false. I cannot name it in Korean starting with the denominator 'five-분 (parts)-의(of)', as this is not equally partitioned into five. In order to make it $^{3}/_{5}$, I might or ask students to track what action was missing in order to set up the assumed context of the fraction (*i.e.,* drawing one line for equal partitions).

I have interpreted the classroom incident by Hewitt and Pimm, built on Sfard's view, and proposed the idea of a fraction as a narrative, based on whole-part thinking in the Korean fraction language. The Korean fractional language shows a different understanding of fractions compared to English: Fractions in Korean as a narrative assume equal partitions, which leads to unit fractions as a starting point of fraction concepts. This is how my native language shapes the way I see the fraction world.

Note

[1] Information about the words' meanings and the Chinese origins of Korean come from The National Institute of the Korean Language, . I also give details based on my Korean knowledge as a Korean native speaker.

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From the archives

Geoffrey Howson died on 1 November 2022. He contributed to curriculum reforms in the UK and internationally, was active within ICMI and is well known for his contributions to the history of mathematics education. The following is an edited excerpt from his 1984 article 'The questions remain the same: only the solutions change' in *FLM* **4**(2), 14–17.

Mathematics educators would not in general appear to have either a great knowledge of, or even concern for, the history of their subject. This I believe to be unfortunate for a number of reasons. A real understanding of the position in which we find ourselves today presupposes an understanding of how we arrived at this position. Moreover, this lack of a knowledge of the past can lead to a continual "reinvention of the wheel"—there is a need for a shared stock of knowledge which mathematics educators can take for granted and on which they can build. Yet a facile view of history can lead to the argument that "It has all been said before" and to despair. If we wish to gain help and encouragement from history it is essential that we probe more deeply and study in detail the gradual evolution and elaboration of responses to key problems. (Thus, for example, there is a considerable difference between our present-day understanding of the difficulties of presenting mathematics *via* a text and that of Robert Recorde in the mid-sixteenth century. Nevertheless, Recorde succeeded in identifying some key issues concerned with the writing of texts which will always face the author—and the reader.) Also, it is in such studies that we shall identify that basic "foundation" knowledge. For comparison we note how within the sphere of mathematics proper it is the good expository survey article which in tracing the historical development of a subject effectively defines what has now become the professional's basic knowledge within that particular area. Mathematicians appear to have accepted this fact, and are beginning to recognise how highly the ability to write comprehensive and comprehensible survey articles should be ranked. Mathematics educators have yet to come to terms with the idea.

[…]

As if to reassure us that everything in education has been said or done before, it must be remarked that in the past not a great deal of attention has been paid to the reasons which people might have for learning mathematics, and what place students see for it within their "general education" or their process of "growing up."

Robert Recorde, writing in 1551, clearly distinguished between two types of reader who might use his geometry text. There were those "who study principally for learning" and those who wished to acquire the knowledge, for some purpose or other, but who had "no time to travail [work] for exacter knowledge"[1]. In a sense Recorde roughly distinguishes between those who, to use Mellin-Olsen's (1981) terms, wish to learn because they attach personal *significance* to what is being presented to them, and those who see such knowledge as merely *instrumental* in attaining other, possibly non-mathematical, goals. Recorde, however, was not writing primarily for the school pupil, but for the maturer student. It is highly unlikely that Shakespeare would have chanced upon his works at the school he attended in Stratford-upon-Avon. Yet he, too, had something to say on motivation—or rather the lack of it—when he described "the whining school-boy, with his satchel and shining morning face, creeping like a snail unwillingly to school" [2].

For several centuries motivation was provided within the schools primarily by the use of the birch and rod. There was also the power of expulsion—Arnold, the famed nineteenth-century headmaster of Rugby School explained that, "Till a man learns that the first, second and third duty of a schoolmaster is to get rid of unpromising subjects [*i.e.* pupils], a great public school will never be what it might be and what it ought to be" (Ballard, 1969, p. 29). Today's teachers must smile ruefully at such advice—the "unpromising" cannot be removed from state schools quite so easily. The birch, which served Arnold well but which was never very successful in meeting its metaphorical ends—for often riots erupted in the schools and on occasions had to be quelled by the army—has in the past century been displaced as the prime motivator by the examination. Vast and complicated examination systems have been established, educational ladders have been erected, and meritocracies founded on systems where success has frequently been dependent on passing examinations, usually with a mathematical component. In expanding systems, with respect to both educational opportunities and subsequent occupational rewards, the examination has proved a very powerful motivator indeed. Now, however, that period of historical growth appears to be coming to an end—at least, so far as the West is concerned. A motivational vacuum is developing for many students within those educational systems which offer universal secondary education. For the first time it is becoming imperative to distinguish carefully between two questions. The first, asked by society, is *What can teaching mathematics contribute to a person's general education?;* the second, asked by the student, is *What can learning mathematics contribute to my personal growth?*

Editor's notes

[1] I have been unable to trace this quotation. I assume it is somewhere in the Preface to Recorde's 'The Pathway to Knowledg, containing the First Principles of Geometry' which is unpaginated and difficult to read.

[2] From Jaques' 'Seven ages of man' monologue in 'As You Like It' Act II Scene VII.

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Jeremy Kilpatrick died on 17 September 2022. His contributions to mathematics education are too numerous to list here. Suffice to say he was awarded the 2007 Felix Klein Medal and was a member of the *FLM* Advisory Board for twenty-three years, 1980–2003. The following is an edited version of his 1984 short communication on the theme 'Research problems in mathematics education' in *FLM* **4**(1), 45–46. It was written in response to a request by David Wheeler for suggestions of problems that might be the focus of mathematics education, akin to Hilbert's problems posed in his address to the IMU in 1900.

You undoubtedly know that when the program committee for ICME 4 invited Freudenthal to be a plenary speaker at Berkeley in 1980, their hope and expectation was that he might attempt a Hilbertian list for our field. He didn't, and I've about decided that such a list doesn't make sense for mathematics education since our problems are never clearly defined, let alone solved. Each generation of mathematics educators ends up wrestling with many of the same problems the preceding generations thought they had "solved", and I think that's likely to be a permanent condition of our field, not simply a product of our limited history and our lack of agreed-on criteria for what problems are "solvable". We don't solve problems of mathematics education, we *inter* them. Like Dracula, they come back to haunt us because we never quite manage to put a stake through their heart.

Nevertheless, I have attempted to define three problems that, even if not well expressed or solvable, seem to me to be central to our field. All three problems are at the interface between curriculum and instruction.

The first problem concerns skills and "automaticity of response". One reason mathematics teachers provide "drill and practice" for pupils is that they want the pupils to be able to respond automatically to certain questions (*e.g.,* what is the product of 5 and 9?). The argument is that, when such responses are automatic, the pupil's attention is free for consideration of more complex questions (*e.g.,* do I next add the remaining number or multiply by it?) In the January [1983] issue of the *Journal for Research in Mathematics Education,* Bob Gagné argues that "automaticity of skills" has been undervalued by mathematics educators. One can ask, however, what price automaticity? Les Steffe and Rick Blake, in the May *JRME*, contend that too great a stress on automatic responses is likely to leave pupils confused as to the meaning of what they are doing. It's an old debate—how are "meaning" and "automaticity" to be orchestrated? Should one teach for automaticity and let meaning follow—running the risk of finding what Kath Hart found in the CSMS project with respect to ratio and proportion: "No evidence in this topic of rules learned and repeated with understanding" (*JRME*, March 1983, p. 124)? Should one teach for meaning, and let automaticity follow—as some proponents of the "new math" advocated? Or should one phrase the issue as William Brownell did in the title of his 1956 *Arithmetic Teacher* article: "Meaning and Skill—Maintaining the Balance"? The Hilbertian problem might be posed as follows: *For each skill in the school mathematics curriculum, what level of automaticity is optimal for subsequent use of that* *skill, and how can the skill be made meaningful without inhibiting automaticity?* Behind this problem is the old joke in which the centipede becomes selfconscious about where he puts his feet and then trips himself up. Certain skills need to be brought to consciousness—so they can be understood and controlled more precisely—and then made automatic. We know something about how this might be done in training pilots or coaching athletes; do those principles transfer to the mathematics class?

The second problem concerns the hierarchical view of mathematics learning that many people have adopted—how does it affect mathematics teaching? Jere Brophy, an educational psychologist at Michigan State, was quoted recently as saying that mathematics educators are misguided who believe that, because calculators are so easily available, we can drop from the school mathematics curriculum the multiplication of numbers with more than two digits. Brophy argues that "performance must be perfect on low-level objectives if success on higher-level objectives is to be reasonably expected" (*Notes and News,* Institute for Teaching, Michigan State University, 25 February 1983, p. 3). Leaving aside the question of what it might mean to drop a certain kind of multiplication from the curriculum, let us consider Brophy's argument for putting the low-level spinach before the high-level dessert. Certainly, many teachers of mathematics have bought the morality and good sense of this argument. But where is it written that low-level must or should come before high- level? Zoltan Dienes once broached what he called the "deep-end hypothesis"—the idea that learning might be improved if pupils were thrown in at the deep end of a subject, and compelled to sink or swim, rather than being helped along from the shallow end. The Hilbertian problem might be something like: *What are the effects on learning if instruction is aimed at the attainment of certain "higher-level" objectives given imperfect attainment of related "low-level" objectives?* This formulation of the problem begs the question of how one establishes whether and how two objectives are related. It also neglects the issue cited at the beginning of this paragraph—what are the effects on teaching of this low-level/high-level view of objectives?

The third problem concerns transfer. Everyone knows that Thorndike *et al.* showed conclusively that the study of mathematics doesn't make pupils better reasoners, yet teachers remain convinced of its power to do so. One way to reinterpret Thorndike's research is to suggest that perhaps his instruments were insensitive to certain changes that studying mathematics makes in how pupils think. If the teachers are right and Thorndike wrong, it might be worthwhile to develop more sensitive instruments for measuring reasoning—and other intellectual abilities likely to be affected by mathematics learning. The Hilbertian problem: *What general intellectual abilities are affected by the study of mathematics, and how are they affected?* Stated this way, the problem is too broad to be addressed reasonably, but pieces of the problem might be amenable to attack.

I hope this response to your request for problems hasn't affirmed the old saw that a fool can ask more questions than a wise man can answer. Better, perhaps, is James Thurber's observation: It's better to know some of the questions than all of the answers. Best wishes to you in orchestrating the responses you get.

Heinrich Bauersfeld died on 1 December 2022. He was one of the founding directors of the Research Institute for Mathematics Education at Bielefeld University and a pioneer in including social interactions in mathematics education research. In collaboration with Paul Cobb, Terry Wood and Erna Yackel he sought to integrate psychological and social perspectives on mathematics learning. He was a member of the original *FLM* Advisory Board, serving from 1980–1990. The following is an edited excerpt from his article 'Integrating theories for mathematics education' in *FLM* **12**(2), 19–28.

An attempt at integration

> If the kingpin of cognition is its capacity for bringing forth meaning, then information is not pre-established as a given order, but it amounts to regularities that emerge from the cognitive activities themselves (Varela, 1990, p. 121/66)

The outlined positions are very near to the radical constructivist principle (von Glasersfeld, 1991), as well as to fundamental pragmatic linguists' or social interactionists' theses (Mehan and Wood, 1975; Walkerdine, 1988; Coulter, 1990), to fundamentals of discourse analysis (Cazden 1986), and to certain perspectives of systems theory approaches (Luhmann, 1990; Maturana and Varela, 1986). One cannot expect to identify clear boundaries for the region of convergence at this level of abstractness. But it appears to be possible to enlist a few shared core convictions in this area. (The descriptors used will present a mixture, just because it is impossible to describe the deficient parts of an approach with the specific "language game" of the very same approach.)

3.1 *Learning* is a process of personal life forming, a process of an interactive adapting to a culture through active participation (which in parallel also produces and develops the culture itself), rather than a transmission of norms, knowledge and objectified items.

3.2 *Meaning* lies with the use of words, sentences, or signs and symbols rather than in the related sounds, signs or pictures, or even in a related set of such items.

3.3 *Languaging* (the French term *parole* as distinct from that of *langage*) is a social practice, serving in communication for pointing at shared experiences and for orientation in the same culture, rather than as an instrument for the direct transportation of sense or as a carrier of attached meanings.

3.4 *Knowing* or *remembering something* denotes an actual activation of options from experienced actions rather than a storable, treatable, and retrievable object-like item, called *knowledge*, from a loft, called *memory*.

3.5 *Mathematising* is a practice based on social conventions rather than the applying of an universally applicable set of eternal truths; according to Davis and Hersh (1980), this holds for mathematics itself.

3.6 (Internal) *Representations* are individual constructs, emerging through social interaction as a viable balance between the person's actual interests and her realised constraints, rather than an internal one-to-one mapping of something pre-given or a fitting re-construction of "the" world.

3.7 *Using visualisations and embodiments* with the related intention of using them as didactical means depends on taken-as-shared social conventions in classroom practice rather than on a plain reading or the discovering of inherent or inbuilt mathematical structures and meanings.

3.8 *Teaching* is the attempt to organise an interactive and reflexive process, with the teacher engaging in a constantly continuing and mutual differentiating and actualising of activities with the students, and thus the establishing and maintaining of a classroom "culture", rather than the transmission, introduction, or even re-discovery of pre-given and objectively codified knowledge.

The notion of an "integrating perspective" in the following will refer to this set of core convictions. It is quite challenging to extend such an integrated perspective into possible didactical considerations—no inferences, clearly, since what we have mostly refers to single theories, if anything at all. At least in the interest of students and teachers, such an attempt appears to be as necessary as the bustling discussion of compatibilities, of the drawing of boundaries, and of attempting to decide the relative dominance of one model over another.

[...]

Language, languaging and the teacher

In a narrow interchange with the described attitudes, views about language will also have to undergo change. "Learning how to use language involves both learning the culture and learning how to express intentions in congruence with the culture" (Bruner & Haste, 1987, p 89). And "one has to conclude that the subtle and systematic basis upon which linguistic reference itself rests must reflect a natural organization of mind, one into which we *grow* through experience rather than one we achieve by learning" (p. 88).

For many teachers the strength and the generalisability of mathematics is inseparable from the strictness and the precision of the related verbal or other symbolic representations. Similar to priests who celebrate the esoteric language game of their caste, many mathematics teachers permanently insist on saying things as precisely as possible. An observer may find the teacher insists on this technical language. For the students, the emphasis functions as a requirement to say it exactly "as she/he said it".

One may wonder whether many teachers "have it" at all in any other way. That is to say, they know how to talk about "it" in the terminology of the accepted language game but there seems to be not much more beyond this, as the limited availability in other "contexts", the difficulties and shakiness of use in other situations suggests, and is also indicated by an inability to find adequate metaphors for the issues.

To be fair, nobody has trained them in the initial phases to speak about the intended subject matter in everyday language, to "point at" similar issues, *etc*. (Cognitivists may prefer descriptions like: they cannot "translate", or "say it in other words", they cannot "embed" or "visualise", or "refer it to", thus treating what is meant as an object rather than as something emerging from the actually situated processes.) In consequence, many mathematics teachers are quite rigid in their *verbal* aspirations and their related evaluations of students' utterances. But they are quite permissive in the *social* organisation of their class. Under the integrating perspective the opposite way round appears to be a more promising one: to accept and encourage students' mathematical utterances within very wide limits with respect to how it is said, as long as a serious background (reason, argument, *etc*.) can be identified. But to be rigid about keeping the social regulations, namely, insisting on listening to others' inventions and explanations, keeping turn-taking order, taking seriously the others' serious contributions, *etc*.

Analysing many videotapes has convinced me of the all-too-general poverty of classroom communication with respect to this view (in many countries, by the way). If the culture the students inhabit in the classroom is poor in languaging and in presenting models of what is wanted, if it is lacking incentives and challenges, if it is more a non-transparent celebration of technical language rather than a participation in a scaffolding culture, and if it is neither providing resistance to the critical mind nor further orientation for the keen-minded, what then are we to expect from our schools?

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Hysteresis as an authentic mathematics application

AMENDA N. CHOW

Integrating authentic applications in mathematics teaching is an important part of student learning because it supports classroom participation, engagement with assessments and greater retention, which leads to an overall increased interest in the subject (Campbell, Patterson, Busch-Vishniac & Kibler, 2008). To support this, I suggest a real-world practical application called 'hysteresis' as a motivation for learning a variety of mathematics topics. Hysteresis is well-known in engineering and physics because of its connections to physical processes; however, it is not normally discussed

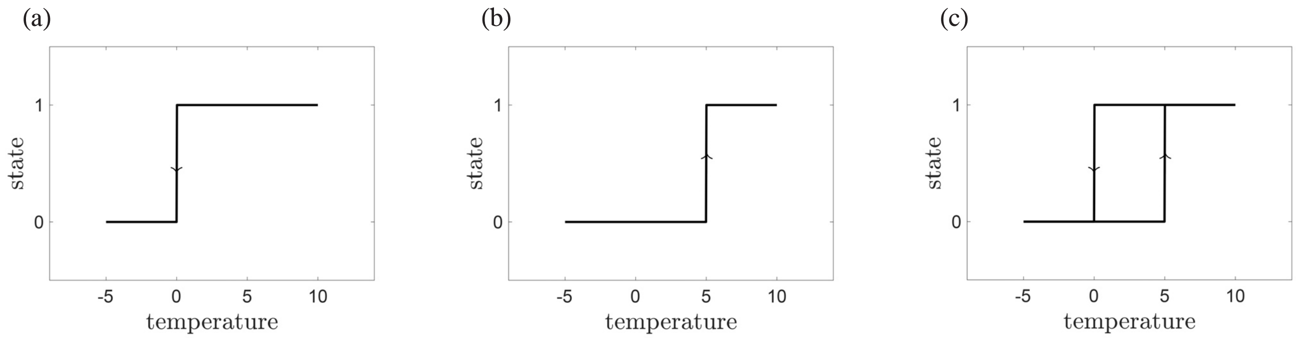

Figure 1. A thermostat with 0 as its off state and 1 as its on state, and its state dependent on the temperature. (a) The thermostat switching from on to off when the temperature is 0. (b) The thermostat switching from off to on when the temperature is 5. (c) The dynamics shown in (a) and (b) plotted together.

in mathematics education. Hysteresis provides an example that is not a function, and such 'non-examples' are often neglected in the context of teaching functions.

An introduction to hysteresis

Consider the dynamics of a thermostat, which is either in a state of being off or in a state of being on. Let 0 represent its off state and 1 represent its on state. In this example, consider the thermostat switching on or off based on the temperature of a refrigerator. Suppose the thermostat turns the refrigerator off if the temperature is less than 0 degrees Celsius. This behaviour is shown in Figure 1a. On the other hand, suppose the thermostat switches from off to on when the temperature is 5 degrees, as depicted in Figure 1b. This means the path from off to on is different from on to off; that is, the dynamics of the thermostat are *path dependent.* This path dependence creates a loop as shown in Figure 1c. This loop is known as a *hysteresis loop,* and we say the performance of the thermostat exhibits hysteresis.

In the case of the thermostat, the presence of hysteresis is a benefit. Consider Figure 1a, which without the presence of hysteresis would mean any slight temperature change above or below zero causes the thermostat to frequently switch off and on. This would quickly wear down the components of the thermostat, and hence, the presence of hysteresis improves the performance and quality of a thermostat. Furthermore, in Figure 1c, the state of the thermostat may be 0 or 1 for temperature values between 0 and 5. That is, for a temperature between 0 and 5, there are two possibilities, namely 0 and 1, so this is not a function. Instead, in the presence of hysteresis, the state of the thermostat is determined by knowing whether it was off or on previously. This dependence on the past is known as the *memory effect* of hysteresis.

In the example of the thermostat, hysteresis appears in a human-made device. Hysteresis is more often observed in natural processes such as freezing-thawing, magnetism, population dynamics, potential energy, and ecosystem changes (Aiki & Minchev 2005; Berdugo, Vidiella, Solé & Maestre, 2022; Morris, 2011; Mukhamadullina, Kornev & Alimov, 1998; Noori, 2014). Since hysteresis is a phenomenon that occurs in physical systems, it is commonly modelled by a differential equation, where $x(\mathrm{t}) \in \mathbb{R}$ is the solution to the differential equation,

$$\dot{x}(t) = f(x(t), u(t))$$

$t \in \mathbb{R}$ is time, and f is a continuous and differentiable mapping of x and u. In the context of determining whether a system exhibits hysteresis, $u(t) \in \mathbb{R}$ is the input of the system, and it affects the behaviour of $x(t)$, which is called the output. The relationship between the input and output is given by the equation above. The plane used to depict a hysteresis loop is an input-output graph as shown in Figure 2. The horizontal axis has been labelled 'input' and the vertical axis has been labelled 'output', but they can also be symbolized by u and x, respectively. The specific values of the input

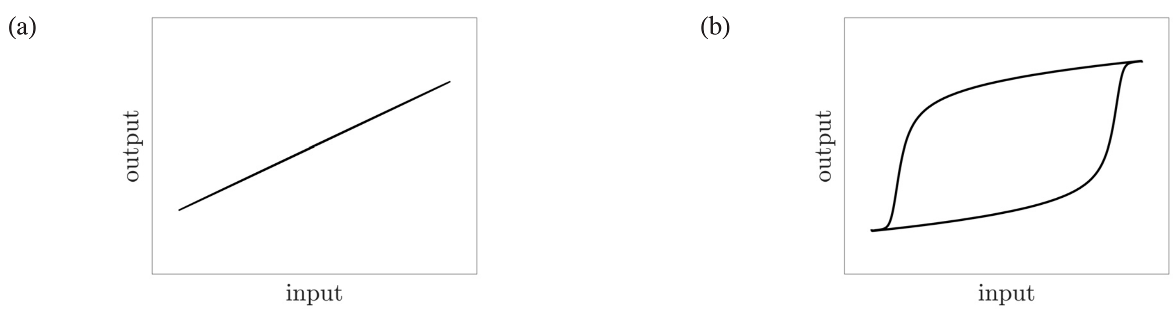

Figure 2. (a) Input-output graph displaying no looping behaviour. (b) Input-output graph displaying a hysteresis loop.

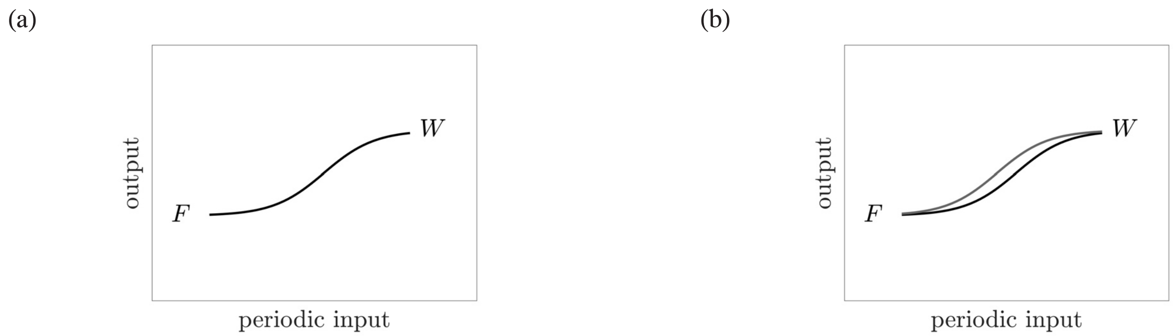

Figure 3. For the process of freezing and thawing, the depiction in (a) illustrates if freezing and thawing are processes that are exactly the reverse of each other, and (b) shows the case when they are not.

and output do not add to the understanding of the discussions presented, so numerical scales along the axes have been omitted.

The thermostat example is discrete in that values of its output are either 1 or 0, and hence not a suitable solution to the equation above, which has continuous solutions for $x(t)$. For a continuous example, consider the process of water freezing and thawing. We can tell how frozen water is by measuring its flow rate, which varies continuously. The output is the flow rate of the water, and this depends on the temperature, which is the input. The input-output graph of these dynamics is shown in Figure 3a if freezing is exactly the reverse of thawing. In other words, the frozen object as it is thawing has initial frozen state labelled by F, and this state (measured by the flow rate of the water, which is the output) changes as the input (*i.e.,* temperature) increases until reaching its final thawed state denoted by W, and reversing these dynamics, freezing is the same curve in Figure 3a but from W to F if freezing is exactly the reverse of thawing.

It has been shown that freezing is not exactly the reverse of thawing in Mukhamadullina, Kornev & Alimov (1998). This may be due to external factors such as different evaporation rates of water between freezing and thawing, and the fact water expands when it freezes. As freezing is not exactly the reverse of thawing, the curve from F to W for thawing cannot be used to represent freezing from W to F. Figure 3b depicts the difference from F to W (in black) as compared with from W to F (in gray). These dynamics result in a hysteresis loop, and we say the freezing-thawing process exhibits hysteresis. In the examples of freezing and thawing, and the performance of the thermostat, there is a repetition between hot and cold temperatures that triggers each process to cycle over time. This repetitive nature is needed to test for hysteresis in a physical system, and consequently, the input is a periodic function (*e.g.,* $u(t)$ = sin(t) or u(t) = 0.1cos(2t)). Additional details about input-output graphs for hysteresis loops can be found in Morris (2011) and Oh & Bernstein (2005).

Despite the differences in shapes of hysteresis loops observed in Figures 1c, 2b and 3b, the structure of all hysteresis loop has the appearance of one path *lagging* behind another. This observation led to the name hysteresis because the etymology of hysteresis means to lag behind (Morris, 2011). Path dependence, the memory effect and lagging offer several ways to characterize hysteresis, and they inspire the following colloquial definitions for hysteresis.

> Hysteresis is a process that follows a different path forward than backward when the process is reversed. This implies hysteresis is a process that exhibits path dependence.
>
> Hysteresis is a phenomenon that depends on its past behaviour to determine its current behaviour. In this context, we say hysteresis has a memory.
>
> Hysteresis describes a system whose output lags behind itself as the system input changes.

An application for non-functions

A typical discussion of non-functions is usually either the graph of a circle or an arbitrary curve failing the vertical line test, which has limited connections to real-world applications, and the main focus is usually about what *is* a function. Hysteresis loops have one particular value of the input leading to two possible values of the output. While hysteresis is a real application that cannot be modelled by a function, it is important to point out that in the exploration of hysteresis, the concept of a function is used. All the graphs in Figures 1a, 1b and 3a can be described by a function. These graphs of functions lead to the construction of the hysteresis loops shown in Figures 1c and 3b.

Inverse functions describe reversible processes (*e.g.,* adding five is precisely reversed by subtracting five), while hysteresis is an application that motivates functions without inverse functions. Hysteresis loops are visual representations of physical processes that are not exactly reversible as discussed in the examples of freezing-thawing and thermostat switching. In the case of the thermostat example, the presence of hysteresis is a benefit; it keeps the thermostat from switching too quickly.

An application for the geometry of curves

Exploring the geometry of curves can be inspired by physical applications. Hysteresis loops can be an example for this. The previous discussions explained why closed curves (*i.e.,* loops) form in the input-output graphs of systems that exhibit hysteresis. Within this, there is quite a bit to investi-

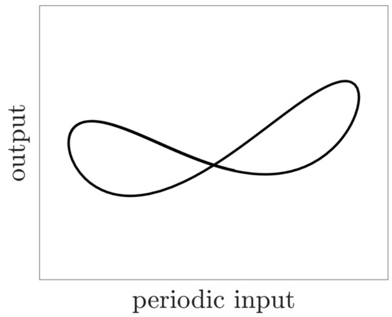

Figure 4. *A pinched hysteresis loop, which is in the shape of a closed curve that is not simple.*

gate about the shape of hysteresis loops. For instance, self-crossing pinched hysteresis loops, as depicted in Figure 4, are closed curves that are not simple. These type of hysteresis loops arise in engineering systems such as circuits and smart materials (Drinčić, Tan & Bernstein, 2011; Wang & Hui, 2017). The pinched hysteresis loop in Figure 4 appears to exhibit some symmetry along its self-crossing; however, this may not always be the case and hence symmetry is another avenue for exploration (Wang & Hui, 2017). By considering the physical explanation of self-crossing in hysteresis loops, it becomes a motivation for why it is worthwhile to understand geometric concepts like symmetry and distinguishing simple curves from non-simple ones.

Additional suggestions for applications

What has been presented in this article is a brief and modest discussion of hysteresis. There are many more facets to hysteresis worth considering, such as using hysteresis as a motivational example to examine the use of mathematics in computer programming. All the figures in this article are created from scripts written in MATLAB, and this requires mathematical knowledge of time scales, graphing and differential equations. Of course, using other software is possible and avenues for learning include writing code to generate hysteresis loops.

Another application is using hysteresis loops to inspire learning about differentiability and the appearance of their corresponding graphs since some hysteresis loops are smooth (*e.g.,* Figure 4) while some have cusps (*e.g.,* Figure 2b). Other topics that can be investigated more deeply are defining hysteresis, modelling hysteresis in differential equations, historical perspectives of the first observations of hysteresis, and reflecting on the use of authentic practical applications as a tool for learning mathematics.

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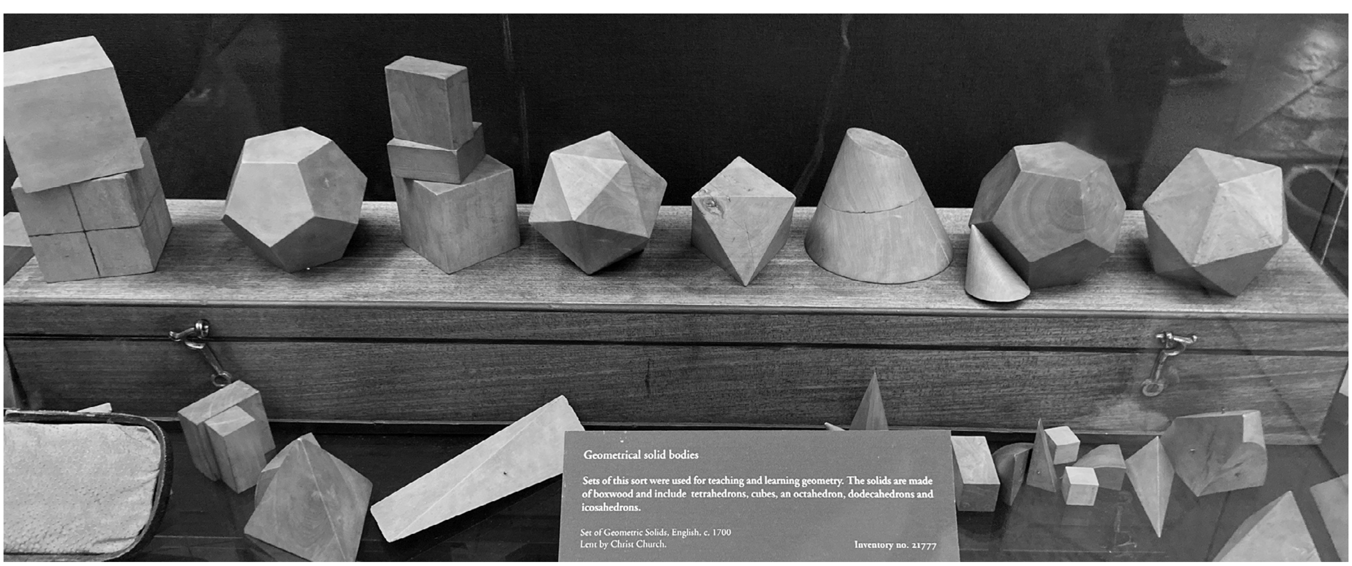

Set of geometric solids made of boxwood, English, c. 1700. History of Science Museum, Oxford.