Some Reflections On and Criticisms of the Principle of Learning Concepts by Abstraction

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In mathematics education literature the process of abstraction appears to play a privileged role in explaining concept formation. Mathematics has often been described by both mathematicians and mathematics educators as a discipline built mainly through successive levels of abstraction. This philosophical belief, supported by evidence from psychological studies on learning processes, has had considerable influence on many of the new methods of teaching mathematics suggested since the late fifties. Theoretical support and extensive applications for what we call in this paper the "learning by abstraction" approach, has been especially provided by Dienes.

Although the fundamental role of abstraction in mathematics and its learning is undeniable, we would like to discuss in this paper the limitations of an exclusive application of the principle of learning concepts by abstraction. Using ideas originating from different fields and analyzing the specific case of the mathematical concept of infinity, we will argue that perhaps such an approach is not suitable for every topic in mathematics. In some cases it could even obscure the different facets of a concept and therefore prevent a full understanding of it, or cause difficulties in its application.

Given the difficult and complex nature of the topic, we do not expect to reach definite answers with this paper, but rather we aim to provide material for reflection by teachers and mathematics educators, and to stimulate further research for the development of alternative approaches to concept formation in mathematics.

An exemplification of the "learning by abstraction" rationale

To describe in more detail the rationale of "learning by abstraction" we will refer to Dienes' work, as it may be considered the most representative of such an approach.

Dienes' psychological studies on how children learn mathematics were strongly influenced by his perception of the nature of mathematics. He describes it as a discipline created through the centuries by means of successive abstractions, starting from the observation of regularities in the real world. The term abstraction is used here in a very technical sense:

> An abstraction is a class of situations and each situation belongs to this class because of a certain property it has and any other properties it might have are, for the moment, considered as irrelevant. [1]

For instance, let us consider the concept of "square." According to Dienes, it is constituted by the properties that are common to all geometrical plane figures with four equal sides and angles. Characteristics like size, colour or position are treated as irrelevant in this case.

Dienes picks out abstraction not only as the main characteristic of mathematics, but also as the key point in the learning of mathematics. It involves the first three stages in which the author subdivides the process of learning a mathematical concept [2]:

1) Free play with situations in which the concept itself is concretized
2) Play structured games in the situation, in order to recognize and use basic properties and rules
3) Make comparisons among the situations, in order to recognize the elements common to them all, and therefore to abstract the concept itself
4) Find representations of the new concept in order to better identify and fix it.
5) Develop a symbolism to express the properties discovered.
6) Formalize the results obtained in a rigorous and possibly axiomatic way.

Educators should help the learner to successfully overcome each stage, with special attention given to the first three stages that involve the process of abstraction. For this purpose, Dienes created structured materials and educational activities based on the following pedagogical principle:

> Play as many variations as possible in different media of the same conceptual theme. This can be done by providing tasks which "look" quite different but have essentially the same conceptual structure. In other words, we can vary the perceptual representation, keeping the conceptual structure constant. [...] Children learn what is in common among these different representations, and it is this common feature which "is" the mathematical concept. [3]

For example, Dienes suggested that the concept of the group of two elements (very important in mathematics—for example, it is the basis of the sign rule in the multiplication of signed numbers) be introduced in many different situations like:

- rotations by 180 and 360 degrees of a square
- turning on and off a light-switch
- moves such as standing still and turning back.

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Role of abstraction in different areas of mathematics

It is interesting to notice that Dienes and other supporters of the principle of "learning by abstraction" mainly developed their examples or projects in the following areas of mathematics:

- arithmetical and algebraic concepts and rules
- algebraic structures (groups, vector spaces, etc.)
- elementary transformation geometry
- propositional calculus

In these specific areas, "abstraction" as described by Dienes plays a fundamental and explicit role, and his six stages appear to be a natural approach to concept formation. However this may not be the case in all areas of mathematics. Elementary calculus and number theory may be considered as examples. During their history a rapid progress through levels of abstraction was never a primary objective, and has occurred only in recent times. Both of them developed rather through an in-depth study of a specific situation—functions of real variables and natural numbers, respectively. In elementary calculus we could argue that many concepts were not so much created by noticing properties common to diverse situations but were rather motivated by potential applications in the real world or in other sciences. For example, consider the concept of derivative and its relation with the pre-existing notion of velocity in physics. It is also interesting to notice that, even if the concept of natural number can be rigorously defined by abstraction today (in terms of the property common to all sets that can be put into one-to-one correspondence with each other), such a result is due to Cantor who developed his set theory more than two thousand years after mathematicians started to study and discover the properties of the numbers 1, 2, 3,... The concept of infinity could also provide a different example of a mathematical notion in which abstraction does not appear to be the most suitable approach. We will discuss this case in detail later in the paper.

The historical development and the nature of certain topics seems therefore to warn against an extensive use of the principle of learning by abstraction and invites us to consider alternative approaches.

Limitations of concentrating only on the "common properties"

In his philosophical analysis of concepts expressed by words in ordinary language, Wittgenstein pointed out that it is not always the case that instances of a specific concept need to share some common properties:

> Consider for example the proceedings that we call "games." I mean board-games, Olympic games, and so on. What is common to all of them? — Don't say: "There must be something in common, or they would not be called 'games' " — but look and see whether there is something in common to all. — For if you look at them you will not see something in common to all, but similarities, relationships, and a whole series of them at that. [...] I think of no better expression to characterize these similarities than "family resemblances" for the various resemblances between members of a family: build, features, colour of eyes, temperament, etc. etc. overlap and criss-cross in the same way. — And I shall say: "games" form a family. [4]

Even if it may be difficult to find a mathematical counterpart to concepts such as "games," Wittgenstein's observations may have interesting consequences from an educational point of view. Properties that belong to some but not all instances of a concept could be very useful for further understanding and developing it. Their consideration could lead to interesting and valuable results in spite of their lack of generality. They might even open new areas of mathematical research by using different definitions and assumptions, and sometimes even suggest the opportunity to modify the original concept.

Concentrating only on the "common properties" may also have negative consequences for the applications of a concept. We can find support for this opinion in one of Kline's criticisms of the "new math" movement:

> Students are asked to learn abstract concepts in the expectation that if they learn these, the concrete realizations will be automatically understood. Thus, if a student learns [...] what a field is, he will know all about the rational numbers, the real numbers, the complex numbers and other mathematical objects. [5]
>
> Would he be able to add fractions, irrational, complex numbers? No. And the reason is that just because the field properties are "common" to all these systems of numbers, they automatically wipe out any distinguishing features. The field concept wipes out just those important processes that are needed to operate with these basic number systems. [6]

Thus the in-depth study of specific instances or models may play a fundamental role in the acquisition of mathematics and at least in some cases should not be superseded by a set of learning by abstraction principles.

Psychological evidence conflicting with the principle of learning by abstraction

A critical point in Dienes' theory of learning is the necessity of considering some properties of the instances examined as "irrelevant" and therefore to be disregarded once the abstraction is completed. It seems unrealistic to believe that they can really be forgotten and have no influence in our dealing with the concept later. Recent research conducted by Tall and Vinner [7] on the process of learning mathematics has produced theoretical support and experimental evidence to substantiate this criticism.

First of all, the authors warn against confusing the "concept definition"—i.e. the formal, rigorous wording of a concept in a mathematical theory—with the "concept image" that the individual learner builds in his or her own mind. The "concept image, " in fact, consists in:

> The total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up

> over the years through experiences of all kinds, changing as the individual meets new stimuli and matures [...] All mental attributes associated with a concept, whether they be conscious or unconscious, should be included in the concept image. [8]

Let us apply these notions to the case of a concept reached by abstraction. We can then see that while the concept definition must contain only the properties common to all the instances considered, the concept image is likely to retain other elements pertaining to those instances which may consciously or unconsciously be used when dealing with the concept, with both positive and negative consequences. In fact, Tall and Vinner also observed that, as not all the elements constituting the concept image need to be evoked every time we make use of it, the concept image itself may be incoherent even without our realizing it. As long as these conflicting aspects are kept unconscious they may cause even more confusion and difficulty for the learner.

It is interesting to notice that Tall and Vinner avoid discussing the "content" of a concept in their theory. Given their result that demonstrates the distinction between concept image and concept definition, we are still left with the problem of identifying the relationships among these three notions. Supporters of learning by abstraction seem to identify concepts with the properties common to all their instances, which generally correspond to the rigorous definition. Recognizing the importance of the concept image from Tall and Vinner's results, we suggest that such a narrow interpretation of a concept may not be very desirable and other alternatives should be considered and studied. At the very least, as teachers we need to attend carefully to the distinction between the concept that the student may have acquired and the mathematical concept that in some sense exists as an abstraction.

We would like now to apply some of the previous observations to the specific case of the mathematical concept of infinity. In order to deal with a rigorously defined mathematical notion we decided to restrict ourselves to the more precise concept of infinite set rather than the vaguer one of infinity. Even with these limitations, such a concept appears to be especially suitable for the analysis of possible shortcomings of an abstraction approach because of its complex and contradictory nature.

Common instances of infinity and their characteristic properties

With the goal of identifying the characteristic properties of infinite sets we will start by considering some of its well-known instances. Even if we operate only in elementary areas of mathematics we cannot avoid encountering infinite sets like:

a) the whole numbers, $W = \{1,2,3,\ldots\}$
b) the square numbers, $S = \{1,4,9,\ldots\}$
c) the integers, $Z = \{\ldots,-2,-1,0,1,2,\ldots\}$
d) the rational numbers, Q
e) the real numbers, R
f) a line of points
g) a plane of points
h) three dimensional space
i) the points of a segment
j) the points of a square.

From consideration of these examples we can identify at least the following properties related to infinity:

I) Each of the above sets is non-finite in the sense that it can never be fully described by enumerating all its constituting elements, but only by means of a generating process that can go on "forever."

II) In most of the cases (a-e) the infinite object is unbounded. That is to say, we cannot identify a first and/or a last element. This is certainly a quality that only instances of infinity can have, but not all of them necessarily do. Segments and squares provide us with simple counterexamples.

III) While some sets consist of "isolated" elements (for example, sets a,b,c are such that there is not necessarily an element of the set "between" every pair of distinct elements), others (d-j) share the amazing property that between any two of the elements there is always another one. A consequence of this property—known as "density"—is that in such sets we can find pairs of elements as close together as we want.

IV) Even among dense infinite sets we can distinguish those for which elements are so close that "no gap exists" (a-j) from those that can still be increased by inserting extra objects in between their elements (as in the case of the natural and rational numbers, but not in the case of the real numbers). This property seems strictly related to the notion of "continuity."

V) In the geometrical examples it seems important to consider the "dimension" of the geometrical figure in question. We have in our list examples of figures of one (f, i), two (g, j) and even three (h) dimensions.

VI) All the examples considered share the possibility that the set can be put into one-to-one correspondence" with some of its proper parts.
Examples:
— N with its subset consisting of the square numbers S:

$$\begin{array}{c}\{1,2,3,\ldots,n,\ldots\}\\ \updownarrow\updownarrow\updownarrow\quad\updownarrow\\ \{1,4,9,\ldots,n^2,\ldots\}\end{array}$$

— a segment AB with a part AC of it, as suggested by the following diagram:

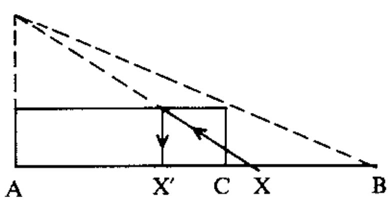

Illustration of some limitations of a "learning by abstraction" approach in this case

If we intend to isolate the concept of infinite set by abstraction, we should now consider only those properties that are common to all instances, i.e. properties I and VI. That was the choice made by Cantor in order to overcome apparent contradictions inherent in the concept of infinite set and to rigorously develop a mathematical theory for infinite sets. While such an approach—overwhelmingly supported in mathematical literature—may be appropriate with regard to the concept definition of infinity, in our opinion it fails to reflect the complexity and richness of its concept image. In the following paragraphs we will present some mathematical and psychological reasons that make us believe that it is not advisable to disregard properties II-V just because they are not shared by some instances.

We notice that, among all the properties of infinity analyzed in the previous section, only V could be found also in finite sets. This fact underscores the importance of properties II, III, IV, as in all likelihood the study of properties unique to infinite sets would only serve to further our knowledge and understanding of the concept of infinity.

Reflections on property II, as well as I, can give us ideas about how we can represent some infinite sets: with a generating rule (a-d) or a "boundary" (i-j).

Even if some infinite sets are not dense or continuous it may be very important to study and use these properties as most of calculus and geometry have been built by virtue of them, and in these areas we can find some of the most useful applications regarding the concept of infinity.

Besides, we must acknowledge that all properties I-VI are part of our original intuition of infinity and can influence our operation with the concept. Recent studies conducted by Fischbein [9,10] with grade 5-9 students produced empirical data that seem to confirm this hypothesis. For example, when asked to compare the number of points of a segment with those of a square, and the number of points of a segment with those of a line, the subjects almost split in half with regard to positive and negative answers, providing explanations like:

- The points of a segment and the points of a line (or square) can be matched "because there is an infinity of points in both."
- "The number of points of the line is bigger than the number of points of the line segment. The line is continuing to infinity and the segment is limited."
- "A line segment, no matter its length, is part of a line and for that reason there are less points on the line segment."
- The correspondence between square and line segment is not possible because the former is bi-dimensional and the latter uni-dimensional.

These results appear even more striking if we consider that most of the subjects had been more or less explicitly taught the one-to-one correspondence criterion to compare the number of elements in two sets, at least in the finite case.

Conclusions

We have tried to illustrate, using the concept of infinity as a specific example, how teaching mathematical concepts by abstraction may not be the best approach on all occasions.

Questioning the validity of the process of abstraction as the definite approach to concept formation, even in a discipline as supposedly formal as mathematics, first of all opens some difficult and intriguing questions with regard to the notions of "concept" and "definition." Perhaps we are in need of a new logic of the concept of definition in mathematics that would soften the need for intersection as the major criterion of "belongingness." Furthermore, once we challenge the fundamental assumption that a particular concept is the collection of properties common to all its realizations, then we are left with the question of what is the relation between the concept and the properties of its realizations, and between the concept and its formal definition.

Besides raising these philosophical issues the criticisms brought to the principle of learning concepts by abstraction can have a direct impact on teaching. Mathematics educators are challenged to identify elements and processes that could help in creating operative and useful "images" of the concepts to be taught, and to devise alternative or complementary approaches to abstraction. According to our previous analysis, we may suggest the consideration of at least the following activities:

- Choose some specific examples as "case-studies" and analyze them in depth, considering also applications.
- Compare several instances of the concept with the goal of isolating interesting properties without the constraint of focusing only on common ones, at least in the beginning.
- Reflect on how these properties may differently contribute to:
    - representations of the concept
    - a rigorous definition of the concept
    - applications involving the concept.
- Try to make explicit conflicting factors that may be part of the concept image, at least when this does not require the introduction of complex material.
- In the analysis of the concept itself, stimulate the creative activity of generating alternative and related concepts, and isolating interesting subconcepts.

They are by no means the only ones possible, and we hope that other useful suggestions will come out by reflecting on the issues raised in the paper.

Finally, we would like to insist once more that we are not discounting the use of abstraction in mathematics education. We are only suggesting that we need a certain degree of flexibility in dealing with different mathematical topics. In this regard we are arguing that several approaches to concept formation should be devised and taken into account, and applied in teaching according to the nature of the specific concept being taught. Research that will determine what features of a concept it might be appropriate to take into account in each case is still needed, and it would be of great help for a more effective teaching of mathematical concepts.

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Notes

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2. Z.P. Dienes, *Six Stages of Learning*. N.F.E.R., London, 1971
3. Z.P. Dienes, *Building up mathematics*. London, Hutchinson, 1960, pp. 43-44
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7. D. Tall, S. Vinner, Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity *Educational Studies in Mathematics*, 1981 pp. 151-169
8. ibid., p. 152
9. E. Fischbein, D. Tirosh, P. Hess, The Intuition of Infinity *Educational Studies in Mathematics*, 1979, pp. 3-40
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Thus, at first, people studied circles empirically, amassing many items of information about them (e.g. that the circumference is approximately 22/7 times the diameter, that certain chords are related to the diameter in certain ways, etc.). This may be called the stage of associative thinking; for the circle is treated simply by associating together a large number of remembered properties that were discovered empirically.

At a certain stage, however, a new geometrical theory was developed and a circle was regarded as a curve traced by a point moving equidistant from a fixed point, while a straight line was regarded as the result of moving a small straight segment in its own direction, etc. Later, when this geometrical theory is explained to a particular individual, at first he tries to bring together the various parts of the argument. Then suddenly he says "I see", meaning by this that he *understands*. Of course, this does not mean that one immediately knows the whole set of properties of a circle exhaustively and in full detail. Rather, it means that one sees the essential process by which the circle is generated, along with its various parts and aspects, which are now treated as sides of this totality, so formed that they are automatically in their proper relationships. In this way, one not only grasps the essential character of *each* circle, but also of *all* circles.

David Bohm

To a scientific mind, all knowledge is an answer to a question. If there has been no question, there can be no scientific knowledge. Nothing is self-evident. Nothing is given. Everything is constructed.

Gaston Bachelard
